HEP World's Classics

On the Study and Difficulties of Mathematics

数学学习及其困难

AUGUSTUS DE MORGAN
奥古斯都·德·摩根

HIGHER EDUCATION PRESS

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内容简介

本书是英国数学家德·摩根在19世纪80年代撰写的著作，主要论述了代数和几何原理中的多个要点，这些要点的重要性在当时一些基本教程中没有得到应有的重视。对于具有通常的代数法则以及欧几里得相关定理的各种知识的读者，本书可以对其在数学教学中疏漏的那些部分进行补充。

本书适合对代数和几何方面感兴趣的读者，虽然书中提到的要点现今大部分可在一般教程中找到，但其重要性依然值得读者予以关注。

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HEP World's Classics

There is a Chinese saying: "It is beneficial to open any book." It is even more fruitful to open and read classic books. The world is keeping on changing, but really fundamental and essential things stay the same since there is nothing new under the sun. Great ideas have been discovered and re-discovered, and they should be learnt and re-learnt. Classic books are our inheritance from all the previous generations and contain the best of knowledge and wisdom of all the people before us. They are timeless and universal. We cannot travel back in time, but we can converse with the originators of current theories through reading their books. Classic books have withstood the test of time. They are reliable and contain a wealth of original ideas. More importantly, they are also books which have not finished what they wanted or hoped to say. Consequently, they contain unearthed treasures and hidden seeds of new theories, which are waiting to be discovered. As it is often said: history is today. Proper understanding of the past work of giants is necessary to carry out properly the current and future researches and to make them to be a part of the history of science and mathematics. Reading classic books is not easy, but it is rewarding. Some modern interpretations and beautiful reformulations of the classics often miss the subtle and crucial points. Reading classics is also more than only accumulating knowledge, and the reader can learn from masters on how they asked questions, how they struggled to come up with new notions and theories to overcome problems, and answers to questions. Above all, probably the best reason to open classic books is the curiosity: what did people know, how did they express and communicate them, why did they do what they did? It can simply be fun!

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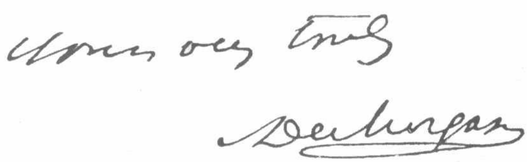
Augustus De Morgan (1806—1871)



德 · 摩根 (Augustus De Morgan, 1806—1871), 英国数学家、逻辑学家, 1806 年 6 月 27 日出生于印度的马都拉, 1871 年 3 月 18 日卒于伦敦。1823 年入剑桥大学三一学院学习, 1827 年毕业, 后在伦敦大学学院任数学教授 (1828—1831; 1836—1866)。1865 年参加筹备伦敦数学会, 并于 1866 年任会长。

他认为: 代数学实际上是一系列 "运算", 这种 "运算", 能在任何符号 (不一定是数字) 的集合上, 根据一定的公设来进行。这一新的数学思想使代数得以脱离算术的束缚。

德 · 摩根在分析学方面给出了形如 $\sum \frac{1}{\varphi(n)}$ 的级数的收敛性判别准则, 即设

$$e = \lim_{n\to\infty} \frac{n\varphi'(n)}{\varphi(n)},$$

则当 $e > 1$ 时, 级数收敛; 当 $e \le 1$ 时, 级数发散。

在逻辑学方面, 德 · 摩根首创了关系逻辑的研究。他提出了论域的概念, 并用代数方法来研究逻辑演算, 建立了著名的德 · 摩根律, 即

$$(A \cap B)' = A' \cup B', \quad (A \cup B)' = A' \cap B'。$$

他还分析了关系的种类和性质, 研究了关系命题和关系推理, 得到了一些逻辑规律和定理, 从而突破了古典的主谓词逻辑的局限性, 这对其后数理逻辑的发展有一定的影响。

德 · 摩根撰写了不少算术、代数、三角等方面的教材, 他在分析学和逻辑学方面的主要著作有《微积分学》(1842)、《形式逻辑》(1847) 等。

ON THE

STUDY AND DIFFICULTIES OF MATHEMATICS

BY

AUGUSTUS DE MORGAN

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The title page of the 1910 version

EDITOR'S NOTE*

No apology is needed for the publication of the present new edition of *The Study and Difficulties of Mathematics*,—a characteristic production of one of the most eminent and luminous of English mathematical writers of the present century. De Morgan, though taking higher rank as an original inquirer than either Huxley or Tyndall, was the peer and lineal precursor of these great expositors of science, and he applied to his lifelong task an historical equipment and a psychological insight which have not yet borne their full educational fruit. And nowhere have these distinguished qualities been displayed to greater advantage than in the present work, which was conceived and written with the full natural freedom, and with all the fire, of youthful genius. For the contents and purpose of the book the reader may be referred to the Author's Preface. The work still contains points (notable among them is its insistence on the study of logic), which are insufficiently emphasised, or slurred, by elementary treatises; while the freshness and naturalness of its point of view contrasts strongly with the mechanical character of the common text-books. Elementary instructors and students cannot fail to profit by the general loftiness of its tone and the sound tenor of its instructions.

The original treatise, which was published by the Society for the Diffusion of Useful Knowledge and bears the date of 1831, is now practically inaccessible, and is marred by numerous errata and typographical solecisms, from which, it is hoped, the present edition is free. References to the remaining mathematical text-books of the Society for the Diffusion of Useful Knowledge now out of print [iv]
have either been omitted or supplemented by the mention of more modern works. The few notes which have been added are mainly bibliographical in character, and refer, for instance, to modern treatises on logic, algebra, the philosophy of mathematics, and pangeometry. For the portrait and autograph signature of De Morgan, which graces the page opposite the title, The Open Court Publishing Company is indebted to the courtesy of Principal David Eugene Smith, of the State Normal School at Brockport, N. Y.

Thomas J. McCormack
La Salle, Ill., Nov. 1, 1898

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*The numbers in square brackets at the margin of the book refer to the page numbers of the 1910 version. Some of which are cited or referenced in the book.

AUTHOR'S PREFACE

In compiling the following pages, my object has been to notice particularly several points in the principles of algebra and geometry, which have not obtained their due importance in our elementary works on these sciences. There are two classes of men who might be benefited by a work of this kind, viz., teachers of the elements, who have hitherto confined their pupils to the working of rules, without demonstration, and students, who, having acquired some knowledge under this system, find their further progress checked by the insufficiency of their previous methods and attainments. To such it must be an irksome task to recommence their studies entirely; I have therefore placed before them, by itself, the part which has been omitted in their mathematical education, presuming throughout in my reader such a knowledge of the rules of algebra, and the theorems of Euclid, as is usually obtained in schools.

It is needless to say that those who have the advantage of university education will not find more in this treatise than a little thought would enable them to collect from the best works now in use [1831], both at Cambridge and Oxford. Nor do I pretend to settle the many disputed points on which I have necessarily been obliged to treat. The perusal of the opinions of an individual, offered simply as such, may excite many to become inquirers, who would otherwise have been workers of rules and followers of dogmas. They may not ultimately coincide in the views promulgated by the work which first drew their attention, but the benefit which they will derive from it is not the less on that account. I am not, however, [vi] responsible for the contents of this treatise, further than for the manner in which they are presented, as most of the opinions here maintained have been found in the writings of eminent mathematicians.

It has been my endeavor to avoid entering into the purely metaphysical part of the difficulties of algebra. The student is, in my opinion, little the better for such discussions, though he may derive such conviction of the truth of results by deduction from particular cases, as no *à priori* reasoning can give to a beginner. In treating, therefore, on the negative sign, on impossible quantities, and on fractions of the form $\frac{0}{0}$, etc., I have followed the method adopted by several of the most esteemed continental writers, of referring the explanation to some particular problem, and showing how to gain the same from any other. Those who admit such expressions as $-a$, $\sqrt{-a}$, $\frac{0}{0}$, etc., have never produced any clearer method; while those who call them absurdities, and would reject them altogether, must, I think, be forced to admit the fact that in algebra the different species of contradictions in problems are attended with distinct absurdities, resulting from them as necessarily as different numerical results from different numerical data. This being granted, the whole of the ninth chapter of this work may be considered as an inquiry into the nature of the different misconceptions, which give rise to the various expressions above alluded to. To this view of the question I have leaned, finding no other so satisfactory to my own mind.

The number of mathematical students, increased as it has been of late years, would be much augmented if those who hold the highest rank in science would condescend to give more effective assistance in clearing the elements of the difficulties which they present. If any one claiming that title should think my attempt obscure or erroneous, he must share the blame with me, since it is through his neglect that I have been enabled to avail myself of an opportunity to perform a task which I would gladly have seen confided to more skilful hands.

Augustus De Morgan

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CHAPTER I [1]

Introductory Remarks on the Nature and Objects of Mathematics

The Object of this Treatise is—(1) To point out to the student of Mathematics, who has not the advantage of a tutor, the course of study which it is most advisable that he should follow, the extent to which he should pursue one part of the science before he commences another, and to direct him as to the sort of applications which he should make. (2) To treat fully of the various points which involve difficulties and which are apt to be misunderstood by beginners, and to describe at length the nature without going into the routine of the operations.

No person commences the study of mathematics without soon discovering that it is of a very different nature from those to which he has been accustomed. The pursuits to which the mind is usually directed before entering on the sciences of algebra and geometry, are such as languages and history, etc. Of these, neither appears to have any affinity with mathematics; yet, in order to see the difference [2]
which exists between these studies,—for instance, history and geometry,—it will be useful to ask how we come by knowledge in each. Suppose, for example, we feel certain of a fact related in history, such as the murder of Cæsar, whence did we derive the certainty? How came we to feel sure of the general truth of the circumstances of the narrative? The ready answer to this question will be, that we have not absolute certainty upon this point; but that we have the relation of historians, men of credit, who lived and published their accounts in the very time of which they write; that succeeding ages have received those accounts as true, and that succeeding historians have backed them with a mass of circumstantial evidence which makes it the most improbable thing in the world that the account, or any material part of it, should be false. This is perfectly correct, nor can there be the slightest objection to believing the whole narration upon such grounds; nay, our minds are so constituted, that, upon our knowledge of these arguments, we cannot help believing, in spite of ourselves. But this brings us to the point to which we wish to come; we believe that Cæsar was assassinated by Brutus and his friends, not because there is any absurdity in supposing the contrary, since every one must allow that there is just a possibility that the event never happened: not because we can show that it must necessarily have been that, at a particular day,

[3] at a particular place, a successful adventurer must have been murdered in the manner described, but because our evidence of the fact is such, that, if we apply the notions of evidence which every-day experience justifies us in entertaining, we feel that the improbability of the contrary compels us to take refuge in the belief of the fact; and, if we allow that there is still a possibility of its falsehood, it is because this supposition does not involve absolute absurdity, but only extreme improbability.

In mathematics the case is wholly different. It is true that the facts asserted in these sciences are of a nature totally distinct from those of history; so much so, that a comparison of the evidence of the two may almost excite a smile. But if it be remembered that acute reasoners, in every branch of learning, have acknowledged the use, we might almost say the necessity, of a mathematical education, it must be admitted that the points of connexion between these pursuits and others are worth attending to. They are the more so, because there is a mistake into which several have fallen, and have deceived others, and perhaps themselves, by clothing some false reasoning in what they called a mathematical dress, imagining that, by the application of mathematical symbols to their subject, they secured mathematical argument. This could not have happened if they had possessed a knowledge of the bounds within which the empire of mathematics is contained. That empire
[4] is sufficiently wide, and might have been better known, had the time which has been wasted in aggressions upon the domains of others, been spent in exploring the immense tracts which are yet untrodden.

We have said that the nature of mathematical demonstration is totally different from all other, and the difference consists in this—that, instead of showing the contrary of the proposition asserted to be only improbable, it proves it at once to be absurd and impossible. This is done by showing that the contrary of the proposition which is asserted is in direct contradiction to some extremely evident fact, of the truth of which our eyes and hands convince us. In geometry, of the principles alluded to, those which are most commonly used are—

I. If a magnitude be divided into parts, the whole is greater than either of those parts.

II. Two straight lines cannot inclose a space.

III. Through one point only one straight line can be drawn, which never meets another straight line, or which is *parallel* to it.

It is on such principles as these that the whole of geometry is founded, and the demonstration of every proposition consists in proving the contrary of it to be inconsistent with one of these. Thus, in Euclid, Book I., Prop. 4, it is shown that two triangles which have two sides and the included angle respectively equal are equal in all respects, by proving that, if they are not equal, two straight lines
[5] will inclose a space, which is impossible. In other treatises on geometry, the same thing is proved in the same way, only the self-evident truth asserted sometimes

differs in form from that of Euclid, but may be deduced from it, thus—

Two straight lines which pass through the same two points must either inclose a space, or coincide and be one and the same line, but they cannot inclose a space, therefore they must coincide. Either of these propositions being granted, the other follows immediately; it is, therefore, immaterial which of them we use. We shall return to this subject in treating specially of the first principles of geometry.

Such being the nature of mathematical demonstration, what we have before asserted is evident, that our assurance of a geometrical truth is of a nature wholly distinct from that which we can by any means obtain of a fact in history or an asserted truth of metaphysics. In reality, our senses are our first mathematical instructors; they furnish us with notions which we cannot trace any further or represent in any other way than by using single words, which every one understands. Of this nature are the ideas to which we attach the terms number, one, two, three, etc., point, straight line, surface; all of which, let them be ever so much explained, can never be made any clearer than they are already to a child of ten years old.

But, besides this, our senses also furnish us with the means of reasoning on the
things which we call by these names, in the shape of incontrovertible propositions, [6]
such as have been already cited, on which, if any remark is made by the beginner in mathematics, it will probably be, that from such absurd truisms as "the whole is greater than its part," no useful result can possibly be derived, and that we might as well expect to make use of "two and two make four." This observation, which is common enough in the mouths of those who are commencing geometry, is the result of a little pride, which does not quite like the humble operation of beginning at the beginning, and is rather shocked at being supposed to want such elementary information. But it is wanted, nevertheless; the lowest steps of a ladder are as useful as the highest. Now, the most common reflection on the nature of the propositions referred to will convince us of their truth. But they must be presented to the understanding, and reflected on by it, since, simple as they are, it must be a mind of a very superior cast which could by itself embody these axioms, and proceed from them only one step in the road pointed out in any treatise on geometry.

But, although there is no study which presents so simple a beginning as that of geometry, there is none in which difficulties grow more rapidly as we proceed, and what may appear at first rather paradoxical, the more acute the student the more serious will the impediments in the way of his progress appear. This necessarily
follows in a science which consists of reasoning from the very commencement, for it [7]
is evident that every student will feel a claim to have his objections answered, not by authority, but by argument, and that the intelligent student will perceive more readily than another the force of an objection and the obscurity arising from an unexplained difficulty, as the greater is the ordinary light the more will occasional

darkness be felt. To remove some of these difficulties is the principal object of this Treatise.

We shall now make a few remarks on the advantages to be derived from the study of mathematics, considered both as a discipline for the mind and a key to the attainment of other sciences. It is admitted by all that a finished or even a competent reasoner is not the work of nature alone; the experience of every day makes it evident that education develops faculties which would otherwise never have manifested their existence. It is, therefore, as necessary to *learn to reason* before we can expect to be able to reason, as it is to learn to swim or fence, in order to attain either of those arts. Now, something must be reasoned upon, it matters not much what it is, provided that it can be reasoned upon with certainty. The properties of mind or matter, or the study of languages, mathematics, or natural history, may be chosen for this purpose. Now, of all these, it is desirable to choose the one which admits of the reasoning being verified, that is, in which we can find out by other means, such as measurement and ocular demonstration of all sorts,
[8] whether the results are true or not. When the guiding property of the loadstone was first ascertained, and it was necessary to learn how to use this new discovery, and to find out how far it might be relied on, it would have been thought advisable to make many passages between ports that were well known before attempting a voyage of discovery. So it is with our reasoning faculties: it is desirable that their powers should be exerted upon objects of such a nature, that we can tell by other means whether the results which we obtain are true or false, and this before it is safe to trust entirely to reason. Now the mathematics are peculiarly well adapted for this purpose, on the following grounds:

1. Every term is distinctly explained, and has but one meaning, and it is rarely that two words are employed to mean the same thing.

2. The first principles are self-evident, and, though derived from observation, do not require more of it than has been made by children in general.

3. The demonstration is strictly logical, taking nothing for granted except the self-evident first principles, resting nothing upon probability, and entirely independent of authority and opinion.

4. When the conclusion is attained by reasoning, its truth or falsehood can be ascertained, in geometry by actual measurement, in algebra by common arith-
[9] metical calculation. This gives confidence, and is absolutely necessary, if, as was said before, reason is not to be the instructor, but the pupil.

5. There are no words whose meanings are so much alike that the ideas which they stand for may be confounded. Between the meanings of terms there is no distinction, except a total distinction, and all adjectives and adverbs expressing difference of degrees are avoided. Thus it may be necessary to say, "A is greater than B;" but it is entirely unimportant whether A is very little or very much greater than B. Any proposition which includes the foregoing assertion will prove

its conclusion generally, that is, for all cases in which A is greater than B, whether the difference be great or little. Locke mentions the distinctness of mathematical terms, and says in illustration: "The idea of two is as distinct from the idea of three as the magnitude of the whole earth is from that of a mite. This is not so in other simple modes, in which it is not so easy, nor perhaps possible for us to distinguish between two approaching ideas, which yet are really different; for who will undertake to find a difference between the white of this paper, and that of the next degree to it?"

These are the principal grounds on which, in our opinion, the utility of mathematical studies may be shown to rest, as a discipline for the reasoning powers. But the habits of mind which these studies have a tendency to form are valuable in the highest degree. The most important of all is the power of concentrating
the ideas which a successful study of them increases where it did exist, and cre- [10]
ates where it did not. A difficult position, or a new method of passing from one proposition to another, arrests all the attention and forces the united faculties to use their utmost exertions. The habit of mind thus formed soon extends itself to other pursuits, and is beneficially felt in all the business of life.

As a key to the attainment of other sciences, the use of the mathematics is too well known to make it necessary that we should dwell on this topic. In fact, there is not in this country any disposition to under-value them as regards the utility of their applications. But though they are now generally considered as a part, and a necessary one, of a liberal education, the views which are still taken of them as a part of education by a large proportion of the community are still very confined.

The elements of mathematics usually taught are contained in the sciences of arithmetic, algebra, geometry, and trigonometry. We have used these four divisions because they are generally adopted, though, in fact, algebra and geometry are the only two of them which are really distinct. Of these we shall commence with arithmetic, and take the others in succession in the order in which we have arranged them.