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Hopf Algebras and Their Actions on Rings

Hopf 代数及其在环上的作用

Susan Montgomery

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图字：01-2016-3509 号

Hopf 代数及其在环上的作用

Hopf Daishu jiqi zai Huan shang de Zuoyong

图书在版编目 (CIP) 数据

Hopf 代数及其在环上的作用 = Hopf Algebras and Their Actions on Rings : 英文 / (美) 苏珊·蒙哥马利 (Susan Montgomery) 著 . — 影印本 . — 北京 : 高等教育出版社 , 2018.8
ISBN 978-7-04-050231-2
Ⅰ . ① H… Ⅱ . ①苏… Ⅲ . ① hopf 代数—英文 Ⅳ . ① O152
中国版本图书馆 CIP 数据核字 (2018) 第 164091 号

策划编辑 李 鹏　　责任编辑 李 鹏
封面设计 张申申　　责任印制 赵义民

出版发行 高等教育出版社
社址 北京市西城区德外大街 4 号
邮政编码 100120
购书热线 010-58581118
咨询电话 400-810-0598
网址 http://www.hep.edu.cn
http://www.hep.com.cn
网上订购 http://www.hepmall.com.cn
http://www.hepmall.com
http://www.hepmall.cn
印刷 北京盛通印刷股份有限公司

开本 787mm × 1092mm 1/16
印张 16
字数 400 千字
版次 2018 年 8 月第 1 版
印次 2018 年 8 月第 1 次印刷
定价 99.00 元

[物 料 号 50231-00]

出版者的话

近年来，我国的科学技术取得了长足进步，特别是在数学等自然科学基础领域不断涌现出一流的研究成果。与此同时，国内的科研队伍与国外的交流合作也越来越密切，越来越多的科研工作者可以熟练地阅读英文文献，并在国际顶级期刊发表英文学术文章，在国外出版社出版英文学术著作。

然而，在国内阅读海外原版英文图书仍不是非常便捷。一方面，这些原版图书主要集中在科技、教育比较发达的大中城市的大型综合图书馆以及科研院所的资料室中，普通读者借阅不甚容易；另一方面，原版书价格昂贵，动辄上百美元，购买也很不方便。这极大地限制了科技工作者对于国外先进科学技术知识的获取，间接阻碍了我国科技的发展。

高等教育出版社本着植根教育、弘扬学术的宗旨服务我国广大科技和教育工作者，同美国数学会（American Mathematical Society）合作，在征求海内外众多专家学者意见的基础上，精选该学会近年出版的数十种专业著作，组织出版了“美国数学会经典影印系列”丛书。美国数学会创建于 1888 年，是国际上极具影响力的专业学术组织，目前拥有近 30000 会员和 580 余个机构成员，出版图书 3500 多种，冯·诺依曼、莱夫谢茨、陶哲轩等世界级数学大家都是其作者。本影印系列涵盖了代数、几何、分析、方程、拓扑、概率、动力系统等所有主要数学分支以及新近发展的数学主题。

我们希望这套书的出版，能够对国内的科研工作者、教育工作者以及青年学生起到重要的学术引领作用，也希望今后能有更多的海外优秀英文著作被介绍到中国。

高等教育出版社

2016 年 12 月

反盗版举报电话　(010) 58581999　58582371　58582488

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高等教育出版社法律事务与版权管理部

邮政编码　100120

Preface

These lecture notes are an expanded version of ten lectures given at the CBMS conference on Hopf Algebras and Their Actions on Rings, which took place at DePaul University in Chicago, August 10-14, 1992.

It was a very good time to have such a conference, for several reasons. The most obvious of these is the current great interest in quantum groups; these are Hopf algebras which arose in statistical mechanics and now have connections to many areas of mathematics. However there have been a number of significant recent developments within Hopf algebras themselves. Several old conjectures of Kaplansky have recently been solved, the most striking of which is a kind of Lagrange's theorem for Hopf algebras. In a different direction there has been a lot of work on actions of Hopf algebras, which unifies earlier results known for group actions, actions of Lie algebras, and graded algebras.

The object of the meeting, and of these notes, was to bring together many of these recent developments; in fact there is a great deal of interconnection between the various directions. The point of view throughout, however, is the algebraic structure of Hopf algebras and their actions and coactions. Quantum groups are treated as an important example rather than as an end in themselves; never-the-less the reader interested in quantum groups should find much basic material here.

Most of Chapters 1 and 2 is old, and in fact appears in the books on Hopf algebras by Sweedler [S] and Abe [A]; this is also true of parts of Chapters 5 and 9. I have included this material in order to be as self-contained as possible; moreover some of the arguments are new. The rest of these notes has not previously appeared in book form. Although many of the proofs are only sketched, and even occasionally omitted (with appropriate references to the literature), enough detail is given so that this book could be used for a graduate level course. In fact these notes grew out of courses I gave at USC in 1989 and in 1992. A standard first-year graduate algebra class should be a sufficient prerequisite.

There are many people I wish to thank. First of all is Jeff Bergen, who organized the conference and made all the arrangements, and second are

the Supporting Lecturers: Miriam Cohen, Yukio Doi, Warren Nichols, Bodo Pareigis, Donald Passman, David Radford, Hans-Jürgen Schneider, Earl Taft, and Mitsuhiro Takeuchi. Many of their lectures are being collected and will appear in the volume [BeM 93].

In writing the notes, my deepest gratitude goes to Maria Lorenz, whose careful reading of the entire manuscript was invaluable, and to Hans Schneider, who provided many historical references as well as simplifying a number of proofs in the literature. I also want to thank Bill Chin, Davida Fischman, and Bodo Pareigis for their comments, Robert Blattner for making available to me his course notes from UCLA and for many conversations about Hopf algebras over the years, and the students in my two classes at USC whose questions on earlier versions of these notes were particlarly helpful: Ioana Boca, Paul Glezen, Michael Jochner, Horia Pop, and Yegan Satik. Finally, thanks to Lesley Newton for typing the manuscript.

Susan Montgomery
Los Angeles, June 1993

Contents

Chapter 1

Definitions and Examples

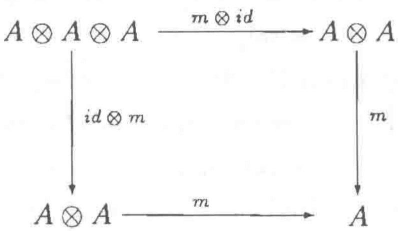
§1.1 Algebras and coalgebras

Throughout we let $\mathbf{k}$ be a field, although much of what we do is valid over any commutative ring. Tensor products are assumed to be over $\mathbf{k}$ unless stated otherwise.

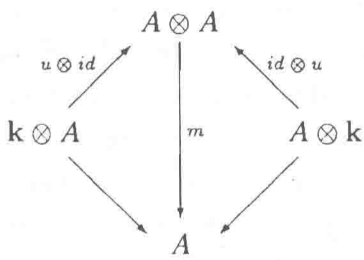
We first express the associative and unit properties of an algebra via maps so that we may dualize them.

1.1.1 DEFINITION. A $\mathbf{k}$-*algebra* (with unit) is a $\mathbf{k}$-vector space A together with two $\mathbf{k}$-linear maps, multiplication $m : A \otimes A \to A$ and unit $u : \mathbf{k} \to A$, such that the following diagrams are commutative:

a) associativity



b) unit



The two lower maps in b) are given by scalar multiplication. 1.1.1, b) gives the usual identity element in A by setting $1_A = u(1_{\mathbf{k}})$.

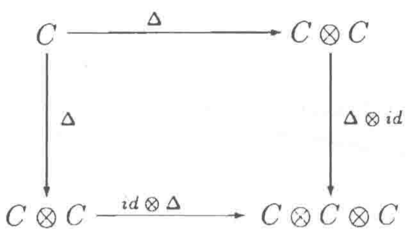
1.1.2 DEFINITION. For any $\mathbf{k}$-spaces V and W, the *twist map* $\tau : V \otimes W \to W \otimes V$ is given by $\tau(v \otimes w) = w \otimes v$.

Note that A is commutative $\Leftrightarrow m \circ \tau = m$ on $A \otimes A$.

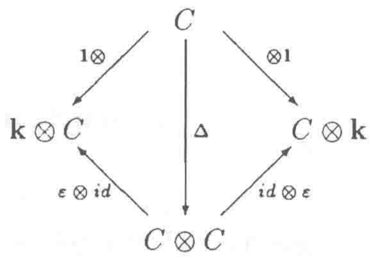
We now dualize the notion of algebra.

1.1.3 DEFINITION. A $\mathbf{k}$-*coalgebra* (with *counit*) is a $\mathbf{k}$-vector space C together with two $\mathbf{k}$-linear maps, *comultiplication* $\Delta : C \to C \otimes C$ and *counit* $\varepsilon : C \to \mathbf{k}$, such that the following diagrams are commutative:

a) coassociativity



b) counit



The two upper maps in 1.1.3.b) are given by $c \mapsto 1 \otimes c$ and $c \mapsto c \otimes 1$, for any $c \in C$. We say C is *cocommutative* if $\tau \circ \Delta = \Delta$.

Note that 1.1.3 b) gives that Δ is injective, just as 1.1.1 b) gives that m is surjective.

1.1.4 DEFINITION. Let C and D be coalgebras, with comultiplications Δ_C and Δ_D, and counits ε_C and ε_D, respectively.

a) A map $f : C \to D$ is a *coalgebra morphism* if $\Delta_D \circ f = (f \otimes f)\Delta_C$ and if $\varepsilon_C = \varepsilon_D \circ f$.

b) A subspace $I \subseteq C$ is a *coideal* if $\Delta I \subseteq I \otimes C + C \otimes I$ and if $\varepsilon(I) = 0$.

It is easy to check that if I is a coideal, then the $\mathbf{k}$-space C/I is a coalgebra with comultiplication induced from Δ, and conversely.

Finally, we may also use the twist map to dualize the notion of opposite algebra. For a given algebra A, recall that A^{op} is the algebra obtained by using A as a vector space, but with new multiplication $a^\circ \cdot b^\circ = (ba)^\circ$, for $a^\circ, b^\circ \in A^{op}$. In terms of maps this new multiplication is given by $m' : A \otimes A \to A$, where $m' = m \circ \tau$.

1.1.5 DEFINITION. Let C be a coalgebra. Then the *coopposite coalgebra* C^{cop} is given as follows: $C^{cop} = C$ as a vector space, with new comultiplication Δ' given by $\Delta' = \tau \circ \Delta$.

It is easy to see that C^{cop} is also a coalgebra.

§1.2. Duals of algebras and coalgebras

We shall now see that there is a very close relationship between algebras and coalgebras, by looking at their dual spaces.

For any $\mathbf{k}$-space V, let $V^* = \mathrm{Hom}_{\mathbf{k}}(V, \mathbf{k})$ denote the linear dual of V.

V and V^* determine a non-degenerate bilinear form $\langle\ ,\ \rangle : V^* \otimes V \to \mathbf{k}$ via $\langle f, v\rangle = f(v)$; we write it as a form since we frequently wish to think of V as acting on V^*. If $\phi : V \to W$ is $\mathbf{k}$-linear, then the *transpose* of ϕ is $\phi^* : W^* \to V^*$, given by

$$\phi^*(f)(v) = f(\phi(v)), \tag{1.2.1}$$

for all $f \in W^*, v \in V$.

1.2.2 LEMMA. *If C is a coalgebra, then C^* is an algebra, with multiplication $m = \Delta^*$ and unit $u = \varepsilon^*$. If C is cocommutative, then C^* is commutative.*

The Lemma is proved simply by dualizing the diagrams; one needs only the additional observation that since $C^* \otimes C^* \subseteq (C \otimes C)^*$, we may restrict Δ^* to get a map $m : C^* \otimes C^* \to C^*$. Explicitly, m is given by $m(f \otimes g)(c) = \Delta^*(f \otimes g)(c) = (f \otimes g)\Delta c$, for all $f, g \in C^*, c \in C$.

If we begin with an algebra A, however, difficulties arise. For, if A is not finite-dimensional, $A^* \otimes A^*$ is a proper subspace of $(A \otimes A)^*$ and thus the image of $m^* : A^* \to (A \otimes A)^*$ may not lie in $A^* \otimes A^*$. Of course if A is finite-dimensional, all is well, and A^* is a coalgebra. For the general case, we require a definition.

1.2.3 DEFINITION. Let A be a $\mathbf{k}$-algebra. The *finite dual* of A is $A^\circ = \{f \in A^* \mid f(I) = 0$, for some ideal I of A such that $\dim A/I < \infty\}$.

1.2.4 PROPOSITION. *If A is an algebra, then A° is a coalgebra, with comultiplication $\Delta = m^*$ and counit $\varepsilon = u^*$. If A is commutative, then A° is cocommutative.*

Explicity, $\Delta f(a \otimes b) = m^* f(a \otimes b) = f(ab)$, for all $f \in A^\circ, a, b \in A$.

We will prove 1.2.4 in Proposition 9.1.2. Some additional characterizations of A° will also be discussed in Chapter 9. In particular A° is the largest subspace V of A^* such that $m^*(V) \subseteq V \otimes V$.

§1.3 Bialgebras

Now we combine the notions of algebra and coalgebra.

1.3.1 DEFINITION. A $\mathbf{k}$-space B is a *bialgebra* if (B, m, u) is an algebra, (B, Δ, ε) is a coalgebra, and either of the following (equivalent) conditions holds:

1) Δ and ε are algebra morphisms
2) m and u are coalgebra morphisms.

As expected, a map $f : B \to B'$ of bialgebras is called a *bialgebra morphism* if it is both an algebra and a coalgebra morphism, and a subspace $I \subseteq B$ is a *biideal* if it is both an ideal and a coideal. The quotient B/I is a bialgebra precisely when I is a biideal of B.

1.3.2 EXAMPLE. Let G be any group and let $B = \mathbf{k}G$ be its group algebra. Then B is a bialgebra via $\Delta g = g \otimes g$ and $\varepsilon(g) = 1$, for all $g \in G$.

1.3.3 EXAMPLE. Let $\mathbf{g}$ be any $\mathbf{k}$-Lie algebra and let $B = U(\mathbf{g})$ be its universal enveloping algebra. Then B becomes a bialgebra by defining $\Delta x = x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 0$, for all $x \in \mathbf{g}$.

Note that examples 1.3.2 and 1.3.3 are cocommutative.

In any coalgebra, elements whose Δ is as in 1.3.2 or 1.3.3 are very important; thus we give them a name.

1.3.4 DEFINITION. Let C be any coalgebra, and let $c \in C$.

a) c is called *group-like* if $\Delta c = c \otimes c$ and if $\varepsilon(c) = 1$. The set of group-like elements in C is denoted by $G(C)$.
b) For $g, h \in G(C), c$ is called *g, h-primitive* if $\Delta c = c \otimes g + h \otimes c$. The set of all g, h-primitives is denoted by $P_{g,h}(C)$. If $C = B$ is a bialgebra and $g = h = 1$, then the elements of $P(B) = P_{1,1}(B)$ are simply called the *primitive* elements of B.

It is not difficult to prove that in any coalgebra, distinct group-like elements are $\mathbf{k}$-independent [S, 3.2.1], [A, 2.1.2]. As a consequence, if $B = \mathbf{k}G$, then $G(B) = G$, the original group.

If $B = U(\mathbf{g})$ and char $\mathbf{k} = 0$, then $P(B) = \mathbf{g}$, the original Lie algebra. However if char $\mathbf{k} = p \neq 0$, then $P(B)$ is the span of all $x^{p^k}, k \geq 0, x \in \mathbf{g}$; it is a restricted p-Lie algebra. See §5.5.

As another example of group-like elements, let A be any algebra and define

$$\mathrm{Alg}(A, \mathbf{k}) = \{f \in A^* \mid f \text{ is an algebra map }\}. \quad (1.3.5)$$

In 9.1.4 we will see that $\mathrm{Alg}(A, \mathbf{k}) = G(A^\circ)$, the set of group-like elements in the coalgebra A°.

We continue with our examples of bialgebras.

1.3.6 EXAMPLE. If B is any bialgebra, then B° is a bialgebra; this is proved in §9.1. In particular, we consider the special case when $B = \mathbf{k}G$. In this case B° is called the set of *representative functions* $R_{\mathbf{k}}(G)$ on G. It can also be described as follows:

$$B^\circ = R_{\mathbf{k}}(G) = \{f \in (\mathbf{k}G)^* \,|\, \dim_{\mathbf{k}} \text{ span } \{G \cdot f\} < \infty\},$$

where G acts on $(\mathbf{k}G)^*$ via $(x \cdot f)(y) = f(yx)$, for all $x, y \in G, f \in (\mathbf{k}G)^*$. The algebra structure on B° (or on B^*) is given by

$$(fg)(x) = \Delta^*(f \otimes g)(x) = (f \otimes g)(x \otimes x) = f(x)g(x),$$

all $x \in G, f, g \in B^*$; that is, it is the usual pointwise multiplication. The coalgebra structure is given, as for any bialgebra B, by

$$\Delta f(x \otimes y) = m^* f(x \otimes y) = f(xy),$$

all $x, y \in B, f \in B^\circ$. However, this does not give an explicit formula for Δf as an element of $B^\circ \otimes B^\circ$. When B is finite-dimensional, that is $|G| < \infty$, we can give such a description, as follows:

Let $\{p_x \mid x \in G\}$ be a basis of $(\mathbf{k}G)^*$ dual to the basis of group elements in $\mathbf{k}G$; that is $p_x(y) = \delta_{x,y}$, all $x, y \in G$. Then

$$\Delta p_x = \sum_{uv=x} p_u \otimes p_v. \tag{1.3.7}$$

1.3.8 EXAMPLE. Let $B = \mathcal{O}(M_n(\mathbf{k})) = \mathbf{k}[X_{ij} \,|\, 1 \leq i, j \leq n]$, the polynomial functions on $n \times n$ matrices. As an algebra, B is simply the commutative polynomial ring in the n^2 indeterminates $\{X_{ij}\}$. For the coalgebra structure, think of X_{ij} as the coordinate function on the ij^{th} entry of the ring $M_n(\mathbf{k})$ of $n \times n$ matrices. Then Δ is the dual of matrix multiplication; that is $\Delta X_{ij} = \sum_{k=1}^n X_{ik} \otimes X_{kj}$. By setting $\varepsilon(X_{ij}) = \delta_{ij}$, B becomes a bialgebra.

If we let $X = [X_{ij}]$, the $n \times n$ matrix with ij^{th} entry X_{ij}, then one may check that $\det X \in G(B)$.

1.3.9 EXAMPLE. The "quantum plane". Choose $0 \neq q \in \mathbf{k}$ and let $B = \mathcal{O}(\mathbf{k}^2) = \mathbf{k}\langle x, y \mid xy = qyx \rangle$. B has a bialgebra structure given by setting $\Delta x = x \otimes x$, $\Delta y = y \otimes 1 + x \otimes y$, $\varepsilon(x) = 1$, $\varepsilon(y) = 0$. Note that $x \in G(B)$ and that $y \in P_{1,x}(B)$, the set of $1, x$ -primitive elements.

1.3.10 SOME QUANTUM GROUPS. The Appendix gives generators and relations for $U_q(\mathbf{g}), \mathbf{g}$ a finite-dimensional semi-simple Lie algebra, and for $\mathcal{O}_q(M_n(\mathbf{k}))$, and describes their coalgebra structures.

1.3.11 EXAMPLE. If B is any bialgebra, we can form a new bialgebra by taking the opposite of either the algebra or coalgebra structure. Thus B^{op} means B has the opposite multiplication but the same comultiplication, B^{cop} has the same multiplication but the opposite comultiplication, and $B^{op,cop}$ has both opposite structures.

§1.4 Convolution and summation notation

Before proceeding to the definition of Hopf algebra, we introduce another definition and a very useful notation.

1.4.1 DEFINITION. Let C be a coalgebra and A an algebra. Then $\mathrm{Hom}_{\mathbf{k}}(C, A)$ becomes an algebra under the *convolution product*

$$(f * g)(c) = m \circ (f \otimes g)(\Delta c)$$

for all $f, g \in \mathrm{Hom}_{\mathbf{k}}(C, A), c \in C$. The unit element in $\mathrm{Hom}_{\mathbf{k}}(C, A)$ is $u\varepsilon$.

A useful formula for $f * g$ is given in 1.4.4, below.

Note that we have already seen an example of convolution; namely, for any coalgebra C, the multiplication $m = \Delta^*$ in $C^* = \mathrm{Hom}(C, \mathbf{k})$ (see 1.2.2).

One can also define the *twist convolution* (or *anti-convolution*) product on $\mathrm{Hom}_{\mathbf{k}}(C, A)$ via

$$(f \times g)(c) = m \circ (f \otimes g)(\tau \circ \Delta(c)).$$

The following notation was introduced by Heyneman and Sweedler.

1.4.2 NOTATION. Let C be any coalgebra with comultiplication $\Delta : C \to C \otimes C$. The *sigma notation* for Δ is given as follows: for any $c \in C$, we write

$$\Delta c = \sum c_{(1)} \otimes c_{(2)}.$$

The subscripts (1) and (2) are symbolic, and do not indicate particular elements of C; this notation is analogous to notation used in physics (where even the $\sum$ may be omitted). In these notes we usually simplify the notation by omitting parentheses.

The power of the notation becomes apparent when Δ must be applied more than once. In particular, the coassociativity diagram 1.1.3 a) gives that $\sum c_{(1)} \otimes c_{(2)_{(1)}} \otimes c_{(2)_{(2)}} = \sum c_{(1)_{(1)}} \otimes c_{(1)_{(2)}} \otimes c_{(2)}$; this element is written as $\sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)} = \Delta_2(c)$. Iterating this procedure gives

$$\Delta_{n-1}(c) = \sum c_{(1)} \otimes \ldots \otimes c_{(n)}$$

where $\Delta_{n-1}(c)$ is the (necessarily unique) element obtained by applying coassociativity $(n-1)$ times.

In this notation, the reader should check that the counit diagram 1.1.3 b) says that, for all $c \in C$,

$$c = \sum \varepsilon(c_{(1)})c_{(2)} = \sum \varepsilon(c_{(2)})c_{(1)} \tag{1.4.3}$$

and that the convolution product in 1.4.1 is given by

$$(f * g)(c) = \sum f(c_{(1)})g(c_{(2)}). \tag{1.4.4}$$

§1.5 Antipodes and Hopf algebras

1.5.1 DEFINITION. Let $(H, m, u, \Delta, \varepsilon)$ be a bialgebra. Then H is a *Hopf algebra* if there exists an element $S \in \operatorname{Hom}_{\mathbf{k}}(H, H)$ which is an inverse to id_H under convolution $*$. S is called an *antipode* for H.

Note that in sigma notation, S satisfies

$$\sum (Sh_1)h_2 = \varepsilon(h)1_H = \sum h_1(Sh_2) \tag{1.5.2}$$

for all $h \in H$.

We also have the obvious definitions of morphisms and ideals: a map $f : H \to K$ of Hopf algebras is a *Hopf morphism* if it is a bialgebra morphism and $f(S_H h) = S_K f(h)$, for all $h \in H$. A subspace I of H is a *Hopf ideal* if it is a biideal and if $SI \subseteq I$; in this situation H/I is a Hopf algebra with structure induced from H.