PHYSICS OF HIGH TEMPERATURE PLASMAS

SECOND EDITION

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Introduction

The heating of a solid or liquid substance leads to *phase transitions.* Molecules or atoms with sufficient energy to overcome the binding potential will evaporate. At temperatures high enough to impart this energy to almost every particle, the substance becomes a gas. It is characteristic of phase transitions that at a fixed pressure they occur at a constant temperature. The amount of energy that must be fed into the system at this temperature to bring about the transition is the *latent heat.*

Further heating of a gas results in additional transitions. For example, a molecular gas dissociates gradually into an atomic gas if the thermal energy of some particles exceeds the molecular binding energy. An even more drastic change takes place as soon as the temperature of the gas is high enough so that some electrons can overcome the atomic binding energy. With increasing temperature, more and more atoms get stripped of their electrons until the gas becomes a mixture of freely moving electrons and nuclei. We shall call this fully ionized substance a *plasma.* Although the transition from a gas to a plasma takes place gradually and is therefore not a phase transition in the thermodynamic sense, plasma is often referred to as a "fourth state of matter."

The investigation of the behavior of plasmas has great importance for the understanding of our universe. Although our earth consists mainly of the first three states of matter, this cannot be said about most of the stars and interstellar matter. All but a tiny fraction of the universe is plasma.

The full scope of possible earthly application of plasmas cannot even be estimated as yet. One of the most important possibilities to be seen at present is the application of the hot plasma as thermonuclear fuel. Nuclear fusion reactions resulting in the release of considerable energy take place in plasmas composed of certain light elements (D, T, etc.) at temperatures of the order of 10 to 100 million degrees Kelvin. This is the energy source of many stars (including the sun), and man is attempting to use similar processes for his own use. The hydrogen bomb applies thermonuclear fusion in a hot plasma for energy production in an uncontrolled explosive form, and there are hopes of finding ways to produce peaceful power by means of controlled thermonuclear fusion.

The latter problem poses many challenging questions. As the extremely hot plasma—necessary to obtain fusion reactions—would cause every material in close contact to evaporate, it must be confined by some kind of field. The plasma in the sun, for instance, is confined by its own gravitational field. This is evidently impossible to achieve on an earthly scale; the most probable answer is confinement by an electromagnetic field. Two questions arise immediately: 1. Do field configurations exist where the internal pressure of the plasma is counterbalanced by electromagnetic forces (equilibrium configuration)? 2. If such configurations were found, how do they behave if the equilibrium is slightly perturbed; are there any stable confined equilibria? The first question can be answered positively, but the second one (not less vital than the first) has not yet been settled.

In the following, while attempting to investigate the laws of plasma physics in general, emphasis will be placed on those phenomena which are likely to prove important for the realization and operation of thermonuclear machines rather than to applications in astrophysics.

Gravitational forces are much smaller than electromagnetic ones on an earthly scale. Therefore we shall deal mainly with electromagnetic forces. Similarly, for conceivable controlled thermonuclear applications, the momentum of the particles is high and the density low enough to keep the de Broglie wavelengths of particles well below the mean particle distance. Except for some cases of particle collisions (the close collisions), quantum effects are therefore negligible.

The complete system of equations, describing the behavior of a plasma under these conditions, can be presented in a straightforward way. The electromagnetic field inside and outside the plasma is completely defined by Maxwell's equations:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{1-1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1-2}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{1-3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1-4}$$

and the additional conditions

$$\mathbf{B} = \mu \mathbf{H} \tag{1-5}$$

and

$$\mathbf{D} = \varepsilon\mathbf{E} \tag{1-6}$$

Inside the plasma the particles move in vacuum, and therefore $\mu = \mu_0$ and $\varepsilon = \varepsilon_0$ (the mks system is used). If the charge $\rho(\mathbf{r},t)$ and current density $\mathbf{J}(\mathbf{r},t)$ are given, the electromagnetic field as a function of space and time is uniquely defined, provided the initial and boundary conditions are known. In a plasma, however, the charge and current densities are unknown, as the particles moving in the field (to be determined from the solution of Maxwell's equations) give rise to charge accumulations and currents

If, on the other hand, the electromagnetic fields were known, the equations of motion of each particle could be calculated, and the resulting charge and current densities computed. The equation of motion of a nonrelativistic particle with charge q_i and mass m_i in an **E** (electric) and **B** (magnetic) field is

$$m_i\ddot{\mathbf{r}}_i = q_i(\mathbf{E} + \dot{\mathbf{r}}_i \times \mathbf{B}) \tag{1-7}$$

If the total number of plasma particles is N, we have N equations of this type. With known $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ fields, these equations (with given initial conditions) can also be uniquely solved. To "plug back" the results of these solutions into (1-1) to (1-4), we express the plasma charge and current density in the form

$$\rho_{\mathrm{pl}} = \frac{\sum_{\Delta V} q_i}{\Delta V} \tag{1-8}$$

and

$$\mathbf{J}_{\mathrm{pl}} = \frac{\sum_{\Delta V} \dot{\mathbf{r}}_i q_i}{\Delta V} \tag{1-9}$$

where the summation is carried out over a "suitably chosen" small volume element ΔV. As the positions and velocities of particles as a function of time are given as solutions of (1–7), ρ_{pl} and $\mathbf{J}_{\mathrm{pl}}$ in (1–8) and (1–9) can be computed.

Obviously the set of equations (1–1) to (1–9) is complete. The solutions are self-consistent in the sense that the particle motions obtained create the appropriate electromagnetic fields necessary to produce just the particle motions with which one started (Fig. 1–1).

An important approximation is hidden behind (1–8) and (1–9). Since we are dealing with point charges, ρ and $\mathbf{J}$ should be described by δ functions. In other words, carrying out the $\lim_{\Delta V \to 0}$ transition in these equation we find either nothing or a single particle in our volume element. If, however, one keeps ΔV big enough to contain a fairly large number of particles, we obtain "smooth" functions for ρ and $\mathbf{J}$, which are suitable for analytical calculations. This is physically equivalent to "smearing out" the point particles and forgetting about their individuality. When looking for interactions of individual particles, such as collisions, more refined expressions shall be invoked.

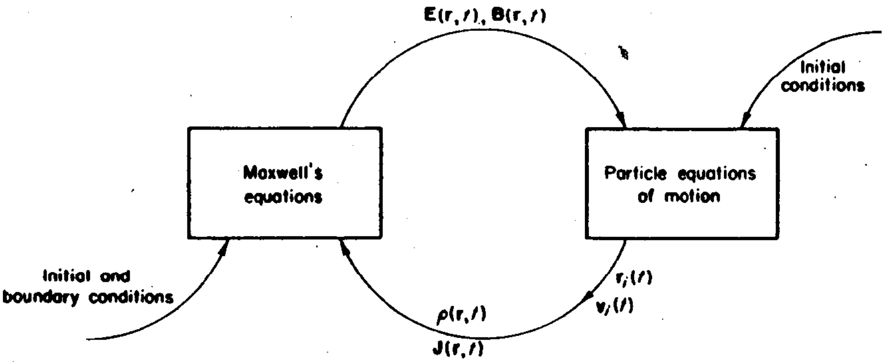

FIG. 1-1. Self-consistent plasma equations.

In addition to the plasma, known charges and currents might be present, associated with external conductors. In this case,

$$\rho = \rho_{\text{pl}} + \rho_{\text{ext}} \tag{1-10}$$

and

$$\mathbf{J} = \mathbf{J}_{\text{pl}} + \mathbf{J}_{\text{ext}} \tag{1-11}$$

Owing to the very large number of equations (1-7), it is practically inconceivable to carry out the above-outlined program of solution, even with the fastest computers available. We have the choice of approximate solution of the equations, or of finding precise solution for simplified models resembling (more or less) the real plasma.

II

Motion of Charged Particles in Electromagnetic Fields

2-1. The Static Magnetic Field

For one particle moving in a magnetic field with velocity $\mathbf{v}$, (1-7) reduces to

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{v} \times \mathbf{B}) \tag{2-1}$$

As the force is perpendicular to the velocity, no work is done by the magnetic field. Indeed, the scalar multiplication of (2-1) by $\mathbf{v}$ yields

$$m\mathbf{v}\frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\tfrac{1}{2}mv^2) = 0 \tag{2-2}$$

showing that the kinetic energy of the particle, in an arbitrary magnetic field, is a constant of motion.

Let us restrict ourselves for the moment to the special case where the magnetic field lines are straight and parallel (but the field is not necessarily uniform). Denoting vector components parallel to the field with the subscript $\parallel$ and those perpendicular to it with $\perp$, we obtain

$$\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp} \tag{2-3}$$

and (2-1) becomes

$$\frac{d\mathbf{v}_{\parallel}}{dt} + \frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}(\mathbf{v}_{\perp} \times \mathbf{B}) \tag{2-4}$$

since $\mathbf{v}_{\parallel} \times \mathbf{B}$ vanishes. Equation (2-4) splits into a $\parallel$-component equation and a $\perp$-component equation

$$\frac{d\mathbf{v}_{\parallel}}{dt} = 0 \qquad (\mathbf{v}_{\parallel} = \text{const}) \tag{2-5}$$

and

$$\frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}(\mathbf{v}_{\perp} \times \mathbf{B}) \tag{2-6}$$

Since the right side of (2-6) is perpendicular to $\mathbf{v}_\perp$, the left side is a centripetal acceleration. It can be written

$$\frac{v_\perp^{\,2}}{r^2}(-\mathbf{r}) = \frac{q}{m}(\mathbf{v}_\perp \times \mathbf{B}) \tag{2-7}$$

where $\mathbf{r}$ is the local radius of curvature of the particle path (Fig. 2-1). Its

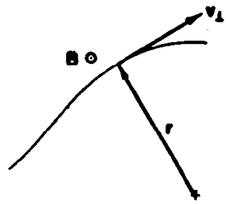

FIG. 2-1. Particle moving in a magnetic field of straight and parallel field lines. The field intensity varies in the plane perpendicular to **B**.

value is, from (2-7),

$$r = \left|\frac{mv_\perp}{qB}\right| \tag{2-8}$$

In the special case of a uniform magnetic field, $B =$ const, and considering the constancy of $v_\perp$ from (2-2) and (2-5), the radius of curvature

$$R = \left|\frac{mv_\perp}{qB}\right| \tag{2-9}$$

is also a constant. In a uniform magnetic field, therefore, the particle moves in a circle with the so-called *cyclotron* or *gyroradius* R in the perpendicular plane, while it moves with a constant velocity along the field lines. The resulting path is a *helix*.

The angular frequency of the circular motion is

$$\omega_c = \frac{v_\perp}{R} = \frac{q}{m}B \tag{2-10}$$

which is often called the *cyclotron frequency*, as its constancy for nonrelativistic velocities ($m =$ const) enables the operation of the cyclotron. For $v_\parallel = 0$ (which does not represent any restriction, just a suitable choice of coordinate system moving with the particle in the parallel direction), the particle moves in a circular path, giving rise to a magnetic field of its own. The time average of this field, over many gyration periods, is that of a ring current with the intensity

$$I = \frac{q\omega_c}{2\pi} = \frac{1}{2\pi}\frac{q^2B}{m} \tag{2-11}$$

The corresponding magnetic moment is

$$\mu_m = I\pi R^2 = \frac{1}{2}\frac{q^2R^2B}{m} \tag{2-12}$$

Denoting the magnetic flux surrounded by the path by ϕ, (2-12) becomes

$$\mu_m = \frac{1}{2\pi}\frac{q^2}{m}\phi \tag{2-13}$$

which shows that the magnetic moment is proportional to the flux enclosed. Inserting R from (2-9) into (2-12) leads to another form of the magnetic moment,

$$\mu_m = \frac{\frac{1}{2}mv_\perp^2}{B} \tag{2-14}$$

or, using (2-9) again, we obtain

$$\mu_m = |\tfrac{1}{2}qv_\perp R| \tag{2-15}$$

The latter is often put into vector form,

$$\boldsymbol{\mu}_m = \tfrac{1}{2}q\mathbf{R} \times \mathbf{v}_\perp \tag{2-16}$$

Note that the direction of μ_m does not depend on the sign of particle charge in a given field.

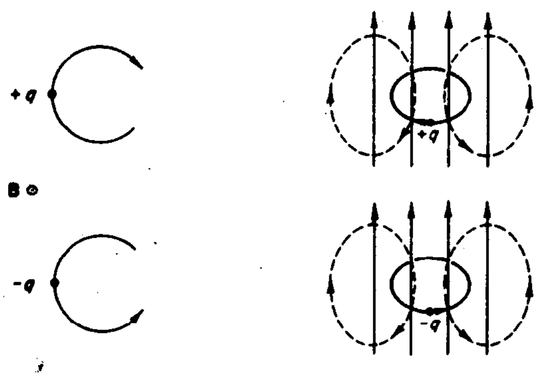

FIG. 2-2. The gyrating particle in the magnetic field generates a field like that of a diamagnetic dipole.

The magnetic field generated by a ring current at a distance much larger than R is similar to that of a magnetic dipole with the same moment. Figure 2-2 shows that the magnetic field associated with both a $+q$ and $-q$ particle moving in a uniform magnetic field opposes the external field inside the path, corresponding to a diamagnetic dipole. This is the source of the diamagnetic properties of plasmas to be studied later.

We are now going to investigate the motion of a charged particle in a uniform magnetic field with an additional constant force present. The equation of motion then becomes

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}) + \frac{\mathbf{F}}{m} \tag{2-17}$$

This splits again into component equations:

$$\frac{d\mathbf{v}_{\parallel}}{dt} = \frac{\mathbf{F}_{\parallel}}{m} \tag{2-18}$$

and

$$\frac{d\mathbf{v}_{\perp}}{dt} = \frac{q}{m}(\mathbf{v}_{\perp} \times \mathbf{B}) + \frac{\mathbf{F}_{\perp}}{m} \tag{2-19}$$

Equation (2-18) represents a constant acceleration along the field line. The external force term in (2-19) results in a drift velocity perpendicular to both the magnetic field and **F**. As shown in Fig. 2-3, the particle accelerated by

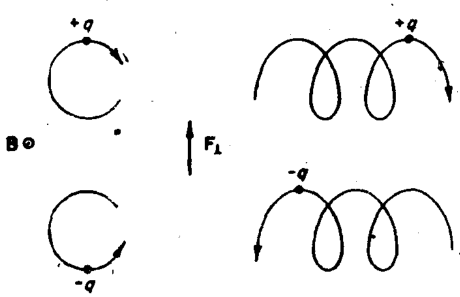

FIG. 2-3. A constant force $\mathbf{F}_{\perp}$ acting on a particle gyrating in a uniform magnetic field results in a drift motion perpendicular to $\mathbf{F}_{\perp}$ and **B**.

this force gains velocity, which in turn increases the radius of curvature according to (2-8). Reaching the turning point, the particle moves backward on a symmetrical path, now decelerated by the force, with decreasing radius of curvature to the opposite turning point.

Equation (2-19) can be resolved by introducing the drift velocity $\mathbf{w}^D$ and writing

$$\mathbf{v}_{\perp} = \mathbf{w}^D + \mathbf{u} \tag{2-20}$$

We shall now show that a suitable choice of the drift velocity, namely,

$$\mathbf{w}^D = \frac{1}{q}\frac{\mathbf{F}_{\perp} \times \mathbf{B}}{B^2} \tag{2-21}$$

"transforms away" the force term in (2-19). With the substitution of (2-20) and (2-21), (2-19) becomes

$$\frac{d\mathbf{u}}{dt} = \frac{1}{m}\left[\frac{(\mathbf{F}_\perp \times \mathbf{B}) \times \mathbf{B}}{B^2}\right] + \frac{q}{m}(\mathbf{u} \times \mathbf{B}) + \frac{\mathbf{F}_\perp}{m} \tag{2-22}$$

The first term on the right side can be written

$$\frac{(\mathbf{F}_\perp \times \mathbf{B}) \times \mathbf{B}}{B^2} = \frac{\mathbf{B}(\mathbf{F}_\perp \cdot \mathbf{B}) - \mathbf{F}_\perp(\mathbf{B} \cdot \mathbf{B})}{B^2} = -\mathbf{F}_\perp \tag{2-23}$$

Substitution in (2-22) leads to cancellation of the force term. The remainder is simply

$$\frac{d\mathbf{u}}{dt} = \frac{q}{m}(\mathbf{u} \times \mathbf{B}) \tag{2-24}$$

which means that the particle motion in a coordinate system moving with velocity $\mathbf{w}^D$ is governed entirely by the magnetic field and therefore moves on a circular path. The constant drift velocity superposed on this motion yields a *cycloid*, such as the one in Fig. 2-3. Note that the drift velocity depends on the particle charge.

We conclude by establishing the following rule: The motion of a charged particle in a uniform magnetic field, under the influence of an external force, can be described as the superposition of a gyration around the so-called *guiding center* with the cyclotron frequency and the motion of this guiding center. The guiding-center motion does not follow the laws of particle mechanics; it responds differently to the external force parallel and perpendicular to **B**. It is accelerated in the parallel direction according to (2-18), as if no magnetic field were present, while it drifts in the perpendicular plane with the constant velocity $\mathbf{w}^D$ as given by (2-21).

2-2. The Guiding-Center Approximation; Dipole-Like Motion

Using the conclusion of the previous section, we are now going to attack several more complicated cases. The combination of a homogeneous magnetic field and an electric field leads, for instance, to the additional force

$$\mathbf{F} = q\mathbf{E} \tag{2-25}$$

and the drift velocity

$$\mathbf{w}^E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{2-26}$$

which is independent of the charge and mass as well. The electric drift is identical for every particle; consequently the electric field can be entirely transformed away. The special theory of relativity shows that this can indeed be done. The Lorentz transformation from a coordinate system, where a homogeneous electric and a magnetic field are present, to another one, which moves with velocity $\mathbf{w}^E$ ($w^E \ll c$) with respect to the first, transforms away the electric field, leaving the magnetic field unchanged (see Exercise 2-1).

A homogeneous gravitational field, however, with gravitational acceleration **g**, gives rise to a force

$$\mathbf{F} = m\mathbf{g} \tag{2-27}$$

and the drift velocity

$$\mathbf{w}^g = \frac{m}{q}\frac{\mathbf{g} \times \mathbf{B}}{B^2} \tag{2-28}$$

depends on the m/q ratio. The gravitational field cannot, therefore—in this context—be transformed away.

An important example is the motion of a charged particle in a slightly inhomogeneous magnetic field, "slightly" meaning that the variation of the magnetic field inside the particle orbit is small compared to the magnitude of the field. If $\mathbf{B}_0$ is the field at the guiding center and $\mathbf{r}$ represents the momentary particle position in the guiding-center coordinate system, the magnetic field at the particle can be expressed by the Taylor expansion,

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0 + (\mathbf{r} \cdot \nabla_0)\mathbf{B} + \cdots \tag{2-29}$$

where ∇_0 means that the differentiation is to be performed at point 0. [Actually the momentary guiding center also varies slightly during a gyration period, while we hold point 0 fixed for this time (see Fig. 2-4)].

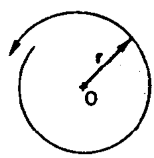

FIG. 2-4. Particle motion in a slightly nonuniform magnetic field.

In our case the higher-order terms can be neglected, and, in addition,

$$|\mathbf{B}_0| \gg |(\mathbf{r} \cdot \nabla_0)\mathbf{B}| \tag{2-30}$$

is true; thus the magnetic field "felt" by the particle differs but little from that prevailing at the guiding center. The particle path differs little, therefore, from a helix corresponding to $\mathbf{B}$ = const, or in the case of $\mathbf{v}_\parallel = 0$ from a circle. The latter case will be assumed in the following calculation.

In this approximation the motion of the particle can be described in the following way. The ring current represented by its dipole moment is located in an inhomogeneous magnetic field. The force exerted by the field inhomogeneity makes the particle drift as described in the previous section. The equation of motion (2-1) becomes, when inserting (2-29),

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\{[\mathbf{v} \times \mathbf{B}_0] + [\mathbf{v} \times (\mathbf{r} \cdot \nabla_0)\mathbf{B}]\} \tag{2-31}$$

As the last term is a small first-order one compared to the first zero-order term, we write the velocity as a superposition:

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 \tag{2-32}$$

where $\mathbf{v}_0$ is the solution of the zero-order equation

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}[\mathbf{v} \times \mathbf{B}_0] \tag{2-33}$$

and $\mathbf{v}_1$ is a perturbation of the first order. Since we neglect second-order terms, the last term of (2-31) can be written in the form

$$\frac{q}{m}[\mathbf{v}_0 \times (\mathbf{r} \cdot \nabla_0)\mathbf{B}] \tag{2-34}$$

Similarly, up to the first order we can put $\mathbf{r} = \mathbf{R}$ (the cyclotron radius corresponding to $\mathbf{B}_0$), and (2-31) becomes

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{v} \times \mathbf{B}_0) + \frac{q}{m}[\mathbf{v}_0 \times (\mathbf{R} \cdot \nabla_0)\mathbf{B}] \tag{2-35}$$

The second right-hand term constitutes the external force of (2-17). It is, however, not a constant, since it depends on the momentary particle position $\mathbf{R}$. We calculate the average of this quantity over a gyration period

$$\mathbf{F} = \langle q\mathbf{v}_0 \times (\mathbf{R} \cdot \nabla_0)\mathbf{B}\rangle \tag{2-36}$$

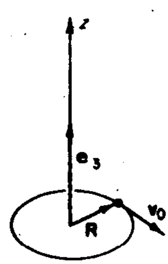

FIG. 2-5. Illustration to aid computation of F.

In a local cylindrical coordinate system with the z coordinate pointing in the $\mathbf{B}_0$ direction (Fig. 2-5), (2-36) becomes

$$\mathbf{F} = \left\langle q\mathbf{v}_0 \times R\frac{\partial_0}{\partial r}\mathbf{B}\right\rangle \tag{2-37}$$