电动力学解题指南

Electrodynamics Solving Guide

编著 王大伦 曾凡金

湘潭大學出版社

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序

电动力学是物理学的四大力学之一。《电动力学》课程是物理学专业重要的理论必修课,其与《量子力学》同属物理学专业的理论基础课程,也是目前高等学校理工科和师范院校本科物理类专业的一门重要基础理论课。由于该课程涉及的数学知识较多,如复变函数、格林函数、特殊函数、数理方程、矩阵、矢量分析和张量分析等,这给学习《电动力学》课程的学生带来了一定的难度。为了给学习《电动力学》课程的学生提供一定的帮助,作者将自己从 20 世纪 80 年代初以来收集到的 269 道题进行了较为详细的解答和整理,历经数年才得以完成此书。

本书主要是对王大伦主编的《电动力学教程》(湘潭大学出版社,2017 年 8 月)的全部习题进行解答,以及对郭硕鸿先生著《电动力学》第三版的部分经典习题进行解答。另外,本书还对一些往年的研究生入学考试试题进行解答。出版本书旨在对学习该门课程的学生有所帮助,且对他们参加研究生入学考试也有所帮助,同时也希望为初讲授《电动力学》这门课程的年轻教师提供一定的教学参考。本书附有数学附录和部分基本物理常数,列出了所需的数学知识和计算所需的一些基本物理常数值。

本书在编写的过程中,得到了安顺学院领导和同事们的支持与帮助,特别是得到了陈琳副教授的鼎力支持,湘潭大学出版社为本书的出版做了大量的工作,在此谨致以衷心感谢。

由于作者学识水平有限,错误之处在所难免,敬请读者批评指正。

编　者

2017 年 11 月

目　录

第 1 章　电磁现象的基本规律

1.1　证明下列恒等式

(1) $(\boldsymbol{a}\cdot\boldsymbol{b})^2+(\boldsymbol{a}\times\boldsymbol{b})^2=a^2b^2$

(2) $(\boldsymbol{a}+\boldsymbol{b})\cdot[(\boldsymbol{a}+\boldsymbol{c})\times\boldsymbol{b}]=-\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$

(3) $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})=0$

(4) $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})=\boldsymbol{c}[\boldsymbol{d}\cdot(\boldsymbol{a}\times\boldsymbol{b})]-\boldsymbol{d}[\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})]$

$\qquad=\boldsymbol{b}[\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{d})]-\boldsymbol{a}[\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{d})]$

【证明】　Ⅰ.方程的左边为

$(\boldsymbol{a}\cdot\boldsymbol{b})^2+(\boldsymbol{a}\times\boldsymbol{b})^2=(ab\cos\theta)^2+(ab\sin\theta)^2=a^2b^2(\cos^2\theta+\sin^2\theta)=a^2b^2$

方程左边等于方程右边,证毕。

Ⅱ.方程的左边为

$$\begin{aligned}(\boldsymbol{a}+\boldsymbol{b})\cdot[(\boldsymbol{a}+\boldsymbol{c})\times\boldsymbol{b}]&=(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}\times\boldsymbol{b})+(\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{b})\\&=\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{b})+\boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{b})+\\&\quad\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{b})+\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{b})\end{aligned}$$

式中

$$\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{b})=0,\quad \boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{b})=0,\quad \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{b})=0$$

所以

$$(\boldsymbol{a}+\boldsymbol{b})\cdot[(\boldsymbol{a}+\boldsymbol{c})\times\boldsymbol{b}]=\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{b})=-\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$$

方程左边等于方程右边,证毕。

Ⅲ.方程的左边为

$$\begin{aligned}&\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{c}\times(\boldsymbol{a}\times\boldsymbol{b})\\&\quad=[\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})+\boldsymbol{b}(\boldsymbol{c}\cdot\boldsymbol{a})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})-\boldsymbol{c}(\boldsymbol{b}\cdot\boldsymbol{a})]+\\&\qquad[\boldsymbol{c}(\boldsymbol{b}\cdot\boldsymbol{a})+\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})-\boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})-\boldsymbol{a}(\boldsymbol{c}\cdot\boldsymbol{b})]+\\&\qquad[\boldsymbol{a}\cdot(\boldsymbol{c}\cdot\boldsymbol{b})+\boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})-\boldsymbol{b}(\boldsymbol{c}\cdot\boldsymbol{a})-\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})]=0\end{aligned}$$

方程左边等于方程右边,证毕。

Ⅳ.将第一个括弧看成一矢量,根据矢量代数公式[附录 1 式(1.17)],方程的左边为

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})=\boldsymbol{c}[\boldsymbol{d}\cdot(\boldsymbol{a}\times\boldsymbol{b})]-\boldsymbol{d}[\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b})]$$

将第二个括弧看成一个矢量,同理有

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{c}\times\boldsymbol{d})=\boldsymbol{b}[\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{d})]-\boldsymbol{a}[\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{d})]$$

方程左边等于方程右边,证毕。

1.2 根据算符∇的微分性与矢量性证明下列公式:

(1) $\nabla(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}+\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}$

(2) $\boldsymbol{A}\times(\nabla\times\boldsymbol{A})=\dfrac{1}{2}\nabla\boldsymbol{A}^2-(\boldsymbol{A}\cdot\nabla)\boldsymbol{A}$

【证明】 Ⅰ.方程的左边为

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B})=\nabla_A(\boldsymbol{A}\cdot\boldsymbol{B})+\nabla_B(\boldsymbol{A}\cdot\boldsymbol{B}) \tag{1}$$

根据公式 $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}(\boldsymbol{c}\cdot\boldsymbol{a})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$,即

$$\boldsymbol{b}(\boldsymbol{c}\cdot\boldsymbol{a})=\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$

注意$\nabla_A\to\boldsymbol{b},\boldsymbol{A}\to\boldsymbol{c},\boldsymbol{B}\to\boldsymbol{a}$,则有

$$\nabla_A(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A} \tag{2}$$

同理

$$\nabla_B(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \tag{3}$$

将(2)式和(3)式代入(1)式得

$$\nabla(\boldsymbol{A}\cdot\boldsymbol{B})=\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{B}\cdot\nabla)\boldsymbol{A}+\boldsymbol{A}\times(\nabla\times\boldsymbol{B})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B} \tag{4}$$

方程的左边等于方程右边,证毕。

Ⅱ.在(4)式中令 $\boldsymbol{A}=\boldsymbol{B}$,则有

$$\begin{aligned}\nabla A^2&=\boldsymbol{A}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{A}+\boldsymbol{A}\times(\nabla\times\boldsymbol{A})+(\boldsymbol{A}\cdot\nabla)\boldsymbol{A}\\&=2\boldsymbol{A}\times(\nabla\times\boldsymbol{A})+2(\boldsymbol{A}\cdot\nabla)\boldsymbol{A}\end{aligned} \tag{5}$$

即

$$\boldsymbol{A}\times(\nabla\times\boldsymbol{A})=\frac{1}{2}\nabla A^2-(\boldsymbol{A}\cdot\nabla)\boldsymbol{A} \tag{6}$$

证毕。

1.3 设 u 是空间坐标 x,y,z 的函数,证明:

(1) $\nabla f(u)=\dfrac{\mathrm{d}f}{\mathrm{d}u}\nabla u$; (2) $\nabla\cdot\boldsymbol{A}(u)=\nabla u\cdot\dfrac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u}$; (3) $\nabla\times\boldsymbol{A}(u)=\nabla u\times\dfrac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u}$。

【证明】 Ⅰ.方程的左边为

$$\begin{aligned}\nabla f(u)&=\frac{\partial f(u)}{\partial x}\boldsymbol{e}_x+\frac{\partial f(u)}{\partial y}\boldsymbol{e}_y+\frac{\partial f(u)}{\partial z}\boldsymbol{e}_z=\frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\boldsymbol{e}_x+\frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}y}\boldsymbol{e}_y+\frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}z}\boldsymbol{e}_z\\&=\frac{\mathrm{d}f}{\mathrm{d}u}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\boldsymbol{e}_x+\frac{\mathrm{d}u}{\mathrm{d}y}\boldsymbol{e}_y+\frac{\mathrm{d}u}{\mathrm{d}z}\boldsymbol{e}_z\right)=\frac{\mathrm{d}f}{\mathrm{d}u}\nabla u\end{aligned}$$

方程的左边等于方程的右边,证毕。

Ⅱ.方程的左边为

$$\begin{aligned}\nabla\cdot\boldsymbol{A}(u)&=\left(\frac{\partial}{\partial x}\boldsymbol{e}_x+\frac{\partial}{\partial y}\boldsymbol{e}_y+\frac{\partial}{\partial z}\boldsymbol{e}_z\right)\cdot[A_x(u)\boldsymbol{e}_x+A_y(u)\boldsymbol{e}_y+A_z(u)\boldsymbol{e}_z]\\&=\frac{\partial A_x(u)}{\partial x}+\frac{\partial A_y(u)}{\partial y}+\frac{\partial A_z(u)}{\partial z}\end{aligned}$$

$$= \frac{\mathrm{d}A_x(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}A_y(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}y} + \frac{\mathrm{d}A_z(u)}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}z} = \nabla u \cdot \frac{\mathrm{d}\boldsymbol{A}(u)}{\mathrm{d}u}$$

方程的左边等于方程右边，证毕。

Ⅲ. 方程的左边为

$$\nabla\times\boldsymbol{A}(u) = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x(u) & A_y(u) & A_z(u) \end{vmatrix}$$

$$= \left[\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right]\boldsymbol{e}_x + \left[\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right]\boldsymbol{e}_y + \left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]\boldsymbol{e}_z$$

$$= \left[\frac{\mathrm{d}A_z}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}y} - \frac{\mathrm{d}A_y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}z}\right]\boldsymbol{e}_x + \left[\frac{\mathrm{d}A_x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}z} - \frac{\mathrm{d}A_z}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\right]\boldsymbol{e}_y +$$

$$\left[\frac{\mathrm{d}A_y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}A_x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}y}\right]\boldsymbol{e}_z$$

方程的右边为

$$\nabla u \times \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}u} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \dfrac{\mathrm{d}u}{\mathrm{d}x} & \dfrac{\mathrm{d}u}{\mathrm{d}y} & \dfrac{\mathrm{d}u}{\mathrm{d}z} \\ \dfrac{\mathrm{d}A_x}{\mathrm{d}u} & \dfrac{\mathrm{d}A_y}{\mathrm{d}u} & \dfrac{\mathrm{d}A_z}{\mathrm{d}u} \end{vmatrix}$$

$$= \left[\frac{\mathrm{d}A_z}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}y} - \frac{\mathrm{d}A_y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}z}\right]\boldsymbol{e}_x + \left[\frac{\mathrm{d}A_x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}z} - \frac{\mathrm{d}A_z}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}\right]\boldsymbol{e}_y +$$

$$\left[\frac{\mathrm{d}A_y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{\mathrm{d}A_x}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}y}\right]\boldsymbol{e}_z$$

方程的左边等于方程的右边，证毕。

1.4　设 $r = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$ 源点 $\boldsymbol{x}'$ 到场点 $\boldsymbol{x}$ 的距离，r 的方向规定为源点指向场点。

(1) 证明下列结果，并体会对源变数求微商 $\left(\nabla' = \boldsymbol{e}_x\dfrac{\partial}{\partial x'} + \boldsymbol{e}_y\dfrac{\partial}{\partial y'} + \boldsymbol{e}_z\dfrac{\partial}{\partial z'}\right)$ 与对场变数求微商 $\left(\nabla = \boldsymbol{e}_x\dfrac{\partial}{\partial x} + \boldsymbol{e}_y\dfrac{\partial}{\partial y} + \boldsymbol{e}_z\dfrac{\partial}{\partial z}\right)$ 的关系：

$$\nabla r = -\nabla' r = \frac{\boldsymbol{r}}{r},\ \nabla\frac{1}{r} = -\nabla'\frac{1}{r} = -\frac{\boldsymbol{r}}{r^3},\ \nabla\times\frac{\boldsymbol{r}}{r^3} = 0,\ \nabla\cdot\frac{\boldsymbol{r}}{r^3} = -\nabla'\cdot\frac{\boldsymbol{r}}{r^3} = 0,$$

$(r \neq 0)$　(最后一式在 $r = 0$ 点不成立)。

(2) 求 ∇r，$\nabla\cdot\boldsymbol{r}$，$\nabla\times\boldsymbol{r}$，$(\boldsymbol{a}\cdot\nabla)\boldsymbol{r}$，$\nabla(\boldsymbol{a}\cdot\boldsymbol{r})$，$\nabla\cdot[\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})]$ 及 $\nabla\times[\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})]$，其中 $\boldsymbol{a}$、$\boldsymbol{k}$ 及 $\boldsymbol{E}_0$ 均为常矢量。

【解】　Ⅰ. 证明如下：

(1) 设 $\boldsymbol{r} = (x-x')\boldsymbol{e}_x + (y-y')\boldsymbol{e}_y + (z-z')\boldsymbol{e}_z$，则

$$r=[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{1}{2}}$$

所以有

$$\nabla r=\frac{\partial r}{\partial x}\boldsymbol{e}_x+\frac{\partial r}{\partial y}\boldsymbol{e}_y+\frac{\partial r}{\partial z}\boldsymbol{e}_z=\frac{(x-x')}{r}\boldsymbol{e}_x+\frac{(y-y')}{r}\boldsymbol{e}_y+\frac{(z-z')}{r}\boldsymbol{e}_z=\frac{\boldsymbol{r}}{r}$$

而

$$\begin{aligned}\nabla' r&=\frac{\partial r}{\partial x'}\boldsymbol{e}_x+\frac{\partial r}{\partial y'}\boldsymbol{e}_y+\frac{\partial r}{\partial z'}\boldsymbol{e}_z\\&=-\frac{(x-x')}{r}\boldsymbol{e}_x-\frac{(y-y')}{r}\boldsymbol{e}_y-\frac{(z-z')}{r}\boldsymbol{e}_z=-\frac{\boldsymbol{r}}{r}\end{aligned}$$

所以有

$$\nabla r=-\nabla' r$$

(2) 同理

$$\nabla\frac{1}{r}=-\frac{1}{2}\frac{1}{r^3}[2(x-x')\boldsymbol{e}_x+2(y-y')\boldsymbol{e}_y+2(z-z')\boldsymbol{e}_z]=-\frac{\boldsymbol{r}}{r^3}$$

而

$$\nabla'\frac{1}{r}=-\frac{1}{2}\frac{1}{r^3}[-2(x-x')\boldsymbol{e}_x-2(y-y')\boldsymbol{e}_y-2(z-z')\boldsymbol{e}_z]=\frac{\boldsymbol{r}}{r^3}$$

所以有

$$\nabla\frac{1}{r}=-\nabla'\frac{1}{r}=-\frac{\boldsymbol{r}}{r^3}$$

(3) 同理

$$\begin{aligned}\nabla\times\frac{\boldsymbol{r}}{r^3}&=\left(\nabla\frac{1}{r^3}\right)\times\boldsymbol{r}+\frac{1}{r^3}\nabla\times\boldsymbol{r}\\&=-\frac{3\boldsymbol{r}}{r^5}\times\boldsymbol{r}+\frac{1}{r^3}\begin{vmatrix}\boldsymbol{e}_x&\boldsymbol{e}_y&\boldsymbol{e}_z\\\dfrac{\partial}{\partial x}&\dfrac{\partial}{\partial y}&\dfrac{\partial}{\partial z}\\(x-x')&(y-y')&(z-z')\end{vmatrix}=0+0=0\end{aligned}$$

而

$$\begin{aligned}\nabla'\times\frac{\boldsymbol{r}}{r^3}&=\left(\nabla'\frac{1}{r^3}\right)\times\boldsymbol{r}+\frac{1}{r^3}\nabla'\times\boldsymbol{r}\\&=\frac{3\boldsymbol{r}}{r^5}\times\boldsymbol{r}+\frac{1}{r^3}\begin{vmatrix}\boldsymbol{e}_x&\boldsymbol{e}_y&\boldsymbol{e}_z\\\dfrac{\partial}{\partial x'}&\dfrac{\partial}{\partial y'}&\dfrac{\partial}{\partial z'}\\(x-x')&(y-y')&(z-z')\end{vmatrix}=0+0=0\end{aligned}$$

所以有

$$\nabla\times\frac{\boldsymbol{r}}{r^3}=-\nabla'\times\frac{\boldsymbol{r}}{r^3}=0$$

(4) 同理

$$\nabla\cdot\frac{\boldsymbol{r}}{r^3}=\left(\nabla\frac{1}{r^3}\right)\cdot\boldsymbol{r}+\frac{1}{r^3}\nabla\cdot\boldsymbol{r}=-\frac{3\boldsymbol{r}}{r^5}\cdot\boldsymbol{r}+\frac{3}{r^3}=0$$

而

$$\nabla'\cdot\frac{\boldsymbol{r}}{r^3}=\left(\nabla'\frac{1}{r^3}\right)\cdot\boldsymbol{r}+\frac{1}{r^3}\nabla'\cdot\boldsymbol{r}=\frac{3\boldsymbol{r}}{r^5}\cdot\boldsymbol{r}-\frac{3}{r^3}=0$$

所以有

$$\nabla\cdot\frac{\boldsymbol{r}}{r^3}=-\nabla'\cdot\frac{\boldsymbol{r}}{r^3}=0$$

证毕。从以上证明看出，矢量微分算符∇是对场变数求微分，即对场点坐标(x,y,z)进行微分作用，而∇'是对源变数求微分，即对源点坐标(x',y',z')进行微分作用，两种作用相差一负号。

Ⅱ.计算如下：

(1) 设 $r=(x-x_0)\boldsymbol{e}_x+(y-y_0)\boldsymbol{e}_y+(z-z_0)\boldsymbol{e}_z$，则

$$r=[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2]^{\frac{1}{2}}$$

所以有

$$\nabla r=\frac{\partial r}{\partial x}\boldsymbol{e}_x+\frac{\partial r}{\partial y}\boldsymbol{e}_y+\frac{\partial r}{\partial z}\boldsymbol{e}_z=\frac{(x-x_0)}{r}\boldsymbol{e}_x+\frac{(y-y_0)}{r}\boldsymbol{e}_y+\frac{(z-z_0)}{r}\boldsymbol{e}_z=\frac{\boldsymbol{r}}{r}$$

(2) 同理

$$\nabla\cdot r=\frac{\partial}{\partial x}(x-x_0)+\frac{\partial}{\partial y}(y-y_0)+\frac{\partial}{\partial z}(z-z_0)=1+1+1=3$$

(3) 同理

$$\nabla\times\boldsymbol{r}=\begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ (x-x_0) & (y-y_0) & (z-z_0)\end{vmatrix}$$

$$=\left[\frac{\partial(z-z_0)}{\partial y}-\frac{\partial(y-y_0)}{\partial z}\right]\boldsymbol{e}_x+\left[\frac{\partial(x-x_0)}{\partial z}-\frac{\partial(z-z_0)}{\partial x}\right]\boldsymbol{e}_y+$$

$$\left[\frac{\partial(y-y_0)}{\partial x}-\frac{\partial(z-z_0)}{\partial y}\right]\boldsymbol{e}_z=0$$

(4) $(\boldsymbol{a}\cdot\nabla)\boldsymbol{r}=(a_x\nabla_x+a_y\nabla_y+a_z\nabla_z)(x\boldsymbol{e}_x+y\boldsymbol{e}_y+z\boldsymbol{e}_z)$

$$=a_x\boldsymbol{e}_x+a_y\boldsymbol{e}_y+a_z\boldsymbol{e}_z=\boldsymbol{a}$$

或

$$(\boldsymbol{a}\cdot\nabla)\boldsymbol{r}=\boldsymbol{a}\cdot\nabla\boldsymbol{r}=\boldsymbol{a}\cdot\mathscr{I}=\boldsymbol{a}$$

(5) $\nabla(\boldsymbol{a}\cdot\boldsymbol{r})=(\nabla_x+\nabla_y+\nabla_z)(a_xx+a_yy+z_zz)$

$$=a_x\boldsymbol{e}_x+a_y\boldsymbol{e}_y+a_z\boldsymbol{e}_z=\boldsymbol{a}$$

(6) $\nabla\cdot[\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})]=(\nabla_x+\nabla_y+\nabla_z)\cdot[(E_{0x}\boldsymbol{e}_x+E_{0y}\boldsymbol{e}_y+E_{0z}\boldsymbol{e}_z)$

$$\sin(k_xx+k_yy+k_zz)]$$

$$= (\nabla_x E_{0x} + \nabla_y E_{0y} + \nabla_z E_{0z})\sin(k_x x + k_y y + k_z z)]$$
$$= (E_{0x}k_x + E_{0y}k_y + E_{0z}k_z)\cos(k_x x + k_y y + k_z z)$$
$$= (\boldsymbol{E}_0 \cdot \boldsymbol{k})\cos(\boldsymbol{k}\cdot\boldsymbol{r})$$

或

$$\begin{aligned}\nabla\cdot[\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})] &= (\nabla\cdot\boldsymbol{E}_0)\sin(\boldsymbol{k}\cdot\boldsymbol{r}) + [\nabla\sin(\boldsymbol{k}\cdot\boldsymbol{r})]\cdot\boldsymbol{E}_0\\ &= 0 + \cos(\boldsymbol{k}\cdot\boldsymbol{r})\nabla(\boldsymbol{k}\cdot\boldsymbol{r})\cdot\boldsymbol{E}_0\\ &= \cos(\boldsymbol{k}\cdot\boldsymbol{r})\nabla(xk_x + yk_y + zk_z)\cdot\boldsymbol{E}_0\\ &= \cos(\boldsymbol{k}\cdot\boldsymbol{r})(k_x\boldsymbol{e}_x + k_y\boldsymbol{e}_y + k_z\boldsymbol{e}_z)\cdot\boldsymbol{E}_0\\ &= (\boldsymbol{E}_0\cdot\boldsymbol{k})\cos(\boldsymbol{k}\cdot\boldsymbol{r})\end{aligned}$$

(7) $$\nabla\times[\boldsymbol{E}_0\sin(\boldsymbol{k}\cdot\boldsymbol{r})] = \begin{vmatrix}\boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z\\ \nabla_x & \nabla_y & \nabla_z\\ E_{0x}\sin(\boldsymbol{k}\cdot\boldsymbol{r}) & E_{0y}\sin(\boldsymbol{k}\cdot\boldsymbol{r}) & E_{0z}\sin(\boldsymbol{k}\cdot\boldsymbol{r})\end{vmatrix}$$

$$\begin{aligned}&= [(\nabla_y E_{0z} - \nabla_z E_{0y})\sin(\boldsymbol{k}\cdot\boldsymbol{r})]\boldsymbol{e}_x +\\ &\quad [(\nabla_z E_{0x} - \nabla_x E_{0z})\sin(\boldsymbol{k}\cdot\boldsymbol{r})]\boldsymbol{e}_y +\\ &\quad [(\nabla_x E_{0y} - \nabla_y E_{0x})\sin(\boldsymbol{k}\cdot\boldsymbol{r})]\boldsymbol{e}_z\\ &= (E_{0z}k_y - E_{0y}k_z)\cos(\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{e}_x +\\ &\quad (E_{0x}k_z - E_{0z}k_x)\cos(\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{e}_x +\\ &\quad (E_{0y}k_x - E_{0x}k_y)\cos(\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{e}_x\\ &= -(\boldsymbol{E}_0\times\boldsymbol{k})\cos(\boldsymbol{k}\cdot\boldsymbol{r}) = \boldsymbol{k}\times\boldsymbol{E}_0\cos(\boldsymbol{k}\cdot\boldsymbol{r})\end{aligned}$$

1.5　计算下列各式：

$$\nabla\cdot\left(\frac{\boldsymbol{r}}{r}\right)\quad \nabla\times(r^n\boldsymbol{r})\quad \nabla\cdot(r^n\boldsymbol{r})\quad (r\neq 0)$$

【解】 (1) 设 $\boldsymbol{r} = (x-x_0)\boldsymbol{e}_x + (y-y_0)\boldsymbol{e}_y + (z-z_0)\boldsymbol{e}_z$，则有

$$\nabla\cdot\left(\frac{\boldsymbol{r}}{r}\right) = \nabla_{\boldsymbol{r}}\cdot\left(\frac{\boldsymbol{r}}{r}\right) + \nabla_r\cdot\left(\frac{\boldsymbol{r}}{r}\right) = \frac{1}{r}\nabla\cdot\boldsymbol{r} + \left(\nabla\frac{1}{r}\right)\cdot\boldsymbol{r}$$
$$= 3\frac{1}{r} + \left(-\frac{\boldsymbol{r}}{r^3}\cdot\boldsymbol{r}\right) = 3\frac{1}{r} - \frac{1}{r} = 2\frac{1}{r}$$

(2) 同理，设 $\boldsymbol{r} = (x-x_0)\boldsymbol{e}_x + (y-y_0)\boldsymbol{e}_y + (z-z_0)\boldsymbol{e}_z$，而

$$r^n = [(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{n}{2}}$$

则有

$$\begin{aligned}\nabla\times(r^n\boldsymbol{r}) &= \nabla_{\boldsymbol{r}}\times(r^n\boldsymbol{r}) + \nabla_{r^n}\times(r^n\boldsymbol{r})\\ &= r^n\nabla_{\boldsymbol{r}}\times\boldsymbol{r} + (\nabla_{r^n}r^n)\times\boldsymbol{r} = (\nabla_{r^n}r^n)\times\boldsymbol{r}\\ &= \frac{n}{2}[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{\frac{n}{2}-1}\\ &\quad 2[(x-x_0)\boldsymbol{e}_x + (y-y_0)\boldsymbol{e}_y + (z-z_0)\boldsymbol{e}_z]\times\boldsymbol{r}\\ &= nr^n\frac{\boldsymbol{r}}{r^2}\times\boldsymbol{r} = 0\end{aligned}$$

(3) 同理,设 $\boldsymbol{r} = (x - x_0)\boldsymbol{e}_x + (y - y_0)\boldsymbol{e}_y + (z - z_0)\boldsymbol{e}_z$,而

$$r^n = [(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{\frac{n}{2}}$$

则有

$$\begin{aligned}\nabla\cdot(r^n\boldsymbol{r}) &= \nabla_r\cdot(r^n\boldsymbol{r}) + \nabla_{r^n}\cdot(r^n\boldsymbol{r}) = r^n(\nabla_r\cdot\boldsymbol{r}) + (\nabla_{r^n}r^n)\cdot\boldsymbol{r}\\ &= 3r^n + nr^n = (3+n)r^n\end{aligned}$$

1.6　若 $\boldsymbol{a}$ 为常矢量,证明除 $\boldsymbol{r} = 0$ 的点以外有:

$$\nabla\left(\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r^3}\right) = -\nabla\times\left(\frac{\boldsymbol{a}\times\boldsymbol{r}}{r^3}\right)$$

【证明】　方程左边为

$$\begin{aligned}\nabla\left(\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r^3}\right) &= \nabla_{r^3}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r^3}\right) + \nabla_a\left(\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r^3}\right) + \nabla_r\left(\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r^3}\right)\\ &= \nabla_{r^3}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r^3}\right) + \nabla_r\left(\frac{\boldsymbol{a}\cdot\boldsymbol{r}}{r^3}\right) = \left(\nabla_r\frac{1}{r^3}\right)(\boldsymbol{a}\cdot\boldsymbol{r}) + \left(\frac{1}{r^3}\nabla_r\right)(\boldsymbol{a}\cdot\boldsymbol{r})\\ &= \boldsymbol{a}\times\left(\nabla_{r^3}\frac{1}{r^3}\times\boldsymbol{r}\right) + \left(\boldsymbol{a}\cdot\nabla_{r^3}\frac{1}{r^3}\right)\boldsymbol{r} +\\ &\quad \boldsymbol{a}\times\left(\frac{1}{r^3}\nabla_r\times\boldsymbol{r}\right) + \left(\boldsymbol{a}\cdot\frac{1}{r^3}\nabla_r\right)\boldsymbol{r}\\ &= 0 + \left(\boldsymbol{a}\cdot\nabla_{r^3}\frac{1}{r^3}\right)\boldsymbol{r} + 0 + \left(\boldsymbol{a}\cdot\frac{1}{r^3}\nabla_r\right)\boldsymbol{r}\\ &= \left(\boldsymbol{a}\cdot\nabla_{r^3}\frac{1}{r^3}\right)\boldsymbol{r} + \left(\boldsymbol{a}\cdot\frac{1}{r^3}\nabla_r\right)\boldsymbol{r}\end{aligned}$$

方程右边为

$$\begin{aligned}-\nabla\times\left(\frac{\boldsymbol{a}\times\boldsymbol{r}}{r^3}\right) &= -\nabla_{r^3}\times\left(\frac{\boldsymbol{a}\times\boldsymbol{r}}{r^3}\right) - \nabla_r\times\left(\frac{\boldsymbol{a}\times\boldsymbol{r}}{r^3}\right)\\ &= -\nabla_{r^3}\frac{1}{r^3}\times(\boldsymbol{a}\times\boldsymbol{r}) - \frac{1}{r^3}\nabla_r\times(\boldsymbol{a}\times\boldsymbol{r})\\ &= -\left[\left(\nabla_{r^3}\frac{1}{r^3}\cdot\boldsymbol{r}\right)\boldsymbol{a} - \left(\nabla_{r^3}\frac{1}{r^3}\cdot\boldsymbol{a}\right)\boldsymbol{r}\right] -\\ &\quad \left[\left(\frac{1}{r^3}\nabla_r\cdot\boldsymbol{r}\right)\boldsymbol{a} - \left(\frac{1}{r^3}\nabla_r\cdot\boldsymbol{a}\right)\boldsymbol{r}\right]\\ &= -\left(-3\frac{\boldsymbol{r}}{r^5}\cdot\boldsymbol{r}\right)\boldsymbol{a} + \left(\boldsymbol{a}\cdot\nabla_{r^3}\frac{1}{r^3}\right)\boldsymbol{r} - 3\frac{1}{r^3}\boldsymbol{a} + \left(\boldsymbol{a}\cdot\frac{1}{r^3}\nabla_r\right)\boldsymbol{r}\\ &= \left(\boldsymbol{a}\cdot\nabla_{r^3}\frac{1}{r^3}\right)\boldsymbol{r} + \left(\boldsymbol{a}\cdot\frac{1}{r^3}\nabla_r\right)\boldsymbol{r}\end{aligned}$$

方程左边等于右边,证毕。

1.7　设 $\boldsymbol{c}$ 为恒矢量,$\boldsymbol{r}$ 为位置矢量,计算下列各式:

(1) $\nabla\times(\boldsymbol{c}\times\boldsymbol{r})$;(2) ∇r^2;(3) $\nabla^2\left(\frac{1}{r}\right)(r \neq 0)$。

【解】　(1) 将 $\boldsymbol{c}$ 和 $\boldsymbol{r}$ 写为分量式

$$\boldsymbol{c} = c_x\boldsymbol{e}_x + c_y\boldsymbol{e}_y + c_z\boldsymbol{e}_z$$

$$\boldsymbol{r} = (x - x_0)\boldsymbol{e}_x + (y - y_0)\boldsymbol{e}_y + (z - z_0)\boldsymbol{e}_z$$

所以

$$\boldsymbol{c} \times \boldsymbol{r} = \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ c_x & c_y & c_z \\ x & y & z \end{vmatrix} = (c_y z - c_z y)\boldsymbol{e}_x + (c_z x - c_x z)\boldsymbol{e}_y + (c_x y - c_y x)\boldsymbol{e}_z$$

$$\begin{aligned}\nabla \times (\boldsymbol{c} \times \boldsymbol{r}) &= \begin{vmatrix} \boldsymbol{e}_x & \boldsymbol{e}_y & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (c_y z - c_z y) & (c_z x - c_x z) & (c_x y - c_y x) \end{vmatrix} \\ &= 2(c_x\boldsymbol{e}_x + c_y\boldsymbol{e}_y + c_z\boldsymbol{e}_z) = 2\boldsymbol{c}\end{aligned}$$

(2) 同理

$$\nabla r^2 = \frac{\mathrm{d}}{\mathrm{d}r}r^2 = 2r\boldsymbol{e}_r = 2\boldsymbol{r}$$

(3) 同理

$$\begin{aligned}\nabla^2\left(\frac{1}{r}\right) &= \nabla \cdot \nabla\frac{1}{r} = \nabla\left(-\frac{\boldsymbol{r}}{r^3}\right) = -\frac{1}{r^3}\,\nabla \cdot \boldsymbol{r} - \nabla\left(\frac{1}{r^3}\right) \cdot \boldsymbol{r} \\ &= -\frac{3}{r} + 3\,\frac{\boldsymbol{r}}{r^3} \cdot \boldsymbol{r} = 0\end{aligned}$$

1.8 直接由式 $\boldsymbol{E}(\boldsymbol{x}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho(\boldsymbol{x}')\boldsymbol{r}}{r^3}\mathrm{d}V'$ 证明：

(1) $\nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = \dfrac{1}{\varepsilon_0}\rho(\boldsymbol{x})$；

(2) $\nabla \times \boldsymbol{E}(\boldsymbol{x}) = 0$。

其中 $\boldsymbol{x}$ 为场点坐标，$\boldsymbol{x}'$ 为源点坐标。

【证明】 Ⅰ. $\nabla \cdot \boldsymbol{E}(\boldsymbol{x}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \rho(\boldsymbol{x}')\,\nabla \cdot \left(\frac{\boldsymbol{r}}{r^3}\right)\mathrm{d}V' = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{x}')\,\nabla \cdot \left(-\nabla\frac{1}{r}\right)\mathrm{d}V'$

$$= \frac{1}{\varepsilon_0}\int \rho(\boldsymbol{x}')\,\frac{1}{4\pi}\,\nabla^2\left(\frac{1}{r}\right)\mathrm{d}V' \tag{1}$$

因为

$$\nabla^2\frac{1}{r^2} = -4\pi\delta(\boldsymbol{x} - \boldsymbol{x}') \tag{2}$$

所以(1) 式为

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{\varepsilon_0}\int \rho(\boldsymbol{x}')\delta(\boldsymbol{x} - \boldsymbol{x}')\mathrm{d}V' = \frac{\rho(\boldsymbol{x}')}{\varepsilon_0} \tag{3}$$

证毕。

Ⅱ. $\nabla \times \boldsymbol{E}(\boldsymbol{x}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \rho(\boldsymbol{x}')\,\nabla \times \left(\frac{\boldsymbol{r}}{r^3}\right)\mathrm{d}V' = \frac{1}{4\pi\varepsilon_0}\int \rho(\boldsymbol{x}')\,\nabla \times \left(-\nabla\frac{1}{r}\right)\mathrm{d}V' \quad (4)$

因为

$$\nabla\times\left(-\nabla\frac{1}{r}\right)=0 \tag{5}$$

所以(4) 式为

$$\nabla\times \boldsymbol{E}(\boldsymbol{x})=0 \tag{6}$$

证毕。

1.9　$\nabla^2\varphi$定义为空间的标量函数φ的梯度的散度,即$\nabla^2\varphi=\nabla\cdot(\nabla\varphi)$. 试根据这个定义,推导出在正交曲线坐标系中$\nabla^2\varphi$的表达式。并由此写出$\nabla^2\varphi$在柱坐标系和球坐标系中的表达式。

【解】　根据 1.3 题对$\nabla\varphi$和$\nabla\cdot\boldsymbol{A}$的表达式及$\nabla^2\varphi$的定义,有

$$\nabla^2\varphi=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\varphi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial u_3}\right)\right]$$

而柱坐标系的标度因子、三个分坐标分量和三个基矢为

$$u_1=r,u_2=\phi,u_3=z$$
$$h_1=1,h_2=r,h_3=1$$
$$\boldsymbol{e}_1=\boldsymbol{e}_r,\boldsymbol{e}_2=\boldsymbol{e}_\phi,\boldsymbol{e}_3=\boldsymbol{e}_z$$

所以

$$\begin{aligned}\nabla^2\varphi&=\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial\varphi}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial\varphi}{\partial z}\right)\right]\\&=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\phi^2}+\frac{\partial^2\varphi}{\partial z^2}\end{aligned}$$

在球坐标系中的标度因子、三个分坐标分量和三个基矢为

$$u_1=r,u_2=\theta,u_3=\phi$$
$$h_1=1,h_2=r,h_3=r\sin\theta$$
$$\boldsymbol{e}_1=\boldsymbol{e}_r,\boldsymbol{e}_2=\boldsymbol{e}_\theta,\boldsymbol{e}_3=\boldsymbol{e}_\phi$$

所以有

$$\begin{aligned}\nabla^2\varphi&=\frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\varphi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\varphi}{\partial\phi}\right)\right]\\&=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}\end{aligned}$$

1.10　应用高斯公式证明

$$\int_V\mathrm{d}V\,\nabla\times\boldsymbol{f}=\oint_S\mathrm{d}\boldsymbol{S}\times\boldsymbol{f}$$

【证明】　设$\boldsymbol{c}$为非零的任意常矢量,用$\boldsymbol{c}$左点乘方程的左边有

$$\boldsymbol{c}\cdot\int_V\mathrm{d}V\,\nabla\times\boldsymbol{f}=\int_V\mathrm{d}V[\boldsymbol{c}\cdot(\nabla\times\boldsymbol{f})] \tag{1}$$

由矢量分析公式

$$\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B})=(\nabla\times\boldsymbol{A})\cdot\boldsymbol{B}-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$$

令(1) 式中的 $\boldsymbol{f}=\boldsymbol{A},\boldsymbol{c}=\boldsymbol{B}$,有

$$\nabla\cdot(\boldsymbol{f}\times\boldsymbol{c})=(\nabla\times\boldsymbol{f})\cdot\boldsymbol{c}-\boldsymbol{f}\cdot(\nabla\times\boldsymbol{c})=\boldsymbol{c}\cdot(\nabla\times\boldsymbol{f})$$

因此(1) 式右边为

$$\int_V \mathrm{d}V[\boldsymbol{c}\cdot(\nabla\times\boldsymbol{f})]=\int_V \mathrm{d}V\,\nabla\cdot(\boldsymbol{f}\times\boldsymbol{c})$$

又根据高斯公式有

$$\begin{aligned}\int_V \mathrm{d}V\,\nabla\cdot(\boldsymbol{f}\times\boldsymbol{c})&=\oint_S(\boldsymbol{f}\times\boldsymbol{c})\cdot\mathrm{d}\boldsymbol{S}=\oint_S(\boldsymbol{f}\times c)\cdot\boldsymbol{n}\mathrm{d}S\\&=\oint_S\boldsymbol{c}\cdot(\boldsymbol{n}\times\boldsymbol{f})\mathrm{d}S=\boldsymbol{c}\cdot\oint_S\mathrm{d}\boldsymbol{S}\times\boldsymbol{f}\end{aligned}\tag{2}$$

由(1) 式和(2) 式有

$$\boldsymbol{c}\cdot\oint_V\mathrm{d}V\,\nabla\times\boldsymbol{f}=\boldsymbol{c}\cdot\oint_S\mathrm{d}\boldsymbol{S}\times\boldsymbol{f}$$

因为 $\boldsymbol{c}$ 为非零的任意常矢量,所以得

$$\int_V\mathrm{d}V\,\nabla\times\boldsymbol{f}=\oint_S\mathrm{d}\boldsymbol{S}\times\boldsymbol{f}$$

证毕。

1.11 应用斯托克斯公式证明:

(1) $\int_S\mathrm{d}\boldsymbol{S}\times\nabla\varphi=\oint_L\varphi\mathrm{d}\boldsymbol{l}$;

(2) $\oint_L(\boldsymbol{a}\times\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{l}=2\int_S\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S}$,式中 $\boldsymbol{a}$ 为常矢量。

【证明】 Ⅰ. 设 $\boldsymbol{c}$ 为非零的任意常矢量,令 $\boldsymbol{B}=\varphi\boldsymbol{c}$ 代入斯托克斯公式

$$\int_S\nabla\times\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}=\oint_L\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l}\tag{1}$$

的左边,利用矢量分析公式,有

$$\begin{aligned}\int_S\nabla\times(\varphi\boldsymbol{a})\cdot\mathrm{d}\boldsymbol{S}&=\int_S[\nabla\varphi\times\boldsymbol{a}+\varphi(\nabla\times\boldsymbol{a})]\cdot\mathrm{d}\boldsymbol{S}\\&=\int_S\nabla\varphi\times\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S}=-\int_S\boldsymbol{a}\times\nabla\varphi\cdot\mathrm{d}\boldsymbol{S}\\&=-\int_S\boldsymbol{a}\cdot\nabla\varphi\times\mathrm{d}\boldsymbol{S}=\int_S\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S}\times\nabla\varphi\\&=\boldsymbol{a}\cdot\int_S\mathrm{d}\boldsymbol{S}\times\nabla\varphi\end{aligned}\tag{2}$$

而(1) 式的右边为

$$\oint_L\varphi\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}=\boldsymbol{a}\cdot\oint_L\varphi\mathrm{d}\boldsymbol{l}\tag{3}$$

由(2) 式和(3) 式相等,即(1) 式方程的两边相等,故有

$$\boldsymbol{a}\cdot\left[\int_S\mathrm{d}\boldsymbol{S}\times\nabla\varphi-\oint_L\varphi\mathrm{d}\boldsymbol{l}\right]=0\tag{4}$$

因 $\boldsymbol{a}$ 为非零的任意常矢量,故得

$$\int_S \mathrm{d}\boldsymbol{S} \times \nabla \varphi = \oint_L \varphi \mathrm{d}\boldsymbol{l} \tag{5}$$

证毕。

Ⅱ.根据矢量分析公式有

$$\begin{aligned}\nabla \times (\boldsymbol{a} \times \boldsymbol{r}) &= (\boldsymbol{r} \cdot \nabla)\boldsymbol{a} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{r} + (\nabla \cdot \boldsymbol{r})\boldsymbol{a} - (\nabla \cdot \boldsymbol{a})\boldsymbol{r} \\ &= -\boldsymbol{a} + 3\boldsymbol{a} = 2\boldsymbol{a}\end{aligned} \tag{6}$$

令 $\boldsymbol{B} = \boldsymbol{a} \times \boldsymbol{r}$,根据斯托克斯公式

$$\int_S \nabla \times \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \oint_L \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} \tag{7}$$

有

$$\oint_L \boldsymbol{a} \times \boldsymbol{r} \cdot \mathrm{d}\boldsymbol{l} = \int_S \nabla \times (\boldsymbol{a} \times \boldsymbol{r}) \cdot \mathrm{d}\boldsymbol{S} = 2\int_S \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} \tag{8}$$

证毕。

1.12　已知一个电荷系统的电偶极矩定义为

$$\boldsymbol{P}(t) = \int_V \rho(\boldsymbol{x}', t)\boldsymbol{x}' \mathrm{d}V'$$

利用电荷守恒定律 $\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$,证明 $\boldsymbol{P}$ 的变化率为

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \int_V \boldsymbol{J}(\boldsymbol{x}', t) \mathrm{d}V'$$

【证明】　电偶极子对时间的变化率为

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho(\boldsymbol{r}', t)\boldsymbol{r}' \mathrm{d}V' = \int_V \frac{\partial}{\partial t}[\rho(\boldsymbol{r}', t)\boldsymbol{r}'] \mathrm{d}V' \\ &= \int_V \frac{\partial \rho(\boldsymbol{r}', t)}{\partial t}\boldsymbol{r}' \mathrm{d}V' = \int_V [-\nabla' \cdot \boldsymbol{J}]\boldsymbol{r}' \mathrm{d}V' \\ &= -\int_V (\nabla' \cdot \boldsymbol{J})x' \mathrm{d}V' \boldsymbol{e}_x - \int_V (\nabla' \cdot \boldsymbol{J})y' \mathrm{d}V' \boldsymbol{e}_y - \int_V (\nabla' \cdot \boldsymbol{J})z' \mathrm{d}V' \boldsymbol{e}_z\end{aligned} \tag{1}$$

其中第一项为

$$\begin{aligned}\int_V (\nabla' \cdot \boldsymbol{J})x' \mathrm{d}V' &= \int_V x'(\nabla' \cdot \boldsymbol{J}) \mathrm{d}V' = \int_V [\nabla' \cdot (x'\boldsymbol{J}) - (\nabla' x') \cdot \boldsymbol{J}] \mathrm{d}V' \\ &= \oint_S x'\boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}' - \int_V J_x \mathrm{d}V'\end{aligned} \tag{2}$$

式中封闭曲面 S 为电荷系统的边界面,电流不能流出此边界面,因此

$$\int_S x'\boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}' = 0 \tag{3}$$

所以

$$\int_V (\nabla' \cdot \boldsymbol{J})x' \mathrm{d}V' = -\int_V J_x \mathrm{d}V' \tag{4}$$

同理,可得(1) 式中的第二项和第三项为

$$\int_V (\nabla' \cdot \boldsymbol{J})y' \mathrm{d}V' = -\int_V J_y \mathrm{d}V' \tag{5}$$

$$\int_V (\nabla' \cdot \boldsymbol{J}) z' \mathrm{d}V' = -\int_V J_z \mathrm{d}V' \tag{6}$$

于是

$$-\int_V (\nabla' \cdot \boldsymbol{J}) \boldsymbol{r}' \mathrm{d}V' = -\int_V \boldsymbol{J} \mathrm{d}V' \tag{7}$$

将(7)式代入(1)式即得

$$\frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} = \int_V \boldsymbol{J}(\boldsymbol{r}', t) \mathrm{d}V' \tag{8}$$

证毕。

1.13　若 $\boldsymbol{m}$ 是常矢量，证明除 $\boldsymbol{R}=0$ 点以外，矢量 $\boldsymbol{A}=\dfrac{\boldsymbol{m}\times\boldsymbol{R}}{R^3}$ 的旋度等于标量 $\varphi=\dfrac{\boldsymbol{m}\cdot\boldsymbol{R}}{R^3}$ 的梯度的负值，即

$$\nabla\times\boldsymbol{A} = -\nabla\varphi \quad (R\neq 0)$$

其中 R 为坐标原点到场点的距离，$\boldsymbol{R}$ 方向由原点指向场点。

【证明】　因为 $\nabla\dfrac{1}{R}=-\dfrac{\boldsymbol{R}}{R^3}$，所以有

$$\nabla\times\boldsymbol{A} = \nabla\times\left(\frac{\boldsymbol{m}\times\boldsymbol{R}}{R^3}\right) = -\nabla\times\left[\boldsymbol{m}\times\left(\nabla\frac{1}{R}\right)\right] = \nabla\times\left[\left(\nabla\frac{1}{R}\right)\times\boldsymbol{m}\right]$$

利用矢量公式[数学附录(1.28)]有

$$\begin{aligned}\nabla\times\boldsymbol{A} &= (\nabla\cdot\boldsymbol{m})\nabla\left(\frac{1}{R}\right) + (\boldsymbol{m}\cdot\nabla)\nabla\frac{1}{R} - \left[\nabla\cdot\left(\nabla\frac{1}{R}\right)\right]\boldsymbol{m} - \left[\left(\nabla\frac{1}{R}\right)\cdot\nabla\right]\boldsymbol{m} \\ &= (\boldsymbol{m}\cdot\nabla)\nabla\frac{1}{R} - \left[\nabla^2\frac{1}{R}\right]\boldsymbol{m} \end{aligned} \tag{1}$$

(1) 式中

$$\begin{aligned}\nabla^2\frac{1}{R} &= \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}\frac{1}{R} = \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[-\frac{1}{R^2}\frac{\partial R}{\partial x_i}\right] = -\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[\frac{x_i}{R^3}\right] \\ &= -\sum_{i=1}^{3}\left[\frac{1}{R^3} - \frac{3x_i}{R^4}\frac{\partial R}{\partial x_i}\right] = -\frac{3}{R^3} + \frac{3R^2}{R^5} = 0 \end{aligned} \tag{2}$$

将(2)式代入(1)式有

$$\nabla\times\boldsymbol{A} = (\boldsymbol{m}\cdot\nabla)\nabla\frac{1}{R} \quad (\boldsymbol{R}\neq 0) \tag{3}$$

又因为

$$\begin{aligned}\nabla\varphi &= \nabla\left(\frac{\boldsymbol{m}\cdot\boldsymbol{R}}{R^3}\right) = -\nabla\left[\boldsymbol{m}\cdot\left(\nabla\frac{1}{R}\right)\right] \\ &= -\boldsymbol{m}\times\left[\nabla\times\left(\nabla\frac{1}{R}\right)\right] - \left(\nabla\frac{1}{R}\right)\times(\nabla\times\boldsymbol{m}) - (\boldsymbol{m}\cdot\nabla)\nabla\frac{1}{R} - \left[\left(\nabla\frac{1}{R}\right)\cdot\nabla\right]\boldsymbol{m} \\ &= -(\boldsymbol{m}\cdot\nabla)\nabla\frac{1}{R} \end{aligned} \tag{4}$$

比较(3)式和(4)式得

$$\nabla\times\boldsymbol{A}=-\nabla\varphi\quad(\boldsymbol{R}\neq 0)\tag{5}$$

证毕。

1.14　证明均匀介质内部的体极化电荷密度 ρ_p 总是等于体自由电荷密度的 $-\left(1-\frac{\varepsilon_0}{\varepsilon}\right)$倍。

【证明】　因为

$$\rho_p=-\nabla\cdot\boldsymbol{P}\tag{1}$$

$$\boldsymbol{P}=\chi_e\varepsilon_0\boldsymbol{E}\tag{2}$$

所以

$$\begin{aligned}\rho_p&=-\nabla\cdot\boldsymbol{P}=-\nabla\cdot(\chi_e\varepsilon_0\boldsymbol{E})=-\nabla\cdot[(\varepsilon-\varepsilon_0)\boldsymbol{E}]\\&=\left(\frac{\varepsilon_0}{\varepsilon}-1\right)\nabla\cdot\boldsymbol{D}=\left(\frac{\varepsilon_0}{\varepsilon}-1\right)\rho=-\left(1-\frac{\varepsilon_0}{\varepsilon}\right)\rho\end{aligned}\tag{3}$$

证毕。

1.15　在恒定电流情况下，证明均匀介质内部的磁化电流密度 $\boldsymbol{J}_m$ 总是等于自由电流密度的$\left(\frac{\mu}{\mu_0}-1\right)$倍。

【证明】　因为

$$\boldsymbol{H}=\frac{\boldsymbol{B}}{\mu_0}-\boldsymbol{M}\tag{1}$$

所以两边取旋度有

$$\nabla\times\boldsymbol{H}=\nabla\times\left(\frac{\boldsymbol{B}}{\mu_0}-\boldsymbol{M}\right)=\frac{1}{\mu_0}\nabla\times\boldsymbol{B}-\nabla\times\boldsymbol{M}\tag{2}$$

因为

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}_f+\frac{\partial\boldsymbol{D}}{\partial t}=\boldsymbol{J}_f\tag{3}$$

$$\boldsymbol{J}_m=\nabla\times\boldsymbol{M}\tag{4}$$

所以，将(3)式和(4)式代入(2)式有

$$\boldsymbol{J}_f=\frac{\mu}{\mu_0}\nabla\times\boldsymbol{H}-\boldsymbol{J}_m=\frac{\mu}{\mu_0}\boldsymbol{J}_f-\boldsymbol{J}_m\tag{5}$$

即

$$\boldsymbol{J}_m=\left(\frac{\mu}{\mu_0}-1\right)\boldsymbol{J}_f\tag{6}$$

证毕。

1.16　有一内外半径分别为 r_1 和 r_2 的空心介质球，介质的电容率为 ε。使介质内均匀带静止自由电荷，其体密度为 ρ_f，试求：

(1) 空间各点的电场；

(2) 极化体电荷和极化面电荷分布。