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Albert C.J. Luo

Resonance and Bifurcation to Chaos in Pendulum

单摆：共振层、分岔树到混沌

HIGHER EDUCATION PRESS

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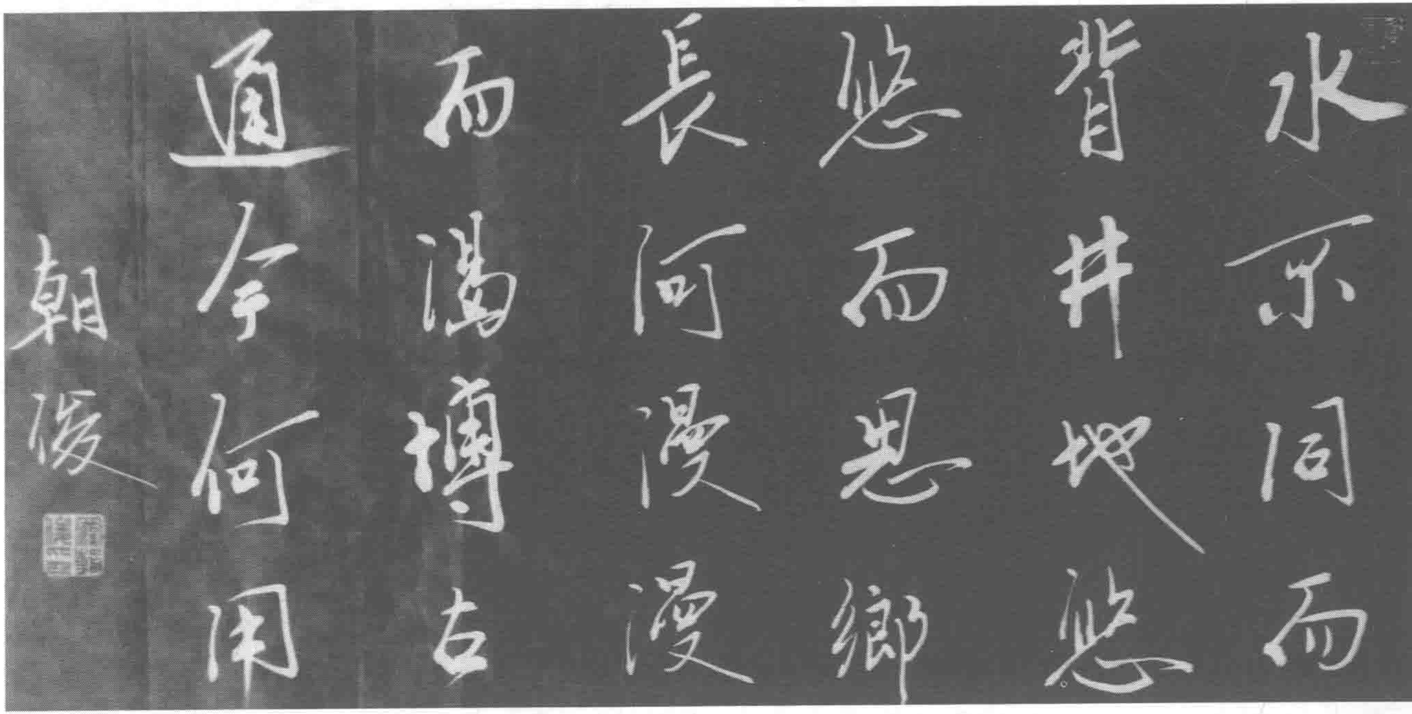

水不同而背井，地悠悠而思乡，长河漫漫而渴，博古通今何用。——《游子吟》，朝俊

Preface

This book discusses Hamiltonian chaos and periodic motions to chaos in pendulums. A periodically forced mathematical pendulum is one of the typical and popular nonlinear oscillators that possess complicated and rich dynamical behaviors. It seems that the periodically forced pendulum is one of the simplest nonlinear oscillators. However, until now, a systematical study of periodic motions to chaos cannot be completed. To know periodic motions and chaos in the periodically forced pendulum, the perturbation method has been adopted. Because the sinusoidal function cannot be easy to be handled in the current mathematical tools, one used the Taylor series to expend the sinusoidal function to the polynomial nonlinear terms, and then the traditional perturbation methods were used to obtain the periodic motions of the approximated differential system. One always emphasized that the periodic solutions in the original pendulum are suitable for the small variation of equilibrium. In fact, once vector fields are modified in nonlinear dynamical systems, the new dynamical systems cannot represent the original systems. For example, one used the softening Duffing oscillator to approximate a pendulum oscillator. Such an investigation should not be acceptable.

For a better understanding of complex motions in nonlinear dynamical systems, one considered the periodically forced pendulum. Herein, a few examples are listed to explain how to use the pendulum oscillator for a better understanding of chaos. Since 1960, the nonlinear dynamics of a particle in a traveling electric field was investigated by a nonlinear pendulum. In 1972, Zaslavsky and Chirikov discussed the stochastic (chaotic) instability of nonlinear oscillation through a periodically forced pendulum, and the resonance overlap was discussed. In 1982, Ben-Jacob et al. used the pendulum to investigate intermittent chaos in Josephson junctions. In 1985, Kadanoff investigated the route from a periodic motion to unbounded chaos by investigation of the simple pendulum, and the Chirikov-Taylor model (or the standard mapping model) was obtained. The scaling analysis for the onset of chaos was completed. Gwinn and Westervelt discussed the intermittent chaos and low frequency noise in the driven damped pendulum through the fourth-order

Runge-Kutta method, and the attraction basin was presented. In 2006, Paula et al. established an experimental nonlinear pendulum to investigate chaotic motions.

It is not easy to find periodic motions to chaos in a pendulum system even though the periodically forced pendulum is one of the simplest nonlinear systems. However, the inherent complex dynamics of the periodically forced pendulum is much beyond our imaginations through the traditional thought of the linear dynamical systems. Until now, we have not known complex motions of pendulum yet. What are the mechanism and mathematics of such complex motions in the pendulum? The results presented in this book will give an alternative view of complex motions in the pendulum. In this book, the resonance-based Hamiltonian chaos and bifurcation trees to chaos in pendulums will be analytically and/or semi-analytically predicted for a better understanding of resonant chaos and bifurcation trees to chaos in nonlinear dynamical systems. Hamiltonian chaos in pendulum will be discussed through periodically and parametrically forced pendulums. The bifurcation trees of travelable and non-travelable periodic motions to chaos will be presented through the periodically forced pendulum.

This book includes six chapters. In Chapter 1, the mechanism and criteria of Hamiltonian chaos in stochastic and resonant layers are discussed. Chapter 2 presents Hamiltonian chaos in stochastic and resonant layers of a periodically excited pendulum. In Chapter 3, parametric chaos in the stochastic and resonant layers of the parametrically excited pendulum are presented. The resonant conditions cannot satisfy the traditional Mathieu equation analysis. In Chapter 4, stability and bifurcation theory of nonlinear discrete system are reviewed. In Chapter 5, the methodology for solutions of periodic motions in continuous dynamical systems is presented through the mapping dynamics of discrete implicit mappings under specific truncated errors. The discrete Fourier series of periodic motions are discussed from discrete nodes of periodic motions, and the corresponding, approximate analytical expression can be obtained. Harmonic amplitude quantity levels can be analyzed for periodic motions in continuous nonlinear systems. Chapter 6 discusses the bifurcation trees of periodic motions to chaos in the periodically forced pendulum. Through the aforementioned materials, one will better understand resonant chaos and complicated bifurcation trees of periodic motions to chaos in pendulums, and such materials may help one to understand resonance and bifurcations to chaos in other nonlinear dynamical systems. I hope this book can throw out a different point of view to look into Hamiltonian chaos and periodic motions to chaos in nonlinear dynamics community.

Finally, I would like to appreciate my former student, Dr. Yu Guo, for completing numerical computations in the last chapter. Herein, I thank my wife (Sherry X. Huang) and my children (Yanyi Luo, Robin Ruo-Bing Luo, and Robert Zong-Yuan Luo) for their understanding and support. This book is also dedicated to my parents for their many years expectation to their son.

This gift is too light to them but everything leaving in the book may last very long.

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Edwardsville, Illinois

Color plots can be found from the website: academic.hep.com.cn

Contents

Chapter 1
Resonance and Hamiltonian Chaos

In this Chapter, nonlinear Hamiltonian chaos including stochastic and resonant layers in 2-dimensional nonlinear Hamiltonian systems will be presented. Chaos and resonance mechanism in the stochastic layer of generic separatrix will be discussed that is formed by the primary resonance interaction in nonlinear Hamiltonians systems. However, chaos in the resonant layer of the resonant separatrix will be presented that is formed by the sub-resonance interaction.

1.1 Stochastic layers

In this section, stochastic layers in 2-dimensional nonlinear Hamiltonian systems will be described, and the approximate criterions for onset and destruction of the stochastic layers will be presented.

1.1.1 Definitions

Consider a 2-dimensional Hamiltonian system with a time periodically perturbed vector field, i.e.,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) + \mu\mathbf{g}(\mathbf{x}, t, \boldsymbol{\pi}); \quad \mathbf{x} = (x, y)^{\mathrm{T}} \in \mathscr{R}^2, \tag{1.1}$$

where $\mathbf{f}(\mathbf{x}, \boldsymbol{\mu})$ is an unperturbed Hamiltonian vector field on $\mathscr{R}^2$ and $\mathbf{g}(\mathbf{x}, t, \boldsymbol{\pi})$ is a periodically perturbed vector field with period $T = 2\pi/\Omega$, and

$$\mathbf{f}(\mathbf{x}, \boldsymbol{\mu}) = (f_1(\mathbf{x}, \boldsymbol{\mu}), f_2(\mathbf{x}, \boldsymbol{\mu})^{\mathrm{T}} \text{ and } \mathbf{g}(\mathbf{x}, t, \boldsymbol{\pi}) = (g_1(\mathbf{x}, t, \boldsymbol{\pi}), g_2(\mathbf{x}, t, \boldsymbol{\pi})^{\mathrm{T}} \tag{1.2}$$

are sufficiently smooth ($C^r, r \geqslant 2$) and bounded on a bounded set $D \subset \mathscr{R}^2$ in phase space. $f_1 = \partial H_0(x, y, \boldsymbol{\mu})/\partial y$, $f_2 = -\partial H_0(x, y, \boldsymbol{\mu})/\partial x$; $g_1 =$

$\partial H_1(x,y,\Omega t,\boldsymbol{\pi})/\partial y$, $g_2 = -\partial H_1(x,y,\Omega t,\boldsymbol{\pi})/\partial x$. If the perturbation (or forcing term) $\mathbf{g}(\mathbf{x},t,\boldsymbol{\pi})$ vanishes, equation (1.1) is a complete nonlinear Hamiltonian system $\dot{\mathbf{x}}^{(0)} = \mathbf{f}(\mathbf{x}^{(0)},\boldsymbol{\mu})$. Thus, the total Hamiltonian of Eq. (1.1) can be expressed by

$$H(x,y,t,\mathbf{p}) = H_0(x,y,\boldsymbol{\mu}) + \mu H_1(x,y,\Omega t,\boldsymbol{\pi}), \tag{1.3}$$

with excitation frequency Ω and strength μ of the perturbed Hamiltonian $H_1(x,y,t,\boldsymbol{\pi})$ as well. To compare with the other approximate analysis, such a perturbation parameter is introduced herein. The Hamiltonian of the integrable system in Eq.(1.1) is $H_0(x,y,\boldsymbol{\mu})$. Once the initial condition is given, the Hamiltonian $H_0(x,y,\boldsymbol{\mu})$ is invariant (i.e., $H_0(x,y,\boldsymbol{\mu}) = E$), which is the first integral manifold.

To restrict this investigation to the 2-dimensional stochastic layer, four assumptions for Eq.(1.1) are introduced as follows:

(H1.1) The unperturbed system of Eq.(1.1) possesses a bounded, closed separatrix $q_0(t)$ with at least one hyperbolic point $p_0 : (x_h, y_h)$.

(H1.2) The neighborhood of $q_0(t)$ for the point $p_0 : (x_h, y_h)$ is filled with at least three families of periodic orbits $q_\sigma(t)(\sigma = \alpha,\beta,\gamma)$ with $\alpha,\beta,\gamma \in (0,1]$.

(H1.3) For the Hamiltonian energy E_σ of $q_\sigma(t)$, its period T_σ is a differentiable function of E_σ.

(H1.4) The perturbed system of Eq.(1.1) possesses a *perturbed* orbit $q(t)$ in the neighborhood of the *unperturbed* separatrix $q_0(t)$.

From the aforementioned hypothesis, the phase portrait of the unperturbed Hamiltonian system in the vicinity of the separatrix is sketched in Fig.1.1. The following point sets and the corresponding Hamiltonian energy are introduced, i.e.,

$$\Gamma_0 \equiv \{(x,y)|(x,y) \in q_0(t), t \in \mathscr{R}\} \cup \{p_0\} \text{ and } E_0 = H_0\left(q_0(t)\right) \tag{1.4}$$

for the separatrix,

$$\Gamma_\sigma \equiv \{(x,y)|(x,y) \in q_\sigma(t), t \in \mathscr{R}\} \text{ and } E_\sigma = H_0\left(q_\sigma(t)\right) \tag{1.5}$$

for the unperturbed, σ-periodic orbit and

$$\Gamma = \{(x,y)|(x,y) \in q(t), t \in \mathscr{R}\} \text{ and } E = H_0\left(q(t)\right) \tag{1.6}$$

for the perturbed orbit $q(t)$.

The Hamiltonian energies in Eqs.(1.4) and (1.5) are constant for any periodic orbit of the unperturbed system but the Hamiltonian energy in Eq.(1.6) varies with $(x,y) \in q(t)$ of the perturbed system. The unperturbed Hamiltonian $H_0(q_\sigma(t))(\sigma = \alpha,\beta,\gamma)$ and $H_0(q_0(t))$ are $C^r(r \geqslant 2)$ smooth (also see, Luo and Han, 2001). The hypotheses (H1.2)–(H1.3) imply that $T_\sigma \to \infty$ monotonically as $\sigma \to 0$ (i.e., the periodic orbit $q_\sigma(t)$ approaches to $q_0(t)$ as $\sigma \to 0$).

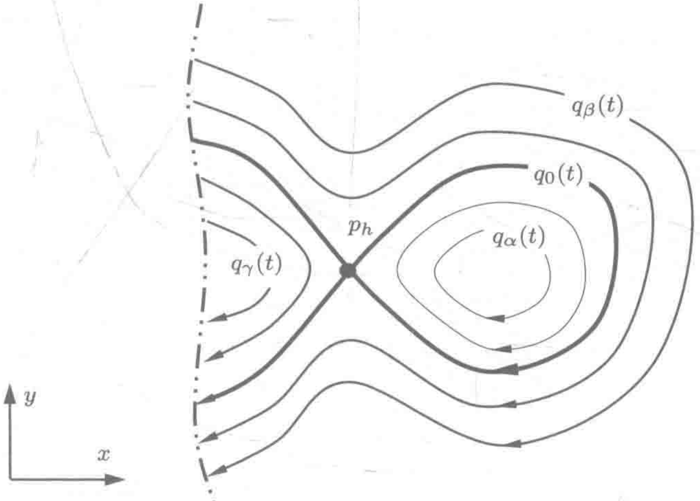

Fig. 1.1 The phase portrait of the unperturbed system of Eq.(1.1) near a hyperbolic point p_h and $q_0(t)$ is a separatrix going through the hyperbolic point and splitting the phase into three parts near the hyperbolic point, and the corresponding orbits $q_\sigma(t)$ are termed the σ-orbit ($\sigma = \{\alpha, \beta, \gamma\}$).

The δ-sets of the first integral quantity (or the energy) of the unperturbed Eq. (1.1) in $\Gamma_\sigma(\sigma = \alpha, \beta, \gamma)$, are defined as

$$N_\sigma^\delta(E_0) = \{E_\sigma \big| |E_\sigma - E_0| < \delta_\sigma, \text{ for small } \delta_\sigma > 0\} \tag{1.7}$$

and the union of the three δ-sets with E_0 is

$$N^\delta(E_0) = \bigcup_\sigma N_\sigma^\delta(E_0) \cup \{E_0\}. \tag{1.8}$$

For some time t, there is a point $\mathbf{x}_\sigma = (x_\sigma(t), y_\sigma(t))^{\mathrm{T}}$ on the orbit $q_\sigma(t)$ and this point is also on the normal $\mathbf{f}^\perp(\mathbf{x}_0) = (-f_2(\mathbf{x}_0), f_1(\mathbf{x}_0))^{\mathrm{T}}$ of the tangential vector of the separatrix $q_0(t)$ at a point $\mathbf{x}_0 = (x_0(t), y_0(t))^{\mathrm{T}}$, as shown in Fig.1.2. Therefore, the distance is defined as

$$\begin{aligned} \|q_\sigma(t) - q_0(t)\| &= \max_{t \in \mathscr{R}} \|\mathbf{x}_\sigma(t) - \mathbf{x}_0(t)\| \\ &= \max_{t \in \mathscr{R}} \sqrt{[x_\sigma(t) - x_0(t)]^2 + [y_\sigma(t) - y_0(t)]^2}. \end{aligned} \tag{1.9}$$

Lemma 1.1 *For* a perturbed Hamiltonian systems in Eq.(1.1) *with* (H5.1)—(H5.4), *for any positive* $\varepsilon > 0$, *there is a positive* $\delta_\sigma > 0(\sigma = \alpha, \beta, \gamma)$ *so that* $\|q_\sigma(t) - q_0(t)\| < \varepsilon$ *for* $E_\sigma \in N^\delta(E_0)$ *at a specific time* t.

Proof. For any positive $\varepsilon > 0$ let $\delta_\sigma = \varepsilon \|H_0\| > 0$ satisfying

$$|E_\sigma - E_0| = |H_0(q_\sigma) - H_0(q_0)| \leqslant \|H_0\| \cdot \|q_\sigma - q_0\| < \delta_\sigma,$$

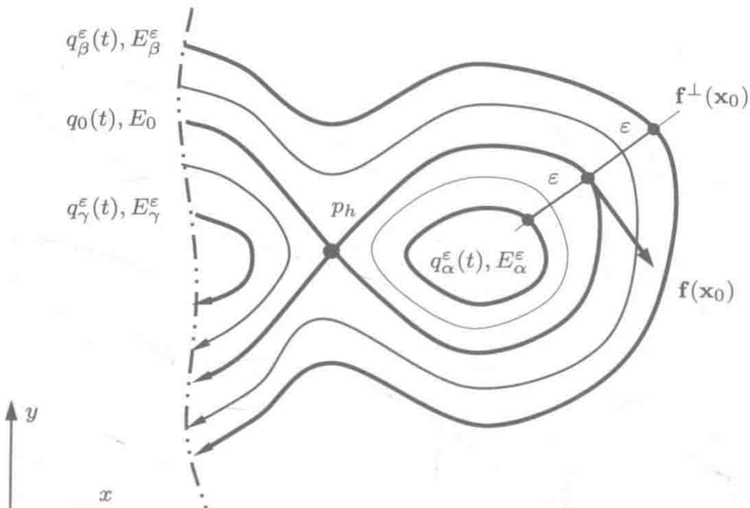

Fig. 1.2 The ε-neighborhood of orbit $q_0(t)$. The bold solid curves represent the separatrix $q_0(t)$ and the ε-neighborhood boundaries $q_\sigma^\varepsilon(t)$ are determined by $\max_{t\in[0,\infty)} \|q_\sigma^\varepsilon(t) - q_0(t)\| = \varepsilon(\sigma = \alpha, \beta, \gamma)$. The solid curves depict all orbits $q_\sigma(t)$ in the ε-neighborhood. The energies on the boundary orbits are given through $E_\sigma^\varepsilon = H_0(q_\sigma^\varepsilon(t))$.

where

$$\|H_0\| \equiv \sup_{\sigma\neq 0} \left[|H_0(q_\sigma) - H_0(q_0)|/\|q_\sigma - q_0\|\right].$$

Since the unperturbed Hamiltonians H_0 of orbits q_σ and q_0 are C^r-smooth $(r \geqslant 2)$ and $0 < \|H_0\| < \infty$ for bounded and closed orbits. Therefore, one obtains

$$\|q_\sigma(t) - q_0(t)\| < \delta_\sigma/\|H_0\| = \varepsilon.$$

This lemma is proved. ■

The ε-neighborhood of orbit $q_0(t)$ is formed by the three ε-sets of Γ_0 for the σ-orbits $(\sigma = \alpha, \beta, \gamma)$, as shown in Fig.1.2. The bold solid curves denote the separatrix $q_0(t)$ and the ε-neighborhood boundaries, $q_\sigma^\varepsilon(t)(\sigma = \alpha, \beta, \gamma)$, are determined through $\max_{t\in[0,\infty)} \|q_\sigma^\varepsilon(t) - q_0(t)\| = \varepsilon$ as $E_\sigma^\varepsilon = H_0(q_\sigma^\varepsilon(t))$. The solid curves represent all the σ-orbits $q_\sigma(t)$ in the ε-neighborhood.

The three ε-sets of Γ_0 for the σ-orbits $(\sigma = \alpha, \beta, \gamma)$ are defined by

$$\Gamma_\sigma^\varepsilon = \{(x, y)|(x, y) \in q_\sigma(t), \|q_\sigma(t) - q_0(t)\| < \varepsilon, t \in \mathscr{R}\}. \tag{1.10}$$

Furthermore, from Eq.(1.8), the unions of the ε-sets with Γ_0 are

$$\Gamma_0^\varepsilon = \cup_\sigma \Gamma_\sigma^\varepsilon \cup \Gamma_0, \quad \Gamma_{\sigma 0}^\varepsilon = \Gamma_\sigma^\varepsilon \cup \Gamma_0. \tag{1.11}$$

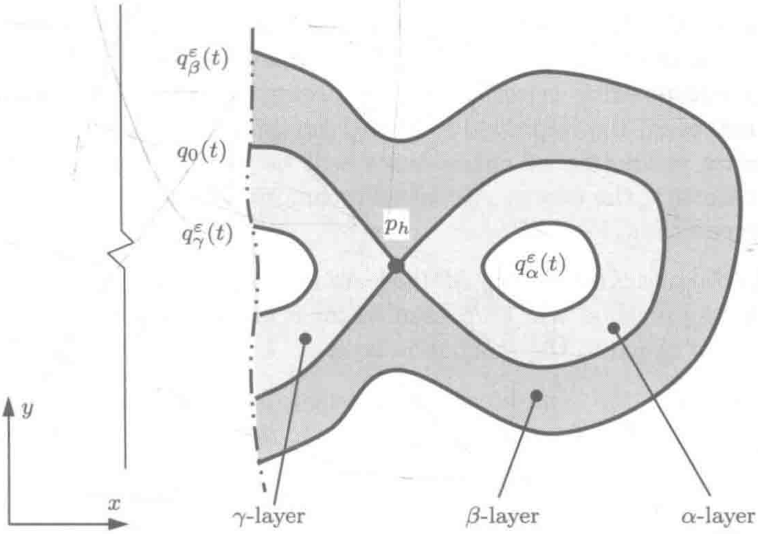

Fig. 1.3 A stochastic layer of Eq.(1.1) formed by the Poincaré mapping set of $q(t)$ in the ε-neighborhood of $q_0(t)$ for $t \in [0, \infty)$. The separatrix separates the stochastic layer into three sub-stochastic layers (i.e., α-layer and β-layer and γ-layer).

The Poincaré map $P : \Gamma^{\mathrm{P}} \to \Gamma^{\mathrm{P}}$, where the Poinaré mapping set in phase space is

$$\Gamma^{\mathrm{P}} \equiv \{(x_N, y_N)|(x_N, y_N) \in q(t), t_N = 2\pi N/\Omega + t_0, N = 0, 1, \cdots\} \subset \Gamma, \tag{1.12}$$

where t_0 is the initial time. Using the above notations, a stochastic layer is defined through the Poincaré mapping set with nonzero measure as follows:

Definition 1.1 The Poincaré mapping set Γ^{P} is termed the stochastic layer in the ε-sense if the compact dense set Γ^{P} belongs to Γ^{ε}_0 (or $\Gamma^{\mathrm{P}} \subset \Gamma^{\varepsilon}_0$) for $t_N = 2N\pi/\Omega + t_0 (N = 0, 1, \cdots)$. Similarly, the Poincaré mapping subset $U_\sigma \subset \Gamma^{\mathrm{P}}$ is the σ-stochastic layer if $U_\sigma \subset \Gamma^{\varepsilon}_{\sigma 0}$ for $t_N = 2N\pi/\Omega + t_0$.

A stochastic layer of system in Eq.(1.1) is formed through the Poincaré mapping set of $q(t)$ in the ε-neighborhood for time $t \in [0, \infty)$, as shown in Fig.1.3. The separation of the stochastic layer by the separatrix gives three sub-stochastic layers shaded. The sub-layers relative to the σ-orbits ($\sigma = \alpha, \beta, \gamma$) are termed the σ-stochastic layer. The more detail description can be referred to Luo (2008).

1.1.2 Approximate criteria

The predictions of resonance in the stochastic layer of a 2-dimensional nonlinear Hamiltonian system will be presented. The incremental energy technique will be presented first from the approximate first integral quantity increment (or approximate energy increment). The whisker mapping will be obtained, and the corresponding criterion will be presented. The linearization of the whicker mapping, the improved standard mapping will be presented and the approximate prediction of chaos onset will be given. From the exact first integral quantity, the energy spectrum technique will be developed for a numerical prediction.

(A) *An Incremental Energy Method.* As in Luo and Han (2001), the incremental energy method will be presented for a understanding of the resonant mechanism of chaos in the stochastic layer.

Lemma 1.2 *For the dynamical system in Eq.*(1.1) *with* (H1.1)—(H1.4), *if a point* $(x, y) \in \Gamma \cap \Gamma_\sigma$ *for some* $\sigma = \{\alpha, \beta, \gamma\}$, *then* $H_0(q(t)) = H_0(q_\sigma(t))$ *for some time* t.

Proof. If the perturbed orbit $q(t)$ in the set Γ is intersected with an unperturbed orbit $q_\sigma(t)$ in the set Γ_σ for some $\sigma \in \{\alpha \in [-1, 0), \beta \in (0, 1], \gamma \in [-1, 0)\}$ at time t, there is a single point $(x, y) \in \Gamma \cap \Gamma_\sigma$. Therefore, for $(x, y) \in \Gamma \cap \Gamma_\sigma$, we have $(x, y) \in q_\sigma(t)$ and $(x, y) \in q(t)$. Thus,$H_0(q(t)) = H_0(x, y) = H_0(q_\sigma(t))$, which implies that the conservative energy is equal for the same point in phase space. This lemma is proved. ■

The detailed discussion is given as follows. Because the conservative energy H_0 is the first integral quantity, for the σ-layer, the map describing the changes of both energy H_0 and phase φ for time transition from t_i to $t_i + T_\sigma$ in Eq.(1.1) is obtained, i.e.,

$$E_{i+1} = E_i + \Delta H^\sigma(\varphi_i) \text{ and } \varphi_{i+1} = \varphi_i + \Delta\varphi^\sigma(E_{i+1}), \tag{1.13}$$

where $E_i = H(q(t_i))$ and $\varphi_i = \varphi(q(t_i))$. For a specific external frequency Ω, the initial phase is defined by $\varphi_i = \Omega t_i$. Notice that the energy relationship in the foregoing can be expressed through the action variable. As in Chirikov (1979) and Lichtenberg and Lieberman (1992), the phase and energy changes, $\Delta\varphi^\sigma(E_{i+1})$ and $\Delta H^\sigma(\varphi_i)$, are approximately computed by

$$\begin{aligned} \Delta\varphi^\sigma(E_{i+1}) &\approx \Omega T_\sigma(E_{i+1}), \text{ and} \\ \Delta H^\sigma(\varphi_i) &\approx \mu \int_{t_i}^{T_\alpha(E_i)+t_i} [H_0, H_1] dt \\ &= \mu \int_{t_i}^{T_\sigma(E_i)+t_i} (f_1 g_2 - f_2 g_1) dt, \end{aligned} \tag{1.14}$$

where $[\cdot, \cdot]$ represents the Poisson bracket. The energy and phase changes in Eq.(1.14) for the system in Eq.(1.1) over one period T_σ of the σ-orbit are sketched in Fig.1.4. If $E_i = E_0$ expresses the energy of the separatrix, equation (1.13) becomes a generalized separatrix map (or a generalized whisker map). When the σ-orbit ($\sigma = \alpha, \beta, \gamma$) is close to the separatrix (i.e., $T_\sigma \to \infty$),