LINEAR ALGEBRA WITH APPLICATIONS

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PREFACE

This book is suitable for either a sophomore-level course or for a junior- or senior-level course. The student should have some familiarity with the basics of differential and integral calculus. This prerequisite can be met by either one semester or two quarters of elementary calculus. If the text is to be taught at the sophomore level, one should probably spend more time on the earlier chapters and omit many of the sections in the later chapters. On the other hand, if the course is to be taught on the junior or senior level, the instructor should probably spend less time on the early chapters and cover more of the later material. The explanations in the text are given in sufficient detail so that students at either level will have little trouble reading and understanding. To further aid the student, a large number of examples have been worked out completely. Applications have been scattered throughout the book rather than put together in a chapter at the end. In this way they can be used to motivate new material and to illustrate the relevance of the material that has just been presented. When applications are included at the end of a book, they are more likely to be omitted because of lack of time.

WHAT'S NEW IN THE SECOND EDITION?

During the past six years, the author has taught from the book nine times. He has had ample opportunity to carefully review and revise each section. Many improvements have been made in order to make the second edition more accessible to students.

The following are some of the major changes in the new edition.

1. All of the exercise sets have been expanded, some more than doubled. There are many new exercises at all levels. The exercises have generally been ordered so that the more routine problems are presented first and the more difficult theoretical exercises follow afterward.
2. There are more worked examples and more applications. Abstract concepts such as linear independence (Chapter 3) and similarity (Chapter 4) are introduced by worked examples. All of the applications in the first edition have been retained and additional applications have been included that show how to use linear algebra in cryptography, digital imaging, and numerical integration.
3. Chapter 4 on linear operators has been expanded from three sections to four. Much of the chapter has been rewritten. The subject is presented from a less abstract point of view. There are many more worked examples.
4. A discussion of the matrix exponential and its application to solving systems of linear differential equations has been added to Section 3 of Chapter 6. This is the student's first exposure to matrix functions and I believe it will be well received.
5. The discussion of positive definite matrices in Chapter 6 has been greatly expanded. Indeed, all of the material on positive definite matrices in the first edition has been moved up from Section 6 to Section 5. The new Section 6 deals with the properties of positive definite matrices in much more depth. Positive definite matrices arise in many modern applications of mathematics. This is an important subject; it deserves the expanded coverage.

GUIDE TO THE CHAPTERS

The book gives fairly complete coverage of a very broad subject. Consequently, there is probably more material included than can possibly be covered in a one-quarter or one-semester course. The instructor then has some freedom in the choice of topics and can design the course to meet the needs of the class. Some instructors may decide to emphasize the mathematical theory in Chapters 3 and 4 and others may decide to skip over these chapters and spend more time on the applied topics in Chapters 5 and 6. As a general rule, if you ask n mathematicians what should go into a course, you will get n different answers. Even if many of the topics in the book are omitted, the students should get a feeling for the overall scope of the subject matter. Furthermore, many of the students may use the book later as a reference and consequently may end up learning many of the omitted topics on their own. Sections designated by an asterisk in the table of contents are optional. These sections are not prerequisites to any of the following sections—they can be skipped over without any loss of continuity. The following is a guide to the various chapters in the book.

Chapter 1. The first two sections deal with systems of equations and the last three sections are concerned with matrices and matrix algebra. The block multiplication introduced in Section 5 is not used in the book until Chapter 6. Consequently, in sophomore level courses it may be possible to skip this section.

Chapter 2. This is a short chapter on determinants. The concept of the determinant is introduced by first studying the 2×2 case and then generalizing. In the first section the standard definition using permutations is presented. Those preferring the cofactor definition can skip over that material and go directly to the next subsection, where the cofactor expansion is presented. The properties of determinants presented in Section 2 can all be derived from the cofactor definition. Some instructors believe that Cramer's rule generally receives more attention than it merits. Therefore, Section 3 has been listed as optional.

Chapter 3. The basic theory of vector spaces is presented in this chapter. All five sections should be covered.

Chapter 4. Chapter 4 is devoted to linear transformations. Instructors wishing to present a more applied course may omit all or part of this chapter.

Chapter 5. This is a long chapter on orthogonality and its applications. Most of Sections 1, 2, 3, 5, 6, and 7 should be covered if at all possible. Section 4 on matrix norms could probably be omitted if one is pressed for time. This section did not receive an asterisk, since it is a prerequisite for Section 7 of Chapter 6 and the last five sections of Chapter 7. The last section is an optional section on orthogonal polynomials. Orthogonal polynomials play an important role in many areas of mathematics, but they never seem to get the attention they deserve in the standard courses.

Chapter 6. This chapter treats one of the most important subjects in linear algebra, eigenvalues. Sections 1 and 3 should definitely be covered. Section 2 presents one of the main applications of eigenvalues. If the instructor does not wish to cover the entire section, we recommend that the material through Example 1 be presented. Sections 4, 5, and 6 should be covered in junior or senior level courses but are optional for sophomore-level courses. Section 7 is optional.

Chapter 7. In a sense this whole chapter is optional, since in many cases there will not be enough time left in the semester to cover it. This is unfortunate because it may well be the most important chapter for students planning to work in industry. If there is only time to cover one topic, I would strongly recommend Section 7 on the singular value decomposition. This section is included in Chapter 7 because of its importance to numerical linear algebra, but it could just as well be covered in Chapter 6. Part of Section 9 is theoretical and could also be covered earlier, along with Section 7. Instructors who want to integrate numerics into the course may consider teaching the first three sections of this chapter immediately after completing Chapter 1. Section 4 could then be taught with the section on matrix norms in Chapter 5.

SUGGESTED COURSE OUTLINES

I. Two-Semester Sequence

In a two-semester sequence it is possible to cover all 41 sections of the text. I generally average about two days per section and only cover 35 sections. I omit the optional sections in Chapters 5, 6, and 7. I also skip the first three sections of Chapter 7, as they are usually covered in numerical analysis courses.

II. One-Semester Course

In a one-semester course most of the sections can be covered in one lecture. The longer sections require two lectures.

A. Sophomore-Level Course

Chapter 1	Sections 1–4	6 lectures
Chapter 2	Sections 1–3	3 lectures
Chapter 3	Sections 1–5	8 lectures
Chapter 4	Sections 1–4	5 lectures

Chapter 5	Sections 1, 2, 3, 6, 7		8 lectures
Chapter 6	Sections 1, 2, 3		5 lectures
		Total	35 lectures

B. Junior- or Senior-Level Courses
The coverage in an upper division course is dependent on the background of the students. Below are two possible courses with 35 lectures each.

Course 1

Chapter 1	Sections 1–5	6 lectures
Chapter 2	Sections 1, 2	2 lectures
Chapter 3	Sections 1–5	6 lectures
Chapter 5	Sections 1–7	10 lectures
Chapter 6	Sections 1–6 Section 7 if time allows	9 lectures
Chapter 7	Section 7 Part of Section 9 if time allows	2 lectures

Course 2

Review of Topics in Chapters 1–3		4 lectures
Chapter 4	Sections 1–4	4 lectures
Chapter 5	Sections 1–7	10 lectures
Chapter 6	Sections 1–6 Section 7 if time allows	9 lectures
Chapter 7	Sections 4, 5, 7, 8, 9 If time allows, Sections 1, 2, 3	8 lectures

SUPPLEMENTARY MATERIALS

The Solutions Manual contains complete, worked-out solutions to all of the nonroutine exercises in the book. The manual also contains answers to all elementary exercises that were not already listed in the answer key in the textbook.

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adopted in the second edition, and I believe they have resulted in a significant improvement in the book. A number of additional exercises for this edition were also provided by Wayne Barrett. Thanks to Warren Ferguson and others who taught from the first edition for their many helpful suggestions and to all those who helped uncover errata in the first edition. The author would again like to acknowledge the influence that Gene Golub and J. H. Wilkinson have had on this book. Thanks to Gerald Vieira and Judith Leon for their help with the proofreading. The revisions were all typed by Kate MacDougall; I would like to thank her for a very professional job. Finally, I would like to express my gratitude to the editorial, production, and sales staff at Macmillan for the work they have done on both the first and second editions.

S. L.

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1

MATRICES AND SYSTEMS OF EQUATIONS

INTRODUCTION

Probably the most important problem in mathematics is that of solving a system of linear equations. It would not be conservative to estimate that well over 75 percent of all mathematical problems encountered in scientific or industrial applications involve solving a linear system at some stage. Using the methods of modern mathematics, it is often possible to take a sophisticated problem and reduce it to a single system of linear equations. Linear systems arise in applications to such areas as business, economics, sociology, ecology, demography, genetics, electronics, engineering, and physics. It seems appropriate, then, that this book should begin with a section on linear systems.

1. SYSTEMS OF LINEAR EQUATIONS

A *linear equation in n unknowns* is an equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, a_2, \ldots, a_n$ and b are real numbers and $x_1, x_2, \ldots, x_n$ are variables. A *linear system* of m equations in n unknowns is then a system of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

where the a_{ij}'s and the b_i's are all real numbers. We will refer to systems of the form (1) as $m \times n$ linear systems. The following are examples of linear systems:

(a) $\begin{aligned} x_1 + 2x_2 &= 5 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$ (b) $\begin{aligned} x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 - x_3 &= 4 \end{aligned}$ (c) $\begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 1 \\ x_1 \quad &= 4 \end{aligned}$

System (a) is a 2×2 system, (b) is a 2×3 system, and (c) is a 3×2 system.

By a solution to an $m \times n$ system, we mean an ordered n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ that satisfies all the equations of the system. For example, the ordered pair $(1, 2)$ is a solution to system (a), since

$$\begin{aligned} 1 \cdot (1) + 2 \cdot (2) &= 5 \\ 2 \cdot (1) + 3 \cdot (2) &= 8 \end{aligned}$$

The ordered triple $(2, 0, 0)$ is a solution to system (b), since

$$\begin{aligned} 1 \cdot (2) - 1 \cdot (0) + 1 \cdot (0) &= 2 \\ 2 \cdot (2) + 1 \cdot (0) - 1 \cdot (0) &= 4 \end{aligned}$$

Actually, system (b) has many solutions. If α is any real number, it is easily seen that the ordered triple $(2, \alpha, \alpha)$ is a solution. However, system (c) has no solution. It follows from the third equation that the first coordinate of any solution would have to be 4. Using $x_1 = 4$ in the first two equations, we see that the second coordinate must satisfy

$$\begin{aligned} 4 + x_2 &= 2 \\ 4 - x_2 &= 1 \end{aligned}$$

Since there is no real number that satisfies both of these equations, the system has no solution. If a linear system has no solution, we say that the system is *inconsistent*. Thus system (c) is inconsistent, while systems (a) and (b) are both consistent.

The set of all solutions to a linear system is called the *solution set* of the system. If a system is inconsistent, its solution set is empty. A consistent system will have a nonempty solution set. To solve a consistent system, one must find its solution set.

2 × 2 SYSTEMS

Let us examine geometrically a system of the form

$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

Each equation can be represented graphically as a line in the plane. The ordered pair (x_1, x_2) will be a solution to the system if and only if it lies on both lines. For example, consider the three systems

$$\text{(i)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 - x_2 &= 2 \end{aligned} \qquad \text{(ii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ x_1 + x_2 &= 1 \end{aligned} \qquad \text{(iii)}\ \begin{aligned} x_1 + x_2 &= 2 \\ -x_1 - x_2 &= -2 \end{aligned}$$

The two lines in system (i) intersect at the point (2, 0). Thus $\{(2,0)\}$ is the solution set to (i). In system (ii) the two lines are parallel. Therefore, system (ii) is inconsistent and hence the solution set is $\varnothing$. The two equations in system (iii) both represent the same line. Any point on that line will be a solution to the system (see Figure 1.1.1).

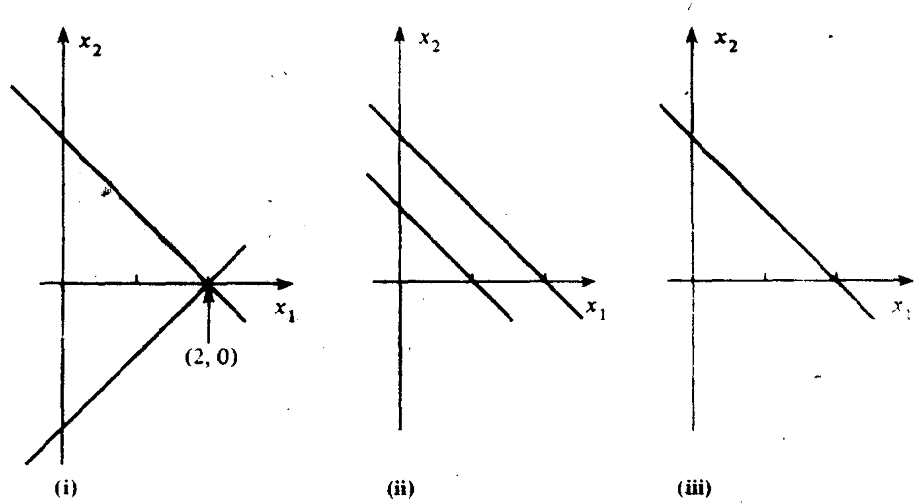

FIGURE 1.1.1

In general, there are three possibilities: the lines intersect at a point, they are parallel, or both equations represent the same line. The solution set then contains either one, zero, or infinitely many points.

The situation is similar for $m \times n$ systems. An $m \times n$ system may or may not be consistent. If it is consistent, then it must either have exactly one solution or infinitely many solutions. These are the only possibilities. We will see why this is so in Section 2 when we study the row echelon form. Of more immediate concern is the problem of finding all solutions to a given system. To tackle this problem, we introduce the notion of *equivalent systems.*

EQUIVALENT SYSTEMS

Consider the two systems

$$\text{(a)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ x_2 \qquad &= 3 \\ 2x_3 &= 4 \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ 3x_1 + 2x_2 + x_3 &= 2 \end{aligned}$$

System (a) is easy to solve because it is clear from the last two equations that $x_2 = 3$ and $x_3 = 2$. Using these values in the first equation, we get

$$\begin{aligned} 3x_1 + 2 \cdot 3 - 2 &= -2 \\ x_1 &= -2 \end{aligned}$$

Thus the solution to the system is $(-2, 3, 2)$. System (b) seems to be more difficult to solve. Actually, system (b) has the same solution as system (a). To see this, add the first two equations of the system

$$\begin{aligned} 3x_1 + 2x_2 - x_3 &= -2 \\ -3x_1 - x_2 + x_3 &= 5 \\ \hline x_2 \qquad &= 3 \end{aligned}$$

If (x_1, x_2, x_3) is any solution to (b), it must satisfy all the equations of the system. Thus it must satisfy any new equation formed by adding two of its equations. Therefore, x_2 must equal 3. Similarly, (x_1, x_2, x_3) must satisfy the new equation formed by subtracting the first equation from the third:

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + 2x_2 - x_3 &= -2 \\ \hline 2x_3 &= 4 \end{aligned}$$

Therefore, any solution to system (b) must also be a solution to system (a). By a similar argument, it can be shown that any solution to (a) is also a solution to (b). This can be done by subtracting the first equation from the second:

$$\begin{array}{rcl} x_2 & = & 3 \\ 3x_1 + 2x_2 - x_3 & = & -2 \\ \hline -3x_1 - x_2 + x_3 & = & 5 \end{array}$$

and by adding the first and third equations:

$$\begin{array}{rcl} 3x_1 + 2x_2 - x_3 & = & -2 \\ 2x_3 & = & 4 \\ \hline 3x_1 + 2x_2 + x_3 & = & 2 \end{array}$$

Thus (x_1, x_2, x_3) is a solution to system (b) if and only if it is a solution to system (a). Therefore, both systems have the same solution set, $\{(-2, 3, 2)\}$.

Definition. Two systems of equations involving the same variables are said to be **equivalent** if they have the same solution set.

Clearly, if we interchange the order in which two equations of a system are written, this will have no effect on the solution set. The reordered system will be equivalent to the original system. For example, the systems

$$\begin{array}{rcl} x_1 + 2x_2 & = & 4 \\ 3x_1 - x_2 & = & 2 \\ 4x_1 + x_2 & = & 6 \end{array} \quad \text{and} \quad \begin{array}{rcl} 4x_1 + x_2 & = & 6 \\ 3x_1 - x_2 & = & 2 \\ x_1 + 2x_2 & = & 4 \end{array}$$

clearly have the same solution set.

If one of the equations of a system is multiplied through by a nonzero real number, this will have no effect on the solution set and the new system will be equivalent to the original system. For example, the systems

$$\begin{array}{rcl} x_1 + x_2 + x_3 & = & 3 \\ -2x_1 - x_2 + 4x_3 & = & 1 \end{array} \quad \text{and} \quad \begin{array}{rcl} 2x_1 + 2x_2 + 2x_3 & = & 6 \\ -2x_1 - x_2 + 4x_3 & = & 1 \end{array}$$

are equivalent.

If a multiple of one equation is added to another equation, the new system will be equivalent to the original system. This follows since the n-tuple $(x_1, \ldots, x_n)$ will satisfy the two equations

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \end{aligned}$$