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Some Curvature Flows into Spheres

(若干演化为球面的曲率流)

Guo Shunzi

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《博士后文库》序言

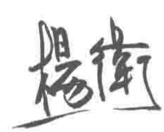
1985年，在李政道先生的倡议和邓小平同志的亲自关怀下，我国建立了博士后制度，同时设立了博士后科学基金。30多年来，在党和国家的高度重视下，在社会各方面的关心和支持下，博士后制度为我国培养了一大批青年高层次创新人才。在这一过程中，博士后科学基金发挥了不可替代的独特作用。

博士后科学基金是中国特色博士后制度的重要组成部分，专门用于资助博士后研究人员开展创新探索。博士后科学基金的资助，对正处于独立科研生涯起步阶段的博士后研究人员来说，适逢其时，有利于培养他们独立的科研人格、在选题方面的竞争意识以及负责的精神，是他们独立从事科研工作的“第一桶金”。尽管博士后科学基金资助金额不大，但对博士后青年创新人才的培养和激励作用不可估量。四两拨千斤，博士后科学基金有效地推动了博士后研究人员迅速成长为高水平的研究人才，“小基金发挥了大作用”。

在博士后科学基金的资助下，博士后研究人员的优秀学术成果不断涌现。2013年，为提高博士后科学基金的资助效益，中国博士后科学基金会联合科学出版社开展了博士后优秀学术专著出版资助工作，通过专家评审遴选出优秀的博士后学术著作，收入《博士后文库》，由博士后科学基金资助、科学出版社出版。我们希望，借此打造专属于博士后学术创新的旗舰图书品牌，激励博士后研究人员潜心科研，扎实治学，提升博士后优秀学术成果的社会影响力。

2015年，国务院办公厅印发了《关于改革完善博士后制度的意见》（国办发〔2015〕87号），将“实施自然科学、人文社会科学优秀博士后论著出版支持计划”作为“十三五”期间博士后工作的重要内容和提升博士后研究人员培养质量的重要手段，这更加凸显了出版资助工作的意义。我相信，我们提供的这个出版资助平台将对博士后研究人员激发创新智慧、凝聚创新力量发挥独特的作用，促使博士后研究人员的创新成果更好地服务于创新驱动发展战略和创新型国家的建设。

祝愿广大博士后研究人员在博士后科学基金的资助下早日成长为栋梁之才，为实现中华民族伟大复兴的中国梦做出更大的贡献。



中国博士后科学基金会理事长

Preface

Curvature flows are powerful tools for solving various problems in geometry and physics, and receive more and more attention in the last decade, starting with the groundbreaking paper [51] of Hamilton who studied the Ricci flow, which describes the evolution of the metric of a manifold by its Ricci curvature tensor. Huisken [55] then considered the mean curvature flow, which describes the normal evolution of convex hypersurface in the Euclidean space by its mean curvature vector. From the viewpoint of partial differential equations (PDEs), one can distinguish different flows by the type of equation used to describe them. Another way to distinguish them arises from the viewpoint of differential geometry by dividing them into extrinsic and intrinsic flows. For example, the Ricci flow is the most important intrinsic curvature flow. For instance, by using the Ricci flow Grigori Pereman solved the Poincaré conjecture. However, the mean curvature flow perhaps is the most important extrinsic curvature flow. The mean curvature flow, originally proposed by Mullins [81], is used to model the formation of grain boundaries of annealed metals. One can also use this flow to classify hypersurfaces satisfying certain curvature condition, to produce minimal surfaces, or to derive isoperimetric inequalities. Many books deal with the theory of mean curvature flow. I cite several of them for the interested readers, which are [35, 76, 109] in the bibliography. In this sense, curvature flows are also called geometric flows or geometric evolution equations. Various methods from the calculus of variations, geometric measure theory, topology and functional analysis enable us to treat the problems involving curvature flows, because these problems arise from interactions among various fields. Therefore, not just in theories of pure mathematics but also in applications of mathematics and other fields, geometric flows have shown their strong vitality and great value.

This book grew up from a collection of my papers for graduate students and researchers interested in geometry and analysis, and I have tried to maintain that spirit. Nevertheless, I have to say a few words about prerequisities. I assume that the readers are familiar with basic concepts of differential geometry, in particular submanifold geometry. In analysis I assume that the reader is familiar with the basic facts from the theory of partial differential equations of second order. Good references for these are [63] and [73] respectively. In the book we concentrate on the fields of extrinsic curvature flows, which describe the evolution of surfaces in the direction of the unit normal with a speed equal to a function of its principle curvature in time. Since extrinsic flows are described by using extrinsic geometric quantities such as those quantities involving the principal curvature. Thus, the manifold under consideration must be embedded (or more generally immersed) into an ambient manifold space to make the extrinsic quantities appear. Therefore one can investigate the extrinsic flows with various choices, such as on one codimension and high codimension, or various ambient spaces, or the various speeds of the evolution.

At some point in a book every person must ask himself the following question: "What have I done in this book?" In this book I investigate the normal evolution of some closed convex hypersurfaces in the Euclidean space and the hyperbolic space by certain much broader class of nonlinear geometric flows, which is a reasonably large class and includes many of the most commonly studied examples such as powers of the mean curvature, Gauss curvature, elementary symmetric functions of curvature and their ratios. More precisely, I mainly study the following four distinct curvature problems which have one thing in common that curvature flows driven by the extrinsic curvature are studied: The first class of problems is concerned with closed convex hypersurfaces of Euclidean space evolving by functions of the mean curvature, including the power mean curvature flow case; the second is concerned with horospherical convex hypersurfaces contracting of the hyperbolic space by functions of the mean curvature; the third is on mixed volume preserving flow by powers of homogeneous curvature functions of degree one, including a class of volume-preserving curvature flow; the last one considers the forced mean curvature flows for submanifolds of high codimension in Euclidean space. It is our main purpose to have a good understanding of the shape and structure of manifolds by the way of analysis coming from the second partial differential equations and differential geometry. In fact, the central results we obtained can be considered as natural extensions to some early closed convex hypersurface theorems in Euclidean space for both the power mean curvature flows and mixed volume preserving flows for speeds with degree one.

In the following, I will give an idea for the remainder of content of the book. More information will be provided at the end in the introduction of each chapter, except at the beginning of Chapter 1.

Chapter 1 sets our notations, recalls definitions, introduces some useful fundamental formulas and summarises some standard facts on the geometry of submanifolds and geometry of graphs. Moreover, we briefly introduce the interior Hölder estimates.

Chapter 2 concerns the evolution of a closed hypersurface of the hyperbolic space $\mathbb{H}_\kappa^{n+1}$ of constant sectional curvature κ, convex by horospheres, in direction of its inner unit normal vector, where the speed equals a positive power β of the positive mean curvature. It is shown that the flow exists on a finite maximal interval, convexity by horospheres is preserved, and the hypersurfaces shrink down to a single point of $\mathbb{H}_\kappa^{n+1}$ as the final time is approached.

Chapter 3 concerns closed hypersurfaces of dimension $n(\geqslant 2)$ in the hyperbolic space $\mathbb{H}_\kappa^{n+1}$ of constant sectional curvature κ evolving in the direction of its normal vector, where the speed equals a power $\beta \geqslant 1$ of the mean curvature. The main result is that if the initial closed, weakly h-convex hypersurface would satify that the ratio of the biggest and smallest principal curvatures everywhere is close enough to 1, depending only on n and β. Then under the flow this is maintained. There exists a unique and smooth solution of the flow which converges to a single point in $\mathbb{H}_\kappa^{n+1}$ in a maximal finite time, and when rescaling appropriately, the evolving exponential hypersurfaces converge to a unit geodesic sphere of $\mathbb{H}_\kappa^{n+1}$.

Chapter 4 concerns closed hypersurfaces of dimension $n(\geqslant 2)$ in the hyperbolic space $\mathbb{H}_\kappa^{n+1}$ of constant sectional curvature κ evolving in the direction of its normal vector, where the speed is given by a power $\beta(\geqslant 1/m)$ of the mth mean curvature plus a volume preserving term, including the case of powers of the mean curvature and of the Gauss curvature. The main result is that if the initial hypersurface satisfies that the ratio of the biggest and smallest principal curvatures is close enough to 1 everywhere, depending only on n, m, β and κ. Then under the flow this is maintained. There exists a unique and smooth solution of the flow for all times, and the evolving hypersurfaces converge exponentially to a geodesic sphere of $\mathbb{H}^{n+1}_{\kappa}$, enclosing the same volume as the initial hypersurface.

Chapter 5 concerns the evolution of a closed convex hypersurface in $\mathbb{R}^{n+1}$, in the direction of its inner unit normal vector, where the speed is given by a smooth function depending only on the mean curvature, and satisfies some further restrictions without requiring homogeneity. It is shown that the flow exists on a finite maximal interval, convexity is preserved and the hypersurfaces shrink down to a single point as the final time is approached. This generalises the corresponding result of Schulze [89] for the positive power mean curvature flows to a much larger possible class of flows by the functions depending only on the mean curvature.

Chapter 6 is dedicated to the study of the evolution of a closed hypersurface of the hyperbolic space, convex by horospheres, in the direction of its inner unit normal vector, where the speed equals a smooth function depending only on the mean curvature, and satisfies some further restrictions, without requiring homogeneity. It is shown that the flow exists on a finite maximal interval, convexity by horospheres is preserved and the hypersurfaces shrink down to a single point as the final time is approached. This generalises the previous result [47] for convex hypersurfaces in the Euclidean space by the author to the setting in the hyperbolic space for the same class of flows.

Chapter 7 considers the evolution of a closed hypersurface of dimension $n(\geqslant 2)$ in the Euclidean space under a mixed volume preserving flow. The speed equals a power $\beta(\geqslant 1)$ of homogeneous, either convex or concave, curvature functions of degree one plus a mixed volume preserving term, including the case of powers of the mean curvature and of the Gauss curvature. The main result is that if the initial hypersurface satisfies a suitable pinching condition, there exists a unique, smooth solution of the flow for all times, and the evolving hypersurfaces converge exponentially to a round sphere, enclosing the same mixed volume as the initial hypersurface. This generalizes the previous results for convex hypersurfaces in the Euclidean space by McCoy [79] and Cabezas-Rivas and Sinestrari [23] to more general curvature flows for convex hypersurfaces with similar curvature pinching condition.

Chapter 8 considers the evolution by mean curvature vector plus a forcing field in the direction of its position vector of a closed submanifold of dimension n ($\geqslant 2$) in $\mathbb{R}^{n+p}$. Suppose that mean curvature vector is nonzero everywhere and that the full norm of the second fundamental form is bounded by a fixed multiple (depending only on n) of the length of the mean curvature vector at every point. It is shown that such submanifolds may contract to a point in finite time if the forcing field is small, or exists for all time and expands to infinity if it is large enough. Moreover, if the evolving submanifolds undergo suitable homotheties and the time parameter is transformed appropriately into a parameter $\tilde{t}$, $0 \leqslant \tilde{t} < \infty$, it is also shown the normalized submanifolds in any case converge smoothly to a round sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+p}$ as $\tilde{t} \to \infty$.

Summing up, this book provides a positive significance as an addition and improvement of the most interesting topics in present-day research on a theory of the generalized curvature flows. I hope that these contributions can be applied to settle problems in a number of areas of geometry. The book is, however, also written for the benefit of the readers who have heard about some of geometric analysis before, and would like to see the newest development so they can research independently after studying this book. Thus, this book is supposed to be a nice textbook for graduate students and researchers interested in differential geometry and general relativity.

My thanks go to China Postdoctoral Science Foundation Grant (2015M582546), the Fujian Provincial Natural Science Foundation Grants (2016J01672 and 2013J01030), and the Natural Science Foundation of China (11761080) and the open foundation of Hubei Key Laboratory of Applied Mathematics (Hubei University) for

supporting this work.

This book was partially written when I was a Ph.D student at Hubei University and was a postdoctoral fellow at Sichuan University. In the two places I found a warm hospitable and fantastic atmosphere to work.

Many people helped me. In particular I wish to thank Professor Li Guanghan for his constant enthusiasm and for his always extremely helpful suggestion, specially thank Professors (in alphabetical order) Guo Zhen, Li Anmin, Li Haizhong, Sheng Li, Wu Chuanxi and Zhao Guosong for their helpful discussions on various aspects of mathematics and many others over the last few years.

Finally, the writing of this book would have been impossible without the understanding and help of my wife Hui and my daughter Xiyun.

Guo Shunzi
Kunming, January 2018

Contents

Chapter 1

Preliminary

In this chapter we review some well-known supporting materials before proving our results. In Section 1.1 we want to give an overview of the conventions and definitions we rely on. In Section 1.2 we provide some useful fundamental formulas and inequalities, which will be needed in the later chapters. In Section 1.3 we summarize some standard facts on the geometry of graphical submanifolds. Moreover, in Section 1.4 we briefly introduce interior Hölder estimates for nonlinear PDEs.

1.1 Notations

The geometry of immersed submanifold Let M^n be a smooth, compact oriented manifold of dimension $n \geqslant 2$ without boundary, $(N^{n+p}, \bar{g})$ be an $(n+1)$-dimensional complete Riemannian manifold, and $X_0 : M^n \to N^{n+p}$ be a smooth immersion. The number p is called the codimension of the immersion. From now on, use the same notation as in [21, 47, 55, 89] in local coordinates $\{x^i\}$, $1 \leqslant i \leqslant n$, near $p \in M^n$ and $\{y^\alpha\}$, $1 \leqslant \alpha, \beta \leqslant n+p$, near $F(p) \in N^{n+p}$. We can agree on the following range of these indexes

$$1 \leqslant i, j, k, \cdots \leqslant n, \quad 1 \leqslant \alpha, \beta, \gamma, \cdots \leqslant n+p, \quad n+1 \leqslant \lambda, \mu, \cdots \leqslant n+p.$$

In this chapter, the repeated index summation follows Einstein summation convention. As in [13, 18], let $T(M \times [0,T))$ denote the tangent bundle of $M \times [0,T)$. The decomposition of the tangential bundle is expressed as $T(M \times [0,T)) = \mathcal{T} \oplus \mathbb{R}\partial t$, where $\mathcal{T} = \{u \in T(M \times [0,T))\mathrm{d}t(u) = 0\}$ is the tangential bundle for space. $F_t^*TN = \bigcup_{p \in M_t} T_{F(p)}N$ is the pullback bundle of M_t, $\bar{g}_F$, ${}^F\bar{\nabla}$ are the restriction metric and pullback connection coming from a Riemannian metric $\bar{g}$ and $\bar{\nabla}$ on F_t^*TN. The tangential map $F_{t*}: \mathcal{T} \to F_t^*TN$ defines a sub-bundle of F^*TN of rank n. The orthogonal complement of $F_*(\mathcal{H})$ in F^*TN is a vector bundle of rank p which we denote by $\mathcal{N}$ and refer to as the (spacetime) normal bundle. The pullback metric $g(=F_t^*\bar{g})= g_{ij}(x,t)\mathrm{d}x^i \otimes \mathrm{d}x^j$ is a metric on M_t, and satisfies $g_{ij} = \bar{g}(F_{t*}(\partial_i), F_{t*}(\partial_j))$. The volume form on M_t is $\mathrm{d}\mu_t = \sqrt{\det(g_{ij})}\mathrm{d}x$. The connection ∇ on the tangent bundle is given by

$$F_{t*}(\nabla_u v) := \left(\bar{\nabla}_{F_{t*}(u)}\overline{F_{t*}(v)}\right)^\top, \quad \forall u, v \in \mathcal{T},$$

where $\top$ denotes the projection onto $F_{t*}(\mathcal{T})$, and $\overline{F_{t*}(v)}$ denotes an arbitrary locally smooth extension on $F_{t*}(v)$. Connection $\nabla^\perp$ on the normal bundle is given by

$$\nabla_u^\perp v := \left(\bar{\nabla}_u^{F_t^*TN} v\right)^\perp, \quad \forall v \in \Gamma(\mathcal{N}) \subset \Gamma(F_t^*TN),$$

where $^\perp$ denotes the projection onto $\mathcal{N}$. Since $\mathcal{T}, \mathcal{T}^*, F_t^*TN, \mathcal{N}$ and connections on their tensor product space can be induced by $\bar{\nabla}$. In the absence of confusion, they are unified as ∇. The second fundamental form $A = (\nabla F_{t*})^\perp \in F_t^*TN \otimes \mathcal{T}^* \otimes \mathcal{T}^*$ is a symmetric multilinear form of $\mathcal{T}$, whose local representation is $A = A_{ij}^\alpha \dfrac{\partial}{\partial y^\alpha} \otimes \mathrm{d}x^i \otimes \mathrm{d}x^j$. $H = \mathrm{trace} A \in \Gamma(\mathcal{N})$ denotes the mean curvature vector field of M_t. $\mathring{A} = A - \dfrac{1}{n} g \otimes H$ denotes the traceless part of second fundamental form. $\forall u, v \in \mathcal{T}$, we have Gauss equation

$$^F\bar{\nabla}_u(F_{t*}v) = F_{t*}(\nabla_u v) + A(u,v), \tag{1.1.1}$$

and $\forall \xi \in \Gamma(\mathcal{N})$, $\mathcal{W}(u,\xi) = -(\nabla_u \xi)^\top$ gives the Weingarten map $\mathcal{W} \in \Gamma(\mathcal{T}^* \otimes \mathcal{N} \otimes \mathcal{T})$, we have Weingarten equation

$$^F\bar{\nabla}_u \xi = \nabla_u^\perp \xi - F_{t*}(\mathcal{W}(u,\xi)). \tag{1.1.2}$$

Let E be a bundle on M_t, ∇ be a connection on E, and we define the curvature tensor on E,

$$\mathcal{R}^{E,\nabla}(u,v)\sigma := \left(\nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}\right)\sigma, \quad \forall \sigma \in \Gamma(E).$$

Furthermore, if E admits a metric bundle $\langle \cdot, \cdot \rangle$,

$$R^{E,\nabla}(\mu, \sigma, u, v) := \left\langle \mu, \mathcal{R}^{E,\nabla}(u,v)\sigma \right\rangle, \quad \forall \mu, \sigma \in \Gamma(E).$$

we can denote the first covariant derivative of A by

$$\nabla_i A_{jk} = \nabla_i^{\perp}(A_{jk}) - A_{lk}\Gamma_{ji}^l - A_{jl}\Gamma_{ki}^l,$$

where Γ_{jk}^i denotes the Christoffel symbol for the induced connection on M_t. Similarly, we can also give the definition of the second covariant derivative of A. The Laplace ΔT of a tensor T on M_t is given by $\Delta T = g^{kl}\nabla_k\nabla_l T$, and g^{kl} denotes the inverse matrix of g_{kl}. In our setting adapted to submanifold, the local expressions for Gauss equation, Codazzi equation, and Ricci equation are displayed respectively as follows.

$$R_{ijkl} = R_{\alpha\beta\gamma\delta}\partial_i F^\alpha \partial_j F^\beta \partial_k F^\gamma \partial_l F^\delta + g_{\alpha\beta}\left(A_{ik}^\alpha A_{jl}^\beta - A_{il}^\alpha A_{jk}^\beta\right), \quad (1.1.3)$$

$$\nabla_i A_{jk}^\alpha - \nabla_j A_{ik}^\alpha = R_{\beta\gamma\delta}^\alpha \partial_k F^\beta \partial_i F^\gamma \partial_j F^\delta - R_{kij}^l \partial_l F^\alpha, \quad (1.1.4)$$

$$(R^\perp)_{\lambda ij}^\mu \nu_\mu^\alpha = R_{\beta\gamma\delta}^\alpha \nu_\lambda^\beta \partial_i F^\gamma \partial_j F^\gamma - g^{kl} R_{\beta\gamma\delta}^\epsilon g_{\epsilon\sigma} \nu_\lambda^\beta \partial_i F^\gamma \partial_j F^\delta \partial_k F^\sigma \partial_l F^\alpha$$
$$- g_{\beta\gamma} g^{kl}\left(\nu_\lambda^\beta A_{ik}^\gamma A_{jl}^\alpha - \nu_\lambda^\beta A_{jk}^\gamma A_{il}^\alpha\right), \quad (1.1.5)$$

where $g_{\alpha\beta}$ is a local expression for $\bar{g}$ in the local system $\{y^\alpha\}$ on N, $R_{\beta\gamma\delta}^\alpha = g^{\alpha\epsilon}R_{\epsilon\beta\gamma\delta}$, and $\nu_\lambda = \nu_\lambda^\alpha \partial_\alpha$ is a local trivialization of $\mathcal{N}$. Note that the Codazzi equation is useless in dimension one (i.e. for curves) and that Ricci equation is useless for hypersurfaces (i.e. in codimension one).

Special situations: hypersurfaces If $X_0 : M^n \to N^{n+1}$ is an immersion of a hypersurface, that is $p = 1$ and one can define a number of scalar curvature quantities related to the second fundamental tensor of M. For simplicity assumes that both M and N are orientable (otherwise the following computations are only local). Then there exists a unique unit normal vector field $\nu \in \mathcal{N}$ which is called the principle normal at $p \in M$. Then further important quantities are the second fundamental form $A(p) = \{h_{ij}\}$ and the Weingarten map $\mathscr{W} = \{g^{ik}h_{kj}\} = \{h_j^i\}$ as a symmetric operator and a self-adjoint operator respectively. The real eigenvalues $\lambda_1(p) \leqslant \cdots \leqslant \lambda_n(p)$ of $\mathscr{W}$ are called the principal curvatures of $X(M^n)$ at $X(p)$. The scalar mean curvature is given by

$$H := \mathrm{tr}_g \mathscr{W} = h_i^i = \sum_{i=1}^n \lambda_i,$$

the square of the norm of the second fundamental form by

$$|A|^2 := \mathrm{tr}_g(\mathscr{W}^t\mathscr{W}) = h^i_j h^j_i = h^{ij}h_{ij} = \sum_{i=1}^n \lambda_i^2,$$

and the Gauss-Kronecker curvature by

$$K := \det(\mathscr{W}) = \det\{h^i_j\} = \frac{\det\{h_{ij}\}}{\det\{g_{ij}\}} = \prod_{i=1}^n \lambda_i.$$

More generally, the mixed mean curvatures $E_r, 1 \leqslant r \leqslant n$, are given by the elementary symmetric functions of the λ_i

$$E_r(\lambda) = \sum_{1\leqslant i_1 \leqslant \cdots \leqslant i_r \leqslant n} \lambda_{i_1}\cdots\lambda_{i_r} = \frac{1}{r!}\sum_{i_1,\cdots,i_r} \lambda_{i_1}\cdots\lambda_{i_r}, \quad \text{for } \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n,$$

and their quotients are

$$Q_r(\lambda) = \frac{E_r(\lambda)}{E_{r-1}(\lambda)}, \quad \text{for } \lambda \in \Gamma_{r-1},$$

where $E_0 \equiv 1$, and $E_l \equiv 0$, if $r > n$, $\Gamma_r := \{\lambda \in \mathbb{R}^n | E_i > 0, i = 1, \cdots, r\}$. Denote the sum of all terms in $E_r(\lambda)$ not containing the factor λ_i by $E_{r;i}(\lambda)$. It is clear that H, K, E_m, Q_m may be viewed as functions of λ, or as functions of A, or as functions of $\mathscr{W}$, or also as functions of space and time on M_t. We sum over repeated indices from 1 to n unless otherwise indicated. In computations on the hypersurface M_t, raised indices indicate contraction with the metric.

When the codimension is one, the local expressions for Gauss equation, and Codazzi equation become change respectively as follows:

$$R_{ijkl} = R_{\alpha\beta\gamma\delta}\partial_i F^\alpha \partial_j F^\beta \partial_k F^\gamma \partial_l F^\delta + h_{ik}h_{jl} - h_{il}h_{jk}, \tag{1.1.6}$$

$$\nabla_i h_{jk} - \nabla_j h_{ik} = \bar{R}^\alpha_{\beta\gamma\delta}\partial_k \nu^\beta \partial_i F^\gamma \partial_j F^\delta - R^l_{kij}\partial_l F^\alpha. \tag{1.1.7}$$

Note that since the codimension is one, we do not have a Ricci equation in this case.

1.2 Some useful properties

The following identities for E_r and the properties on the quotients Q_r were proved by Huisken and Sinestrari in [61].