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To Kirill, Vanya, Lena, and Andrei

Preface

The goal of these lectures is to describe the essence of the orbit method for non-experts and to attract the younger generation of mathematicians to some old and still unsolved problems in representation theory where I believe the orbit method could help.

It is said that to become a scientist is the same as to catch a train at full speed. Indeed, while you are learning well-known facts and theories, many new important achievements happen. So, you are always behind the present state of the science. The only way to overcome this obstacle is to "jump", that is, to learn very quickly and thoroughly some relatively small domain, and have only a general idea about all the rest.

So, in my exposition I deliberately skip many details that are not absolutely necessary for understanding the main facts and ideas. The most persistent readers can try to reconstruct these details using other sources. I hope, however, that for the majority of users the book will be sufficiently self-contained.

The level of exposition is different in different chapters so that both experts and beginners can find something interesting and useful for them. Some of this material is contained in my book [**Ki2**] and in the surveys [**Ki5**], [**Ki6**], and [**Ki9**]. But a systematic and reasonably self-contained exposition of the orbit method is given here for the first time.

I wrote this book simultaneously in English and in Russian. For several reasons the English edition appears later than the Russian one and differs from it in the organization of material.

Sergei Gelfand was the initiator of the publication of this book and pushed me hard to finish it in time.

Craig Jackson read the English version of the book and made many useful corrections and remarks.

The final part of the work on the book was done during my visits to the Institut des Hautes Études Scientifiques (Bures-sur-Yvette, France) and the Max Planck Institute of Mathematics (Bonn, Germany). I am very grateful to both institutions for their hospitality.

In conclusion I want to thank my teachers, friends, colleagues, and especially my students, from whom I learned so much.

Introduction

The idea behind the orbit method is to unite harmonic analysis with symplectic geometry. This can be considered as a part of the more general idea of the unification of mathematics and physics.

In fact, this is a *post factum* formulation. Historically, the orbit method was proposed in [**Ki1**] for the description of the unitary dual (i.e. the set of equivalence classes of unitary irreducible representations) of nilpotent Lie groups. It turned out that the method not only solves this problem but also gives simple and visual solutions to all other principal questions in representation theory: topological structure of the unitary dual, the explicit description of the restriction and induction functors, the formulae for generalized and infinitesimal characters, the computation of the Plancherel measure, etc.

Moreover, the answers make sense for general Lie groups and even beyond, although sometimes with more or less evident corrections. I already mentioned in [**Ki1**] the possible applications of the orbit method to other types of Lie groups, but the realization of this program has taken a long time and is still not accomplished despite the efforts of many authors.

I cannot mention here all those who contributed to the development of the orbit method, nor give a complete bibliography: Mathematical Reviews now contains hundreds of papers where coadjoint orbits are mentioned and thousands of papers on geometric quantization (which is the physical counterpart of the orbit method). But I certainly ought to mention the outstanding role of Bertram Kostant and Michel Duflo.

As usual, the faults of the method are the continuations of its advantages. I quote briefly the most important ones.

MERITS VERSUS DEMERITS

Merits	Demerits
1. Universality: the method works for Lie groups of any type over any field.	1. The recipes are not accurately and precisely developed.
2. The rules are visual, and are easy to memorize and illustrate by a picture.	2. Sometimes they are wrong and need corrections or modifications.
3. The method explains some facts which otherwise look mysterious.	3. It could be difficult to transform this explanation into a rigorous proof.
4. It provides a great amount of symplectic manifolds and Poisson commuting families of functions.	4. Most of the completely integrable dynamical systems were discovered earlier by other methods.
5. The method introduces new fundamental notions: coadjoint orbit and moment map.	5. The description of coadjoint orbits and their structures is sometimes not an easy problem.

For the reader's convenience we formulate the ideology of the orbit method here in the form of a "User's Guide" where practical instructions are given as to how to get answers to ten basic questions in representation theory.

These simple rules are applicable literally for all connected and simply connected nilpotent groups. For groups of general type we formulate the "ten amendments" to these rules in the main text of the book.

Throughout the User's Guide we use the following notation:

G – a connected simply connected Lie group;

$H \subset G$ – a closed connected subgroup;

$\mathfrak{g}$, $\mathfrak{h}$ – Lie algebras of G, H, respectively;

$\mathfrak{g}^*$, $\mathfrak{h}^*$ – the dual spaces to $\mathfrak{g}$, $\mathfrak{h}$, respectively;

$p\colon \mathfrak{g}^* \to \mathfrak{h}^*$ – the canonical projection;

σ – the canonical 2-form (symplectic structure) on a coadjoint orbit;

π_Ω – the unirrep of G corresponding to the orbit $\Omega \subset \mathfrak{g}^*$;

$\rho_{F,H}$ – the 1-dimensional unirrep of H given by $\rho_{F,H}(\exp X) = e^{2\pi i \langle F, X \rangle}$;

P_A – the G-invariant polynomial on $\mathfrak{g}^*$ related to $A \in Z(\mathfrak{g})$, the center of $U(\mathfrak{g})$.

For other notation, when it is not self-explanatory, the reader must consult the Index and look for definitions given in the main text or in the Appendices.

USER'S GUIDE

What you want	What you have to do
1. Describe the unitary dual $\widehat{G}$ as a topological space.	Take the space $\mathcal{O}(G)$ of coadjoint orbits with the quotient topology.
2. Construct the unirrep π_Ω associated to the orbit $\Omega \in \mathfrak{g}^*$.	Choose a point $F \in \Omega$, take a subalgebra $\mathfrak{h}$ of maximal dimension subordinate to F, and put $\pi_\Omega = \operatorname{Ind}_H^G \rho_{F,H}$.
3. Describe the spectrum of $\operatorname{Res}_H^G \pi_\Omega$.	Take the projection $p(\Omega)$ and split it into H-orbits.
4. Describe the spectrum of $\operatorname{Ind}_H^G \pi_\omega$.	Take the G-saturation of $p^{-1}(\omega)$ and split it into G-orbits.
5. Describe the spectrum of the tensor product $\pi_{\Omega_1} \otimes \pi_{\Omega_2}$.	Take the arithmetic sum $\Omega_1 + \Omega_2$ and split it into orbits.
6. Compute the generalized character of π_Ω.	$\operatorname{tr} \pi_\Omega(\exp X) = \int_\Omega e^{2\pi i \langle F, X \rangle + \sigma}$ or $\langle \chi_\Omega , \varphi \rangle = \int_\Omega \widetilde{\varphi}(F)\, e^\sigma$.
7. Compute the infinitesimal character of π_Ω.	For $A \in Z(\mathfrak{g})$ take the value of $P_A \in Pol(\mathfrak{g}^*)^G$ on the orbit Ω.
8. What is the relation between π_Ω and $\pi_{-\Omega}$?	They are contragredient (dual) representations.
9. Find the functional dimension of π_Ω.	It is equal to $\frac{1}{2} \dim \Omega$.

10. Compute the Plancherel measure μ on $\widehat{G}$.	The measure on $\mathcal{O}(G)$ arising when the Lebesgue measure on $\mathfrak{g}^*$ is decomposed into canonical measures on coadjoint orbits.

These short instructions are developed in Chapter 3 and illustrated in the worked-out examples in the main text.

Finally, a technical remark. I am using the standard sign $\square$ to signal the end of a proof (or the absence of proof). I also use less standard notation:

$\diamondsuit$ – the end of an example;

$\heartsuit$ – the end of a remark;

$\clubsuit$ – the end of an exercise;

$\spadesuit$ – the end of a warning about a possible mistake or misunderstanding.

The most difficult exercises and parts of the text are marked by an asterisk (*).

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