Contents

1. Introduction

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With 6 Figures

This chapter is a description of the physical processes that govern turbulence and the mathematical equations that in turn govern them. It is self-contained, but the treatment of the mathematics, already available in many other textbooks, has been abbreviated in favor of discussions of the physical consequences of the equations. The chapter is intended to contain all the main results assumed without proof in later chapters. In some cases, a topic is outlined in Chapter 1 and developed in one or more later chapters; in these cases forward references are given.

1.1 Equations of Motion

The one uncontroversial fact about turbulence is that it is the most complicated kind of fluid motion. It is generally accepted that turbulence in simple liquids and gases is described by the Navier-Stokes equations, which express the principle of conservation of momentum for a continuum fluid with viscous stress directly proportional to rate of strain. Although the principle and the stress law are the simplest that can be imagined, some of the possible solutions of the equations, even for simple flow geometries, are too complicated to be comprehended by the human mind.

The Navier-Stokes momentum-transport equations are the second-order Chapman-Enskog approximation to the Boltzmann equation for molecular motion. For a masterly, if slightly inaccessible, review see GOLDSTEIN [1.1]. The first-order approximation, leading to the Euler equations, neglects viscosity altogether, while the more complicated molecular-stress terms yielded by higher-order approximations are not important in common gases at temperatures and pressures of the order of atmospheric. It is easy to show that the smallest turbulent eddies have a wavelength many times the mean free path unless the Mach number (velocity divided by speed of sound) is exceptionally high; the continuum approximation is a good one. Similarly, the constitutive equations of common liquids are close to the linear Newtonian viscous-stress law. Therefore, we shall use the Navier-Stokes equations throughout this book except for the discussion of "non-Newtonian" fluids in Chapter 7.

Since the equations are needed in their most general three-dimensional form, we shall use Cartesian tensor notation for compactness, ignoring the distinction between covariant and contravariant tensors and using the repeated-suffix summation convention so that $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$ (but $a_i + b_i = a_1 + b_1$ *or* $a_2 + b_2$ *or* $a_3 + b_3$). Sometimes $\partial f / \partial x_i$ – say – is denoted by $f_{,i}$; this form is used for brevity in the extensive mathematics of Chapters 5 and 7. In special cases, x, y, z notation will be used. Unless otherwise stated, x or x_1 is the general direction of flow and y or x_2 is normal to the plane of a shear layer. Occasionally vector notation will appear; in particular, $\boldsymbol{u}$ will be used for the velocity vector, whose components are u_1, u_2, u_3 or u, v, w. In general, capital U is used to denote a mean velocity (time-average or ensemble average) and small u denotes a fluctuation about that mean. However, in most of the discussion of Sections 1.1 to 1.5, the presence or absence of a mean velocity is immaterial, so the symbol u is used for simplicity; if desired it can be interpreted as the instantaneous (mean plus fluctuating) velocity, denoted by $U+u$ in later sections. The conventional division into mean and fluctuating components exists for the convenience of technologists. It is not as arbitrary as is sometimes claimed, because it leads to self-consistent equations with useful physical interpretations, but it is artificial because the motion at a given point and time receives no information about mean values, which necessarily depend on averages over large distances or long times (Sect. 1.3). Flows in which the *mean* velocity vector is everywhere parallel to a plane (usually taken as the xy plane) are called "two dimensional"; an example is the flow over a very long cylindrical body such as an unswept, untapered wing, normal to the oncoming stream. Note that this definition does not require the fluctuations to be parallel to the plane; in this book at least, motion which is two dimensional at every instant is not regarded as turbulence.

In words, the principle of conservation of momentum (Newton's second law of motion), as applied to a fluid subjected to any kind of molecular forces, is

acceleration (following the motion of the fluid)

= molecular force per unit mass

+ body force per unit mass . (1.1)

Molecular force is a surface force and reduces to a sum of stress gradients divided by the density, while body force is a volume force and is usually

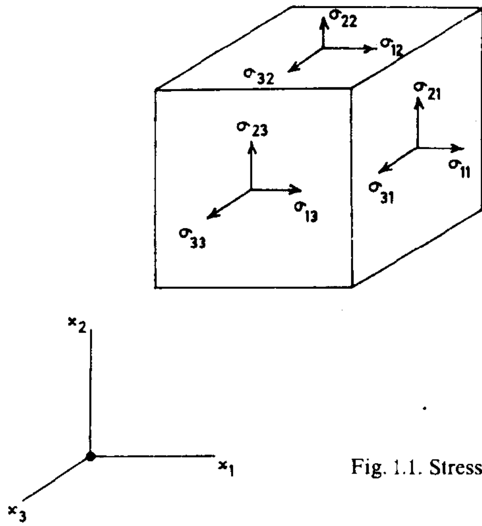

Fig. 1.1. Stresses acting on a fluid element

specified directly as a force per unit mass or equivalent acceleration. The molecular stress is conventionally divided into a scalar pressure p equal to $(-1/3)$ times the sum of the three normal (tensile) stress components, and a stress due to deformation and bulk dilatation which is a second-order tensor with components σ_{il}. For compatibility with the definition of p, the sum of the normal components of the stress due to deformation must be zero, leading to the term in (1.3), below, containing $(2/3)\mu$. However, the individual normal-stress components are *not* zero. According to these definitions the total molecular stress acting in the x_i-direction on a plane normal to the x_l-direction (Fig. 1.1) is $-p\delta_{il}+\sigma_{il}$, where $\delta_{il}=1$ if $i=l$ and $\delta_{il}=0$ if $i\neq l$. Equation (1.1) can now be written for the x_i-component of velocity, u_i (where $i=1$, 2 or 3) as

$$\frac{\partial u_i}{\partial t}+u_l\frac{\partial u_i}{\partial x_l}=\frac{1}{\varrho}\frac{\partial}{\partial x_l}(-p\delta_{il}+\sigma_{il})+f_i\equiv-\frac{1}{\varrho}\frac{\partial p}{\partial x_i}+\frac{1}{\varrho}\frac{\partial \sigma_{il}}{\partial x_l}+f_i$$

or

$$u_{i,t}+u_lu_{i,l}=(1/\varrho)(-p\delta_{il}+\sigma_{il})_{,l}+f_i=-(1/\varrho)p_{,i}+(1/\varrho)\sigma_{il,l}+f_i \quad (1.2)$$

in the compact suffix notation for differentiation. Here f_i is the x_i-component of body force per unit mass and ϱ is the (instantaneous) density. In a gravitational field, $f_i=g_i$ where g_i is a component of the gravitational acceleration. Unless density fluctuations or a free surface is present, this simply leads to an extra pressure gradient, which balances g_i so that both can be forgotten. Gravitational body forces are considered in Chapters 4 and 6, and Coriolis apparent body forces in Chapter 3.

These are "Eulerian" equations, expressed in terms of the velocity components at a fixed point. Corresponding (Lagrangian) equations can be derived in terms of the velocity of a marked particle. The Lagrangian equations are much less convenient for studying ordinary fluid motion and will not be needed in this book, though Lagrangian concepts are used in the discussion of particle-laden flows in Chapter 7. Note that (1.2) applies to any fluid whatever the constitutive law for σ_{il}. It even applies to the mean velocity in turbulent flow if all symbols denote mean quantities and σ_{il} is understood to include apparent turbulent (Reynolds) stresses (Section 1.3) as well as the molecular stresses. For a Newtonian viscous fluid, the instantaneous stress due to deformation is

$$\sigma_{il} = \mu\left(\frac{\partial u_i}{\partial x_l} + \frac{\partial u_l}{\partial x_i}\right) + (\beta - \tfrac{2}{3}\mu)\delta_{il}\frac{\partial u_m}{\partial x_m} \tag{1.3}$$

where the last term enforces $\sigma_{ii}=0$, and β is the bulk viscosity, of the same order as μ [1.1]. In the most general case $\partial\sigma_{il}/\partial x_l$ is quite complicated and forbidding; for a discussion see HOWARTH [Ref. 1.2, pp. 49–51] or SCHLICHTING [Ref. 1.3, Chapter 3], who neglect the bulk viscosity. Clearly $\sigma_{il}=\sigma_{li}$ in Newtonian fluids, so σ_{il} is a "diagonally symmetric" tensor.

Any fluid obeys the law of conservation of mass, obtainable from simple control-volume analysis as

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x_l}(\varrho u_l) = 0\,. \tag{1.4}$$

If the density is constant, (1.2) is unaltered but (1.4) reduces to

$$\frac{\partial u_l}{\partial x_l} = 0 \tag{1.5}$$

in steady or unsteady flow. Thus the last term of (1.3) vanishes in constant-density flow and the remainder of (1.3) implies that the viscous stress due to deformation is

$$\sigma_{il} = 2\mu e_{il} \tag{1.6}$$

where

$$e_{il} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_l} + \frac{\partial u_l}{\partial x_i}\right) \tag{1.7}$$

which defines the rate of strain e_{il}; the factor 1/2 is inserted for compatibility with the usual definition of strain in solid mechanics, but some fluids textbooks omit this and the factor 2 in (1.6). In a pure rotation about an axis normal to the $x_i x_l$-plane $\partial u_i/\partial x_l = -\partial u_l/\partial x_i$, both being numerically equal to the angular velocity, and e_{il} and σ_{il} are zero.

It is sometimes useful to add u_i/ϱ times the "continuity" equation (1.4) to (1.2); the left-hand side of the resulting equation is

$$\frac{1}{\varrho}\left[\frac{\partial \varrho u_i}{\partial t} + \frac{\partial}{\partial x_l}(\varrho u_i u_l)\right]$$

called the "divergence" form, as opposed to the "acceleration" form of (1.2). In this form, the equation shows that the rate of accumulation of x_i-component momentum in a unit control volume, plus the rate at which x_i-component momentum leaves the control volume, equals the force applied to the fluid instantaneously in the control volume by molecular and body forces. The addition of multiples of the continuity equation often helps to simplify or clarify equations.

In constant-property flow (constant density and constant viscosity[1]) the viscous-stress gradients can be simplified by neglect of viscosity gradients and further use of the continuity equation (1.5); then (1.2) becomes

$$\frac{\partial u_i}{\partial t} + u_l \frac{\partial u_i}{\partial x_l} = -\frac{1}{\varrho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_l^2} + f_i \tag{1.8}$$

where ν is the kinematic viscosity μ/ϱ. Note that the three elements of $\partial^2 u_i/\partial x_l^2$ do not individually equal the three elements of $\partial \sigma_{il}/\partial x_l$ for constant-property flow, because part of each term has been removed by using the continuity equation.

Very fortunately, viscosity does not usually affect the larger-scale eddies which are chiefly responsible for turbulent mixing (in fact, as we shall see in Section 7.4, turbulence processes are usually the same in the

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[1] In the study of gas flows the word "incompressible" is used instead of "constant property"; this usage is misleading in liquid flows, which can often be assumed incompressible in the strict sense of constant density but whose viscosity varies very rapidly with temperature. A "constant pressure" flow is one in which $\partial p/\partial x_i$ is negligible in (1.2).

simpler types of non-Newtonian fluids). Equally fortunately, the effects of density fluctuations on turbulence are small if, as is usually the case (Section 2.5), the density fluctuations are small compared to the mean density. Part of the discussion below will be concerned with the two major exceptions to these statements: the effect of viscosity on turbulence in the "viscous sublayer" very close to a solid surface (Sections 1.8, 2.3) and the effect of temporal fluctuations and spatial gradients of density in a gravitational field (Chapters 4 and 6); elsewhere, we will usually neglect the direct effect of viscosity and compressibility on turbulence. Several different definitions of the Reynolds number (velocity scale) × (length scale)/ν will be used below. They fall into two main classes: "bulk" Reynolds numbers in which the scales are those of the mean flow, and "turbulent" or "local" Reynolds numbers, in which the scales are those of the turbulence or even of part of the turbulence.

A fluctuating velocity field will cause fluctuations to develop in an initially smooth spatial variation of a scalar such as enthalpy, or concentration in a two-component flow. It is important to note that the fluctuating velocity field drives the fluctuating scalar field while the effect of the latter on the former, applied via mean gradients and fluctuations of density, is usually weak or even negligible. The conservation equation for a scalar c, equating the rate of change of c with time (following the motion of the fluid) to the sum of molecular diffusion and sources within the fluid, is

$$\frac{\partial c}{\partial t} + u_l \frac{\partial c}{\partial x_l} = \frac{1}{\varrho} \frac{\partial}{\partial x_l} \left(\varrho \gamma \frac{\partial c}{\partial x_l} \right) + S \tag{1.9}$$

where γ is the molecular diffusivity of c (having the same dimensions as the kinematic viscosity ν) and S is the rate of generation of c per unit volume (by chemical reaction, say) at the point considered. Compare (1.2) and (1.9): equations like these, whose left-hand sides contain the "transport operator" $\partial/\partial t + u_l \partial/\partial x_l$, are called "transport equations"; sometimes this name is reserved for equations containing the time- or ensemble-averaged version of this operator. An important case of (1.9) in engineering is when c represents enthalpy and γ represents the thermal diffusivity $k/\varrho c_p$; the dimensionless group of fluid properties $\mu c_p/k \equiv \nu/(k/\varrho c_p) \equiv \sigma$ is the Prandtl number, and, in compressible flow, S is the sum of compression work and viscous dissipation of kinetic energy into heat. If c represents mass concentration, ν/γ is the Schmidt number, Sc. If γ is constant, (1.9) becomes

$$\frac{\partial c}{\partial t} + u_l \frac{\partial c}{\partial x_l} = \gamma \frac{\partial^2 c}{\partial x_l^2} + S. \tag{1.10}$$

The obvious similarities between (1.8), without pressure gradient or body force, and (1.10), without the source term S, are the bases for analogies between momentum transfer and the transfer of heat and other scalars. The similarities between the general equations (1.2) and (1.9) are less obvious. Bearing in mind the relative unimportance of viscosity in turbulent mixing at all but the lowest Reynolds numbers, we can deduce the relative unimportance of molecular diffusivity (constant or otherwise) at least if ν/γ is not much smaller than unity. The presence of pressure gradients in (1.8) and their absence from (1.10) prevent the analogies from being exact in turbulent flow, even if $\nu/\gamma=1$, because pressure fluctuations always accompany velocity fluctuations. To see this, take the divergence of the Navier-Stokes equations [i.e., differentiate (1.8) with respect to x_i] neglecting density variations and body forces; after rearranging and using (1.5) we get the Poisson equation

$$\frac{1}{\varrho}\frac{\partial^2 p}{\partial x_i^2}=-\frac{\partial u_i}{\partial x_l}\frac{\partial u_l}{\partial x_i}=-\frac{\partial^2 u_i u_l}{\partial x_i \partial x_l}. \tag{1.11}$$

However, the analogies between momentum transfer and heat or mass transfer are sufficiently accurate for this introduction to be confined to momentum transfer, leaving the corresponding qualitative results for heat or mass transfer to be inferred by the reader. We return to the heat-transfer equations in Chapter 6: see also Subsection 2.3.9.

Detailed discussions of the equations of motion, some with specific application to turbulence, are given in [1.2–7].

1.2 Shear-Layer Instability and the Development of Turbulence

In steady "inviscid" flow[2] without body forces, the Navier-Stokes equations (1.2) reduces to the requirement that the total pressure

$$P\equiv\varrho\left(\frac{1}{2}|\boldsymbol{u}|^2+\int\frac{dp}{\varrho}\right), \tag{1.12}$$

where $\boldsymbol{u}$ is the velocity vector, shall be constant along a streamline (the envelope of the velocity vector); here the integral is evaluated along a

[2] Meaning a flow with negligible viscous stresses, the result of negligible rate of strain rather than negligible viscosity.

streamline starting from the point where $u=0$. In constant-density flow,

$$P=p+\tfrac{1}{2}\varrho|\boldsymbol{u}|^2. \qquad (1.13)$$

P may vary normal to the streamlines because of the previous influence of viscosity or body forces, and in flow with significant viscous stresses (or turbulent stresses) it will in general vary along and normal to the streamlines. A flow with a total-pressure gradient normal to the streamlines (a working definition of a "shear layer") may be unstable to infinitesimal, or small but finite, time-dependent disturbances. Other kinds of instability may occur, but if turbulence develops it is almost always via the stage of shear layer instability. Naturally, the shear layer is most unstable to the type of disturbance which travels downstream with the fluid, that is, a "traveling-wave" disturbance. Unstable shear flows give rise to complicated flow patterns [1.8, 9] and complicated mathematics [1.10], both of great beauty. Our present concern is with the further development of amplified unstable disturbances, which in the simplest cases can be two-dimensional sinusoidal fluctuations, into the three-dimensional continuous-spectrum fluctuations of turbulence.

The key phenomenon in both the further development of instabilities and the maintenance of fully developed turbulence is the intensification of vorticity in three-dimensional flows [Ref. 1.8, p. 266]. Vorticity is defined as

$$\boldsymbol{\omega}\equiv \operatorname{curl}\boldsymbol{u}\equiv \nabla_\wedge\, \boldsymbol{u} \qquad (1.14a)$$

in vector notation or

$$\omega_i\equiv \varepsilon_{ijk}\left(\frac{\partial u_k}{\partial x_j}-\frac{\partial u_j}{\partial x_k}\right) \qquad (1.14b)$$

in tensor notation. Here the unit alternating tensor ε_{ijk} is defined to be unity if i, j, k are in cyclic order (1 2 3 1 2 3...), -1 if i, k, j are in cyclic order, and zero otherwise (i.e., if two indices are equal); it exists simply to provide a tensor representation of a vector (cross) product. The vorticity of an element of fluid in unstrained ("solid body") rotation about the x_i-axis with angular velocity Ω is $\boldsymbol{\omega}\equiv\omega_i\equiv 2\Omega$. The "transport" equation for vorticity in a Newtonian viscous fluid is obtained by taking the curl of the Navier-Stokes equations. For incompressible, constant-property flow without body forces (or with a body-force vector $\boldsymbol{f}$ that satisfies $\operatorname{curl}\boldsymbol{f}=0$, as is the case for many simple body forces, including

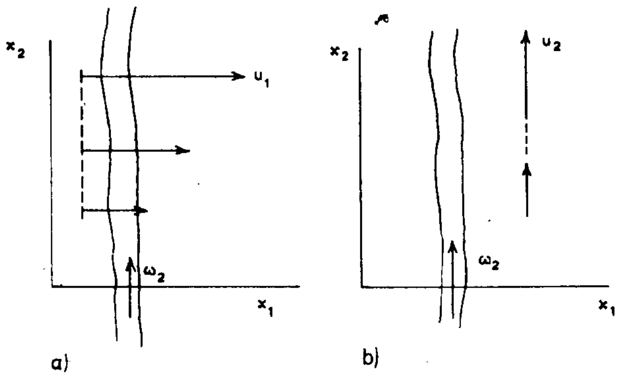

Fig. 1.2a and b. Effect of a velocity gradient on a vortex line. (a) Tilting: vorticity ω_2, velocity gradient $\partial u_1/\partial x_2$. (b) Stretching: vorticity ω_2, velocity gradient $\partial u_2/\partial x_2$

gravity, (1.8), with some use of (1.5), gives

$$\frac{\partial \omega_i}{\partial t} + u_l \frac{\partial \omega_i}{\partial x_l} = \omega_l \frac{\partial u_i}{\partial x_l} + \nu \frac{\partial^2 \omega_i}{\partial x_l^2} . \tag{1.15}$$

Note that the pressure term has disappeared but that a new term appears on the right, in addition to the viscous diffusion term. The vorticity/velocity-gradient interaction term $\omega_l \partial u_i/\partial x_l$, which is a nonlinear term because $\boldsymbol{\omega}$ depends on $\boldsymbol{u}$, has two main effects. It is convenient to discuss these with reference to a slender element of fluid, rotating about its axis (Fig. 1.2). We shall call such elements "vortex lines", the length of the line being nominally infinite and the distribution of vorticity over the cross section being immaterial for present purposes. Except for the effects of viscous diffusion, which tends to increase their cross section, vortex lines move with the fluid, a consequence of Kelvin's circulation theorem [Ref. 1.8, p. 273]. A vortex sheet is an envelope of vortex lines, and a finite body of fluid with vorticity can be regarded as a continuous distribution of vortex lines. Vortex lines induce an irrotational (zero-vorticity) velocity field at other points in space according to the Biot-Savart law, which also governs the magnetic field due to a current-carrying conductor.

The first effect of the term $\omega_l \partial u_i/\partial x_l$ in (1.15) is that if $i \neq l$ (say, $i=1$, $l=2$, see Fig. 1.2a) it represents an exchange of vorticity between components, because a velocity gradient $\partial u_1/\partial x_2$ (say) tilts a vortex line which was initially in the x_2-direction so that it acquires a component in the x_1-direction. Secondly, if $i=l$ ($=2$, say, see Fig. 1.2b) the vortex line is stretched by the rate of tensile strain along its axis, without any change of the direction of that axis. Neglecting viscous diffusion (and, strictly, requiring the cross section of the vortex line to be circular so that pressure gradients cannot apply a torque to it) we see that the vortex line will conserve its angular momentum as its cross-sectional area decreases under the influence of axial stretching; therefore, its vorticity (angular velocity) will increase. If viscous diffusion is small (high Reynolds number) the vorticity/velocity-gradient interaction terms in (1.15) can change an initially simple (but three-dimensional) flow pattern into an unimaginably complicated distribution of vorticity and velocity—turbulence.

Three dimensionality is essential to the genesis and maintenance of turbulence [1.11]: in an instantaneously two-dimensional flow, by definition, the velocity vector would be everywhere parallel to a plane, the vorticity vector would be normal to that plane, and $\omega_l \partial u_i/\partial x_l$ would be zero. It appears that although the most unstable infinitesimal disturbance in a steady two-dimensional shear flow is a two-dimensional traveling wave, amplified disturbances of sufficient amplitude (which can be regarded as packets of vortex lines with spanwise axes) are themselves unstable to infinitesimal three-dimensional perturbations. A small kink in an otherwise straight vortex line is distorted and enlarged by the induced velocity field of the vortex itself, and if viscous diffusion is small enough (i.e., if the vortex Reynolds number is large enough), the distortion will continue indefinitely. Therefore, once the primary unstable disturbances have reached a sufficient amplitude they rapidly become more complicated and unsteady, because of the stretching and tilting by the induced velocity field of the vortex lines themselves, as well as by the basic shear flow. The simplest way of explaining how non-periodic unsteadiness arises is to note that in real life the wavelength of the primary disturbance is bound to be slightly unsteady. The percentage unsteadiness in the wavelength of the first harmonic disturbance will be roughly twice as large, and so on for higher harmonics. Sum-and-difference wave numbers[3] appear because of the nonlinearity of the interaction of different packets of vortex lines via their induced velocity fields, and the wave-number spectrum eventually becomes continuous.

As the motion becomes increasingly complicated the effects are felt of a theorem in random processes, known as the theorem of the random walk or "Drunkard's Walk", which states that a particle subjected to random impulses will, on the average, increase its distance from its

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[3] Wave number $= 2\pi/$(wavelength): it is a vector with the same direction as the wavelength (which is not necessarily the direction of propagation of the wave). Note that small scale = small wavelength = large wave number.

starting point. The phenomenon is known to, and regretted by, cab drivers. An obvious corollary states that the distance between two randomly perturbed particles will, on the average, increase. If those two particles are situated at the ends of a given element of a vortex line, then, in a flow field approximating to random disturbances, the length of the element will on the average increase, and its vorticity will be increased by this stretching. Moreover, the typical length scales of the region of high vorticity—the diameter of the vortex line in our simple model—will also decrease. This is the key mechanism of fully developed turbulence: interaction of tangled vortex lines maintains random fluctuations of the vorticity and velocity, while the random-walk mechanism transfers vorticity to smaller and smaller length scales. It remains to provide the transfer process with a beginning and an end. If there is a mean rate of strain it deforms the fluctuating vorticity field and intensifies vortex lines whose axes are, at any given instant, near the axis of the largest positive principal strain rate. Because of the nonlinearity of the process, this intensification usually predominates over the weakening of those vortex lines whose axes are near that of the negative principal strain rate. Thus the mean strain rate helps to maintain the level of vorticity fluctuation. A more rigorous analysis shows that the main effect of the mean strain rate is on the larger-scale motions, which then distort motions of smaller scale and so on. A limit to the decrease of vortex-line diameter by stretching is set when viscous stress gradients diffuse vorticity away from the axis as fast as stretching reduces the diameter.

If we now consider the kinetic energy of the fluctuating motion, we can see that vortex stretching increases the rotational kinetic energy of the vortex line. Angular momentum is proportional to ωr^2 while energy is proportional to $\omega^2 r^2$, so if the former is conserved while r decreases, the latter increases. The kinetic energy comes from the velocity field that does the stretching, so that kinetic energy passes from the mean flow (if a mean strain rate is present) down through vortex motions of smaller and smaller length scale until it is converted into thermal internal energy via work done against viscous stresses. If there is no mean strain field to do work on the fluctuating motion the latter gradually decays. Now this process of energy transfer to smaller scales, aptly called the "energy cascade", is independent of viscosity except in the final stages, as can be seen from the description above. It therefore follows that the rate of energy transfer to the smallest, viscous-dependent motions that dissipate it into "heat" is *independent* of viscosity. Viscosity causes dissipation but does not control its rate; the intensity and length scale of the small-scale motion adjust themselves so as to dissipate all the energy transferred from larger scales, and the smaller the viscosity the

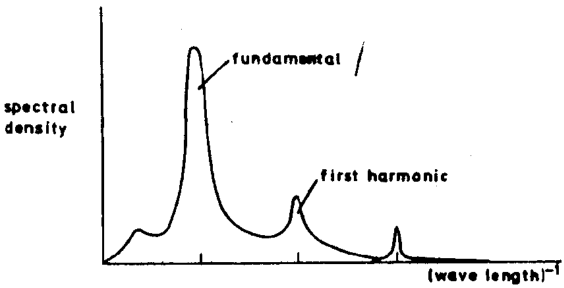

Fig. 1.3. Spectral distribution of velocity fluctuations in a late stage of transition to turbulence

smaller the motions that can survive. The rate of viscous dissipation of energy per unit volume by turbulent fluctuations is the mean product of the fluctuating rate of strain and the fluctuating viscous stress (clearly this is the mean rate at which the turbulence does work against viscous stresses). The viscous stress is equal to 2 μ times the rate of strain, so the dissipation is proportional to the mean square of the rate of strain and is therefore non-negative as required by the second law of thermodynamics.

In summary, the stages in the development of turbulence from an initial unstable shear layer (or from other unstable situations like that of a fluid whose density decreases in the direction of a body-force vector) are:

1) The growth of disturbances with *periodic* fluctuations of vorticity.
2) Their secondary instability to three-dimensional (infinitesimal) disturbance if the primary fluctuations are two dimensional.
3) The growth of three dimensionality and higher harmonics of the disturbance, leading to spectral broadening by vortex-line interaction.
4) The onset of the random-walk mechanism when the vorticity field becomes sufficiently complicated, leading to a general transfer of energy across the spectrum to smaller and smaller scales.

It is not useful to agonize about the exact point at which the motion can properly be called "turbulence". Flows in the later stages of transition from laminar to turbulent (at the spectral state shown in Fig. 1.3, say) are even more difficult to understand and calculate than turbulence in an undoubted state of full development. Therefore, equating "turbulence" and "incalculability"—the unconscious basis of many definitions—is to be deprecated. So is the use of the word "turbulence" in plasma physics to describe miscellaneous instabilities in current-carrying fluids, and in meteorology to describe the synoptic-scale motion (whose horizontal length scales are many times the depth of the atmosphere and which is therefore close to instantaneous two dimensionality). The essential characteristic of turbulence is the transfer of energy to smaller spatial scales across a continuous wave-number spectrum; this is a three-dimensional, nonlinear phenomenon.

Note that we have discussed the main mechanisms of turbulence without any mathematics other than the qualitative use of the vorticity equation. The vortex-line model used above is clearly artificial, but the spatial distribution of vorticity in a real turbulence field is almost discontinuous, the ratio of the width of typical high-vorticity regions to the width of the flow decreasing with increasing Reynolds number. There is some controversy [1.12] over whether the high-vorticity regions are best approximated by rods, strips or sheets (known to some as the "pasta problem"). An alternative concept is that of an "eddy". In qualitative discussion an eddy can be thought of as a typical turbulent flow pattern, covering a moderate range of wavelengths so that large eddies and small eddies can coexist in the same volume of fluid. Flow-visualization experiments (to be heartily recommended to all who work with turbulence and especially those who seek to calculate it) show the usefulness of the concept and the difficulty of a precise definition [1.13]. In quantitative work one uses statistical-average equations based on Fourier analysis of the velocity patterns (Section 1.4), and since sinusoidal modes have no relation to the actual modes (eddies, velocity patterns, high-vorticity regions...) the physical processes are obscured. Note however that an optimum mode for spectral decomposition can be found only *after* the problem has been solved!

1.3 Statistical Averages

The reason for working with statistical averages [1.6, 11, 14, 15] is that one is generally not interested in complete details of the behavior of the three velocity components and the pressure as functions of three space coordinates and time. In most cases, indeed, only very simple statistics, such as the mean rates of transfer of mass or momentum, are required for engineering or geophysical purposes.

The simplest form of statistical average is the mean, with respect to time, at a single point. This is useful only if the mean is independent of the time at which the averaging process is started; in symbols

$$\bar{f} \equiv \underset{T\to\infty}{\mathrm{Lt}} \frac{1}{T}\int_{t_0}^{t_0+T} f\,dt \tag{1.16}$$