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Orthogonal Factorizations in Graphs

图的正交因子分解（英文）

周思中 著

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内 容 简 介

This book mainly discusses the problems with the orthogonal factorizations in graphs, and is divided into five chapters. In Chapter 1, we give the basic terminologies, definitions and notations. In Chapter 2, we study 1-orthogonal factorizations in graphs. In Chapter 3, we obtain some results on 2-orthogonal factorizations in graphs. In Chapter 4, we justify some theorems on randomly r-orthogonal factorizations in graphs. In Chapter 5, we investigate the problems with orthogonal factorizations of digraphs and get some results on orthogonal factorizations of digraphs.

This book is a useful reference for college students and graduate students.

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Preface

Graph theory is one of the branches of modern mathematics which has shown impressive advances in recent years. Graph theory is widely applied in physics, chemistry, biology, network theory, information sciences, computer science and other fields, and so it has attracted a great deal of attention.

Many real-world networks can conveniently be modelled by graphs or digraphs. Examples include a railroad network with nodes presenting railroad stations and links corresponding to railways between two stations, or a communication network with nodes presenting cities, and links presenting communication channels. In particular, a wide variety of systems can be described using complex networks. Such systems include: the World Wide Web, which is a virtual network of Web pages connected by hyperlinks; the cell, where we model the chemicals by nodes and their interactions by edges; and the food chain webs, the networks by which human diseases spread, human collaboration networks etc.

Factor theory of graph is an important branch in graph theory. Factors, factorizations and orthogonal factorizations of graphs have extensive applications in various areas, e.g., combinatorial design, network design, circuit layout, scheduling problems, the file transfer problems and so on. The file transfer problem can be modeled as $(0, f)$-factorizations (or f-colorings) of a graph. Orthogonal factorizations in graphs or digraphs have attracted a great deal of attention due to their applications in combinatorial design, network design, circuit layout, and so on. For example, a pair of orthogonal Latin squares of order n is equivalent to two orthogonal 1-factorizations of a complete bipartite graph $K_{n,n}$, which was first found by Euler. Horton presented that a Room square of order $2n$ was related to the orthogonal 1-factorization of K_{2n}.

In this book, we mainly discuss the problems with the orthogonal factorization of graphs. This book is divided into five chapters.

In Chapter 1, we show definitions, notations and motivation to study orthogonal factorizations in graphs or digraphs.

In Chapter 2, we investigate orthogonal factorizations in some graphs which are natural improvements and generalizations of the previous results. This chapter is divided into two parts. First, We show a result on orthogonal factorizations in $(mg+k-1, mf-k+1)$-graphs. Second, we study (g,f)-factorizations orthogonal to k vertex disjoint subgraphs of G, and obtain some new results on orthogonal (g,f)-factorizations in some graphs.

In Chapter 3, we study 2-orthogonal factorizations in some graphs which are natural generalizations of 1-orthogonal factorizations. This chapter is divided into four parts. First, We investigate 2-orthogonal $(0,f)$-factorizations in $(0, mf-m+1)$-graphs, and put forward a sufficient condition for $(0, mf-m+1)$-graphs to have 2-orthogonal $(0,f)$-factorizations. Second, we study 2-orthogonal $[0,k_j]_1^m$-factorizations in $[0, k_1+k_2+\cdots+k_m-m+1]$-graphs, and obtain a sufficient condition for the existence of 2-orthogonal $[0,k_j]_1^m$-factorizations in $[0, k_1+k_2+\cdots+k_m-m+1]$-graphs. Third, we consider 2-orthogonal (g,f)-factorizations in $(mg+m-1, mf-m+1)$-graphs, and get a new result on 2-orthogonal (g,f)-factorizations in $(mg+m-1, mf-m+1)$-graphs. Finally, we study the existence of subgraphs with 2-orthogonal (g,f)-factorizations in $(mg+k, mf-k)$-graphs, and obtain a new result on this problem.

In Chapter 4, we first investigate the randomly r-orthogonal factorizations in graphs, and present some results on the randomly r-orthogonal factorizations in graphs. Second, we study the factorizations r-orthogonal to vertex disjoint subgraphs, and get some results.

In Chapter 5, we investigate orthogonal factorizations of digraphs. This chapter is divided into three parts. First, we study r-orthogonal $(0,f)$-factorizations of $(0, mf-m+1)$-digraphs. Second, we investigate the existence of subdigraphs with orthogonal (g,f)-factorizations in $(mg+(k-1)r, mf-(k-1)r)$-digraphs. Finally, we consider $(0,f)$-factorizations

r-orthogonal to vertex disjoint subdigraphs, and obtain a result on $(0, f)$-factorizations r-orthogonal to vertex disjoint subdigraphs in $(0, mf-(m-1)r)$-digraphs.

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Contents

Chapter 1 Basic Terminology

This chapter is concerned with the basic notation and terminology of graph theory which will be used throughout this book. We will briefly explain the basic notations and definitions in the first section and will discuss the motivation to study the existence of orthogonal factorizations in graphs or digraphs in the second section. The notations and definitions mainly follows that of Bondy and Murty [1], Chartrand and Lesniak [2] as well as Akiyama and Kano [3] and we direct the reader to these books for any information not given here. Special notations and definitions will be presented where needed.

1.1 Definitions and Notations

In Chapters 2–4 of this book, all graphs considered will be finite undirected graphs without loops or multiple edges. Let G be a graph. We denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called a spanning subgraph of G if $V(H) = V(G)$. For arbitrary $x \in V(G)$, $d_G(x)$ denotes the degree of x in G. Let S and T be two disjoint vertex subsets of G. We denote the set of edges with one end in S and the other in T by $E_G(S,T)$, and write

$$e_G(S,T) = |E_G(S,T)|.$$

For $S \subset V(G)$ and $A \subset E(G)$, $G[S]$ and $G[A]$ are two subgraphs of a graph G induced by S and A, respectively. We write

$$G - S = G[V(G) \setminus S]$$

and

$$G - A = G[E(G) \setminus A].$$

For any function φ defined on $V(G)$, we write

$$\varphi(X) = \sum_{x \in X} \varphi(x)$$

and $\varphi(\emptyset) = 0$, where $X \subseteq V(G)$. Especially

$$d_G(X) = \sum_{x \in X} d_G(x).$$

Let $g, f : V(G) \to N$ be two functions defined on $V(G)$ such that $g(x) \leqslant f(x)$ for arbitrary $x \in V(G)$. If C is a component of $G - (S \cup T)$ with $g(x) = f(x)$ for every $x \in V(C)$, then we call that C is odd or even in terms of

$$e_G(T, V(C)) + f(V(C))$$

being odd or even, respectively. A spanning subgraph F of a graph G with

$$g(x) \leqslant d_F(x) \leqslant f(x)$$

for every $x \in V(G)$ is called a (g, f)-factor of G. Especially, G is said to be a (g, f)-graph if G itself is a (g, f)-factor. A (g, f)-factorization $F = \{F_1, F_2, \cdots, F_m\}$ of a graph G is a partition of $E(G)$ into edge-disjoint (g, f)-factors $F_1, F_2, \cdots, F_m$. A subgraph H of a graph G is called an m-subgraph if H admits m edges in total. Let H be an mr-subgraph of a graph G and $F = \{F_1, F_2, \cdots, F_m\}$ be a (g, f)-factorization of a graph G. A (g, f)-factorization $F = \{F_1, F_2, \cdots, F_m\}$ is r-orthogonal to H if

$$|E(F_i) \cap E(H)| = r \quad for\ 1 \leqslant i \leqslant m.$$

We say that a graph G admits (g, f)-factorizations randomly r-orthogonal to H if for arbitrary partition $\{A_1, A_2, \cdots, A_m\}$ of $E(H)$ satisfying $|A_i| = r$, there is a (g, f)-factorization $F = \{F_1, F_2, \cdots, F_m\}$ of a graph G with $A_i \subseteq E(F_i)$, $i = 1, 2, \cdots, m$. It is easy to see that randomly 1-orthogonal is equivalent to 1-orthogonal and 1-orthogonal is also said to be orthogonal.

In Chapter 5 of this book, we only consider finite digraphs which have neither loops nor parallel arcs. Let G be a digraph with vertex set $V(G)$ and arc set $E(G)$. For any $x \in V(G)$, $d_G^-(x)$ and $d_G^+(x)$ denote the indegree and outdegree of x in G, respectively. We denote by xy the arc with tail x and head y. A subdigraph H of a digraph G is said to be an m-subdigraph if H admits m arcs. Let $g = (g^-, g^+)$ and $f = (f^-, f^+)$ be pairs of nonnegative integer-valued functions defined on $V(G)$ with $g^-(x) \leqslant f^-(x)$ and $g^+(x) \leqslant f^+(x)$ for each $x \in V(G)$. A digraph G satisfying

$$g^-(x) \leqslant d_G^-(x) \leqslant f^-(x)$$

and

$$g^+(x) \leqslant d_G^+(x) \leqslant f^+(x)$$

for any $x \in V(G)$ is called a (g, f)-digraph. A spanning subdigraph F of a digraph G is said to be a (g, f)-factor of G if F itself is a (g, f)-digraph. A (g, f)-factorization $F = \{F_1, F_2, \cdots, F_m\}$ of a digraph G is a partition of $E(G)$ into arc-disjoint (g, f)-factors $F_1, F_2, \cdots, F_m$. Let H be an mr-subdigraph of a digraph G and $F = \{F_1, F_2, \cdots, F_m\}$ be a (g, f)-factorization of a digraph G. If

$$|E(H) \cap E(F_i)| = r \quad for\ 1 \leqslant i \leqslant m,$$

then $F = \{F_1, F_2, \cdots, F_m\}$ is said to be r-orthogonal to H. Especially, 1-orthogonal is abbreviated as orthogonal. Furthermore, such a factorization of a digraph G is called an r-orthogonal factorization of a digraph G. For convenience, we write $g \leqslant f$ if

$$g^-(x) \leqslant f^-(x) \quad and\ g^+(x) \leqslant f^+(x)$$

for each $x \in V(G)$, and write $g \geqslant k$ if

$$\min\{g^-(x), g^+(x)\} \geqslant k$$

for each $x \in V(G)$. Furthermore, we shall write $mf+n$ for $(mf^- + n, mf^+ + n)$.

Let G be a digraph. For any $S, T \subseteq V(G)$, we write

$$E_G(S, T) = \{xy : xy \in E(G), x \in S, y \in T\},$$

and set

$$e_G(S, T) = |E_G(S, T)|.$$

We write

$$\varphi(S) = \sum_{x \in S} \varphi(x)$$

and $\varphi(\emptyset) = 0$, where φ is any function defined on $V(G)$ and $S \subseteq V(G)$.

1.2 Motivation to Study Orthogonal Factorizations in Graphs or Digraphs

Many real-world networks can conveniently be modelled by graphs or networks. Examples include a railroad network with nodes and links modelling railroad stations and railways between two stations, respectively, or a communication network with nodes presenting cities, and links corresponding to communication channels, or the World Wide Web with nodes and links modelling Web pages and hyperlinks between Web pages, respectively. Orthogonal factorizations in graphs or networks have attracted a great deal of attention due to their applications in combinatorial design, network design, circuit layout, and so on. For example, a pair of orthogonal Latin squares of order n is equivalent to two orthogonal 1-factorizations of a complete bipartite graph $K_{n,n}$, which was first found by Euler [4]. Horton [5] presented that a Room square of order $2n$ was related to the orthogonal 1-factorization of K_{2n}. Many other applications in this field can be found in a current survey [6].

Alspach, Heinrich and Liu [6] posed the following problem:

Problem Given a subgraph H, does there exist a factorization F of G with some fixed type orthogonal to H?

Chapter 2 1-Orthogonal Factorizations in Graphs

Let G be a graph, and let g and f be two integer-valued functions defined on $V(G)$ such that

$$0 \leqslant g(x) \leqslant f(x)$$

for each $x \in V(G)$. A (g, f)-factor of G is a spanning subgraph F of G satisfying that

$$g(x) \leqslant d_F(x) \leqslant f(x)$$

for each $x \in V(G)$. In particular, G is called a (g, f)-graph if G itself is a (g, f)-factor. A subgraph H of G is called an m-subgraph if H has m edges in total. A (g, f)-factorization $F = \{F_1, F_2, \cdots, F_m\}$ of G is a partition of $E(G)$ into edge-disjoint (g, f)-factors $F_1, F_2, \cdots, F_m$. Similarly, we can define $[0, k_i]_{i=1}^n$-factorization and $(0, f)$-factorization. Let H be an m-subgraph of a graph G. A (g, f)-factorization $F = \{F_1, F_2, \cdots, F_m\}$ of G is 1-orthogonal to H if

$$|E(H) \cap E(F_i)| = 1$$

for $1 \leqslant i \leqslant m$. 1-orthogonal is also said to be orthogonal. Similarly, we can define orthogonal $[0, k_i]_{i=1}^n$-factorization and orthogonal $(0, f)$-factorization. In this chapter we investigate orthogonal factorizations in some graphs which are natural improvements and generalizations of the previous results. This chapter is divided into two parts. First, we show a result on orthogonal factorizations in $(mg + k - 1, mf - k + 1)$-graphs. Second, we study (g, f)-factorizations orthogonal to k vertex disjoint subgraphs of G, and obtain some new results on orthogonal (g, f)-factorizations in some graphs.

2.1 Orthogonal (g, f)-Factorizations in Graphs

Liu [7] investigated orthogonal (g, f)-factorizations in $(mg+m-1, mf-m+1)$-graphs, and obtained a result on orthogonal factorization problems which is shown in the following.

Theorem 2.1.1 ([7]) Let G be an $(mg+m-1, mf-m+1)$-graph, where g and f are two integer-valued functions defined on $V(G)$ such that $0 \leqslant g(x) \leqslant f(x)$ for each $x \in V(G)$. Then for any m-matching M of G there is a (g, f)-factorization orthogonal to M.

Liu [8] presented a new result on orthogonal (g, f)-factorizations in $(mg+m-1, mf-m+1)$-graphs, which is shown in the following.

Theorem 2.1.2 ([8]) Let G be an $(mg+m-1, mf-m+1)$-graph, where g and f are two integer-valued functions defined on $V(G)$ such that $0 \leqslant g(x) \leqslant f(x)$ for each $x \in V(G)$. Let H be a star of G with m edges, then there exists a (g, f)-factorization of G orthogonal to H.

It is easy to see that m-matching and m-star are both $[1, 2]$-subgraph with m edges. Yan [9] improved Theorems 2.1.1 and 2.1.2, and got the following theorem.

Theorem 2.1.3 ([9]) Let m be a positive integer. Let G be a graph and $g(x) \leqslant f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ for all $x \in V(G)$. If G is an $(mg+m-1, mf-m+1)$-graph and H is a $[1, 2]$-subgraph of G with any m edges of $E(G)$, then there exists a (g, f)-factorization of G orthogonal to H.

Note that a path of G with m edges is a $[1, 2]$-subgraph of G with m edges. Thus, we obtained the following corollary from Theorem 2.1.3.

Corollary 2.1.1 Let m be a positive integer. Let G be a graph and $g(x) \leqslant f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ for all $x \in V(G)$. If G is an $(mg+m-1, mf-m+1)$-graph and P is a path of G with m edges, then there exists a (g, f)-factorization of G orthogonal to P.

For any subgraph H of G with m edges, Yan [10] proved that an $(mg+m-1, mf-m+1)$-graph has a (g, f)-factorization orthogonal to H, which an improvement of Theorems 2.1.1– 2.1.3. This result is shown in the following.

Theorem 2.1.4 ([10]) Let m be a positive integer. Let G be a graph and $g(x) \geqslant 2$ and $f(x) \geqslant 5$ be integer-valued functions defined on $V(G)$ for all $x \in V(G)$. If G is an $(mg+m-1, mf-m+1)$-graph and H is a subgraph of G with m edges of $E(G)$, then there exists a (g, f)-factorization of G orthogonal to H.

Li and Liu [11] improved Theorem 2.1.4, and obtained a better result on orthogonal (g, f)-factorizations in $(mg+m-1, mf-m+1)$-graphs.

Theorem 2.1.5 ([11]) Let G be an $(mg+m-1, mf-m+1)$-graph, $m \geqslant 1$ be an integer, $f(x) \geqslant g(x) \geqslant 0$ be integer-valued functions defined on $V(G)$ for all $x \in V(G)$, and let H be a subgraph of G with m edges. Then G has a (g, f)-factorization orthogonal to H.

Feng [12] got a result on orthogonal $(0, f)$-factorizations in $(0, mf-m+1)$-graphs.

Theorem 2.1.6 ([12]) Let G be a $(0, mf-m+1)$-graph, where $f(x) \geqslant 0$ is an integer-valued function defined on $V(G)$ for all $x \in V(G)$. If H is an arbitrary subgraph with m edges of G, then G has a $(0, f)$-factorization orthogonal to H.

Ma and Bai [13] showed the existence of orthogonal $[0, k_i]_{i=1}^m$-factorizations in $[0, k_1+k_2+\cdots+k_m-m+1]$-graphs.

Theorem 2.1.7 ([13]) Let G be a graph, $k_1, k_2, \cdots, k_m$ be positive integers. If G is a $[0, k_1+k_2+\cdots+k_m-m+1]$-graph, H is a path of m edges in G or a cycle of m edges in G, then G has a $[0, k_i]_{i=1}^m$-factorization orthogonal to H.

Ma and Xu [14] put forward a new result on orthogonal $[0, k_i]_{i=1}^m$-factorizations in $[0, k_1 + k_2 + \cdots + k_m - m + 1]$-graphs.

Theorem 2.1.8 ([14]) Let G be a graph, $k_1, k_2, \cdots, k_m$ be positive integers. If G is a $[0, k_1 + k_2 + \cdots + k_m - m + 1]$-graph, H is a $[1, 2]$-subgraph of m edges in G, then G has a $[0, k_i]_{i=1}^m$-factorization orthogonal to H.

Feng and Liu [15] verified that a $[0, k_1 + k_2 + \cdots + k_m - m + 1]$-graph G has a $[0, k_i]_{i=1}^m$-factorization orthogonal to an arbitrary subgraph of G with m edges. Ma, Xu and Gao [16] verified the same result.

Theorem 2.1.9 ([15, 16]) Let G be a $[0, k_1 + k_2 + \cdots + k_m - m + 1]$-graph, where $m \geqslant 1$ is an integer and $k_1, k_2, \cdots, k_m$ are positive integers. Let H be an arbitrary subgraph of G with m edges. Then G has a $[0, k_i]_{i=1}^m$-factorization orthogonal to H.

Yan and Pan [17] studied orthogonal factorizations in $(mg + k, mf - k)$-graphs, and posed the following result.

Theorem 2.1.10 ([17]) Let G be an $(mg + k, mf - k)$-graph, and H a subgraph of G with k edges, where $1 \leqslant k < m$, g and f be two integer-valued functions defined on $V(G)$ with $g(x) \geqslant 1$ or $f(x) \geqslant 5$ for any $x \in V(G)$. Then there exists a subgraph R of G such that R has a (g, f)-factorization orthogonal to H.

Li, Chen and Yu [18], Dai, Xie and Yu [19] improved Theorem 2.1.10, and got the following result on orthogonal factorizations in $(mg + k, mf - k)$-graphs.

Theorem 2.1.11 ([18, 19]) Let G be an $(mg + k, mf - k)$-graph, and H a subgraph of G with k edges, where $1 \leqslant k < m$, g and f be two integer-valued functions defined on $V(G)$ with $0 \leqslant g(x) \leqslant f(x)$ for every $x \in V(G)$. Then there exists a subgraph R of G such that R has a (g, f)-factorization orthogonal to H.

Wang [20] improved Theorem 2.1.11, and presented orthogonal (g, f)-factorizations in $(mg+k-1, mf-k+1)$-graphs. Furthermore, Wang [20] put forward a conjecture.

Theorem 2.1.12 ([20]) Let G be an $(mg+k-1, mf-k+1)$-graph, $1 \leqslant k \leqslant m$, g and f be two integer-valued functions defined on $V(G)$ with $0 \leqslant g(x) \leqslant f(x)$ for every $x \in V(G)$, $g(x)$ and $f(x)$ be even. If H is an arbitrary subgraph of G with k edges, then there exists a subgraph R of G such that R has a (g, f)-factorization orthogonal to H.

Conjecture 2.1.1 ([20]) Let G be an $(mg+k-1, mf-k+1)$-graph, $1 \leqslant k \leqslant m$, g and f be two integer-valued functions defined on $V(G)$ with $0 \leqslant g(x) \leqslant f(x)$ for every $x \in V(G)$. If H is an arbitrary subgraph of G with k edges, then there exists a subgraph R of G such that R has a (g, f)-factorization orthogonal to H.

Zhou [21] justified that the conjecture above is true for an $(mg+k-1, mf-k+1)$-graph, which is shown in the following.

Theorem 2.1.13 ([21]) Let G be an $(mg+k-1, mf-k+1)$-graph, and H a subgraph of G with k edges, where $1 \leqslant k \leqslant m$, $m-k \neq 1$ and $f(x) > g(x) \geqslant 0$ for each $x \in V(G)$. Then there exists a subgraph R of G such that R has a (g, f)-factorization orthogonal to H.

The proof of Theorem 2.1.13 relies heavily on the following lemmas.

Liu [7] got a necessary and sufficient condition for a graph to have a (g, f)-factor containing a given edge.

Lemma 2.1.1 ([7]) Let G be a graph, $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ with $0 \leqslant g(x) < f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor containing any given edge e of G if and only if

$$\delta_G(S,T) \geqslant f(S) + d_{G-S}(T) - g(T) \geqslant \varepsilon(S,T)$$

for any two disjoint subsets S and T of $V(G)$, where $\varepsilon(S,T)$ is defined as follows:

(1) $\varepsilon(S,T)=2$, if $e=uv,\ u,v\in S$;

(2) $\varepsilon(S,T)=1$, if there exists a neutral component C of $G-(S\cup T)$ such that $e\in E_G(S,V(C))$;

(3) $\varepsilon(S,T)=0$, otherwise.

The following result was proved by Li, Chen and Yu [18].

Lemma 2.1.2 ([18]) Let G be an $(mg+k,mf-k)$-graph, and H a k-subgraph of G, where $1\leqslant k<m$ and $f(x)>g(x)\geqslant 0$ for each $x\in V(G)$. Then there exists a subgraph R of G such that R has a (g,f)-factorization $F=\{F_1,F_2,\cdots,F_k\}$ orthogonal to H, and $G-F_1-F_2-\cdots-F_k$ is an $((m-k)g,(m-k)f)$-graph.

Zhou [21] obtained the following lemma.

Lemma 2.1.3 ([21]) Let G be an (mg,mf)-graph with $m\geqslant 1$ and $m\neq 2$, where $0\leqslant g(x)<f(x)$ for each $x\in V(G)$. Then G has a (g,f)-factor containing any given edge e of G.

Proof Obviously, the result holds for $m=1$. In the following we may assume $m\geqslant 3$. According to Lemma 2.1.1, it suffices to show that for any two disjoint subsets S and T, we have

$$\delta_G(S,T)=f(S)+d_{G-S}(T)-g(T)\geqslant\varepsilon(S,T).$$

Claim 1 $\delta_G(S,T)\geqslant\frac{m-1}{m}d_{G-S}(T)+\frac{1}{m}d_{G-T}(S)$.
Proof Since G is an (mg,mf)-graph, we have

$$\begin{aligned}\delta_G(S,T)&=f(S)+d_{G-S}(T)-g(T)\\&=f(S)+d_G(T)-e_G(S,T)-g(T)\\&=\frac{1}{m}d_G(T)-g(T)+f(S)-\frac{1}{m}d_G(S)\\&\quad+\frac{m-1}{m}d_{G-S}(T)+\frac{1}{m}d_{G-T}(S)\end{aligned}$$