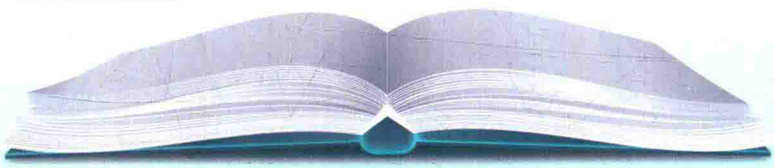
Lie Superalgebras and Related Algebraic Structures

李超代数及相关的代数结构

张润萱 著



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Preface

This monograph has grown from my PhD thesis at Chern Institute of Mathematics of Nankai University, a series of seminars at Northeast Normal University, and research work at Queen's University of Canada during 2014–2016.

The whole book consists of twelve chapters. Part I (the first four chapters) covers some basic concepts and facts on Lie superalgebras and related algebras, without going into detailed proofs that could be found in [52], [26], [22] and [47]. The second part is the core of this book. It consists of five chapters which could be viewed as a continuation of my thesis [64]. This part contains several research topics that my collaborators and I are interested in and a few unsolved, natural questions that measure differences between ordinary algebras and superalgebras. The last part of this book is about my recent works [19, 20, 21] which are concerned with algebraic structures of a kind of non-associative algebras, ω-Lie algebras, that appeared recently in mathematical physics and geometry. I hope this book could be helpful for the graduate students and mathematicians who are working on this subject.

I deeply appreciate my supervisors Professor Yongzheng Zhang at Northeast Normal University, Professor Chengming Bai at Chern Institute of Mathematics and Professor Ivan Dimitrov at Queen's University for their patience, help and guidance. This book is partially supported by NNSF of China (grant number: 11301061).

Northeast Normal University
Runxuan Zhang
January 2018, Changchun

Contents

Part I

Preliminaries on Lie Superalgebas and Related Algebras

In this introductory part, we present some important notions in superalgebraic theory and mathematical physics: graded vector spaces, Lie superalgebras, left-symmetric superalgebras, Hom-type superalgebras and ω-Lie algebras. We also introduce the background and investigate basic properties of these algebraic structures. This part consists of four chapters: graded vector spaces and Lie superalgebras (Chapter 1); left-symmetric superalgebras and their constructions from various well-known algebraic structures (Chapter 2); Hom-type superalgebras (Chapter 3); ω-Lie algebras (Chapter 4).

Chapter 1
Lie Superalgebras

1.1 Graded vector spaces and superalgebras

Let us start with some basic facts of $\mathbb{Z}_2$-graded vector spaces and Lie superalgebras, which could be found in Scheunert [52].

Convention: Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space over a field $\mathbb{F}$. The elements of $V_{\bar{0}}$ are called even and those of $V_{\bar{1}}$ are called odd. If $|x|$ occurs in some expression, we regard x as a $\mathbb{Z}_2$-homogeneous element and $|x|$ as the $\mathbb{Z}_2$-degree of x. All (super) algebras or vector spaces are assumed to be finite-dimensional and the ground field $\mathbb{F}$ has characteristic zero unless stated otherwise.

Let V and W be two $\mathbb{Z}_2$-graded vector spaces over a field $\mathbb{F}$. The direct sum $V \oplus W$ is a $\mathbb{Z}_2$-graded vector space defined by

$$(V \oplus W)_\alpha := V_\alpha \oplus W_\alpha$$

for $\alpha \in \mathbb{Z}_2$. The tensor product space $V \otimes W$ is also a $\mathbb{Z}_2$-graded vector space if we define

$$(V \otimes W)_\gamma := \oplus_{\alpha+\beta=\gamma} V_\alpha \otimes W_\beta$$

for $\alpha, \beta, \gamma \in \mathbb{Z}_2$. The even linear map $\sigma : V \otimes V \longrightarrow V \otimes V$ defined by $\sigma(x \otimes y) = (-1)^{|x||y|} y \otimes x$ for all $x, y \in V$, is called the *twist map* on V. An element $r \in V \otimes V$ is said to be *supersymmetric* if it is fixed under the action of the twist map σ, i.e., $\sigma(r) = r$. For any $\beta \in \mathbb{Z}_2$, a linear map $f : V \longrightarrow W$ is said to be *homogeneous of degree* β if $f(V_\alpha) \subseteq W_{\alpha+\beta}$ for all $\alpha \in \mathbb{Z}_2$. The dual space $V^* = \mathrm{Hom}(V, \mathbb{F})$ of V is $\mathbb{Z}_2$-graded with

$$V^*_\alpha := \{g \in V^* \mid g(V_{\alpha+1}) = \{0\}\}$$

for $\alpha \in \mathbb{Z}_2$. As V is finite-dimensional, we have $(V \otimes V)^* \cong V^* \otimes V^*$ as vectors spaces and we may extend the natural paring $\langle -,- \rangle_V : V^* \times V \longrightarrow \mathbb{F}$ to $\langle -,- \rangle_{V\otimes V} : (V \otimes V)^* \times (V \otimes V) \longrightarrow \mathbb{F}$ by

$$\langle a^* \otimes b^*, u \otimes v \rangle_{V\otimes V} = (-1)^{|b^*||u|} \langle a^*, u \rangle_V \cdot \langle b^*, v \rangle_V$$

for all $a^*, b^* \in V^*$ and $u, v \in V$.

Note that as vector spaces, $V \otimes V \cong \mathrm{Hom}(V^*, V) \cong \mathrm{BF}(V \times V, \mathbb{F})$ (the space of all bilinear forms on V). Now we identify elements of $V \otimes V$ with linear maps from V^* to V via the natural paring on V. Given an element $r \in V \otimes V$, the corresponding linear map $T_r : V^* \longrightarrow V$ is defined by

$$\langle u^*, T_r(v^*) \rangle_V := (-1)^{|r||v^*|} \langle u^* \otimes v^*, r \rangle_{V\otimes V}, \tag{1.1}$$

where $u^*, v^* \in V^*$. We say that r is *non-degenerate* if T_r is an invertible map. Moreover, an invertible linear map $T : V^* \longrightarrow V$ induces a non-degenerate bilinear form $\mathscr{B}_T$ on V defined by

$$\mathscr{B}_T(u, v) := \langle T^{-1}(u), v \rangle_V \tag{1.2}$$

for all $u, v \in V$. We say that T is *supersymmetric* if $\mathscr{B}_T$ is supersymmetric. Note that if $r \in V \otimes V$ is a supersymmetric element, then $\mathscr{B}_{T_r}$ is supersymmetric. Conversely, a supersymmetric bilinear form $\mathscr{B}$ on V also corresponds to a supersymmetric element in $V \otimes V$. Similarly, since $W \otimes V^* \cong \mathrm{Hom}(V, W)$ as vector spaces, a linear map $T : V \longrightarrow W$ of $\mathbb{Z}_2$-graded vector spaces can be identified with the element $r_T \in W \otimes V^*$ defined by

$$\langle w^* \otimes v, r_T \rangle_{W\otimes V^*} := (-1)^{|v||T|} \langle w^*, T(v) \rangle_V, \tag{1.3}$$

where $v \in V$ and $w^* \in W^*$. Thus there exists an embedding $\mathrm{Hom}(V, W) \hookrightarrow (W \oplus V^*) \otimes (W \oplus V^*)$ through the natural injection $W \otimes V^* \hookrightarrow (W \oplus V^*) \otimes (W \oplus V^*)$ which sends every $w \otimes v^* \in W \otimes V^*$ to $(w, 0) \otimes (0, v^*)$. Given a linear map $T \in \mathrm{Hom}(V, W)$, the *dual map* $T^* : W^* \longrightarrow V^*$ is defined by

$$\langle T^*(w^*), v \rangle_V := (-1)^{|T^*||w^*|} \langle w^*, T(v) \rangle_W \tag{1.4}$$

for all $v \in V$ and $w^* \in W^*$.

In what follows, we use $\langle -,- \rangle$ instead of $\langle -,- \rangle_V$ to denote the natural paring on V, unless stated otherwise.

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{F}$. We say that A is a *superalgebra* if A is an algebra (not necessarily associative) over $\mathbb{F}$, and

$$A_\alpha \cdot A_\beta \subseteq A_{\alpha+\beta}$$

for $\alpha, \beta \in \mathbb{Z}_2$.

A superalgebra $A = A_{\bar{0}} \oplus A_{\bar{1}}$ is said to be *nontrivial* if $A_{\bar{1}} \neq \{0\}$. The homomorphisms (endomorphisms, isomorphisms, automorphisms) of superalgebras are always assumed to be homogeneous linear maps of degree zero.

1.2 Lie superalgebras

A superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, [-,-])$ is called a *Lie superalgebra* if

$$[x,y] = -(-1)^{|x||y|}[y,x] \qquad \text{(graded skew-symmetry)}$$

and

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0 \qquad \text{(graded Jacobi identity)}$$

for all homogenous elements $x, y, z \in \mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a finite-dimensional Lie superalgebra over $\mathbb{F}$. For $\theta \in \mathbb{Z}_2$, we define

$$\mathfrak{gl}_\theta(\mathfrak{g}) := \{ D \in \mathrm{Hom}(\mathfrak{g},\mathfrak{g}) \mid D(\mathfrak{g}_\mu) \subseteq \mathfrak{g}_{\mu+\theta} \text{ for all } \mu \in \mathbb{Z}_2 \}$$

to be the set of all $\mathbb{F}$-linear maps of *degree* θ on $\mathfrak{g}$. It is known that the finite-dimensional $\mathbb{Z}_2$-graded vector space $\mathfrak{gl}(\mathfrak{g}) := \mathfrak{gl}_{\bar{0}}(\mathfrak{g}) \oplus \mathfrak{gl}_{\bar{1}}(\mathfrak{g})$ is a Lie superalgebra over $\mathbb{F}$ with respect to the following bracket product:

$$[D_\theta, D_\mu] := D_\theta \circ D_\mu - (-1)^{\theta\mu} D_\mu \circ D_\theta, \tag{1.5}$$

where $D_\theta \in \mathfrak{gl}_\theta(\mathfrak{g})$ and $D_\mu \in \mathfrak{gl}_\mu(\mathfrak{g})$.

For $\theta \in \mathbb{Z}_2$, a *homogeneous derivation of degree* θ of $\mathfrak{g}$ is an element $D \in \mathfrak{gl}_\theta(\mathfrak{g})$ such that

$$[D(x),y] + (-1)^{\theta|x|}[x,D(y)] = D([x,y]) \tag{1.6}$$

for all homogenous elements $x, y \in \mathfrak{g}$. We denote by $\mathrm{Der}_\theta(\mathfrak{g})$ the space of homogeneous derivations of degree θ of $\mathfrak{g}$. We see that $\mathrm{Der}(\mathfrak{g}) := \mathrm{Der}_{\bar{0}}(\mathfrak{g}) \oplus \mathrm{Der}_{\bar{1}}(\mathfrak{g})$ is a Lie subsuperalgebra of $\mathfrak{gl}(\mathfrak{g})$, called the *superalgebra of derivations* of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie superalgebra over $\mathbb{F}$ and V be a $\mathbb{Z}_2$-graded vector space over $\mathbb{F}$. We say that V is a *representation* of $\mathfrak{g}$ (or a $\mathfrak{g}$-*module*) if there exists an even linear map $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ such that

$$\rho([x,y]) = \rho(x)\rho(y) - (-1)^{|x||y|}\rho(y)\rho(x)$$

for all homogenous elements $x, y \in \mathfrak{g}$.

Example 1.1. Let $\mathfrak{g}$ be a Lie superalgebra. Then $\mathrm{ad} : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g})$ $(x \mapsto \mathrm{ad}(x))$ defined by $\mathrm{ad}(x)(y) := [x,y]$ make $\mathfrak{g}$ itself to be a $\mathfrak{g}$-module, where $x, y \in \mathfrak{g}$. We call this module the *adjoint module* of $\mathfrak{g}$.

Example 1.2. Let $\mathfrak{g}$ be a Lie superalgebra and V be a $\mathfrak{g}$-module. Then the dual space V^* is also a $\mathfrak{g}$-module, called the *dual module* of V, if the action of $\mathfrak{g}$ on V^* is defined by

$$(x \cdot f)(v) := -(-1)^{|x||f|} f(x \cdot v),$$

where $x \in \mathfrak{g}, f \in V^*$ and $v \in V$ are homogenous.

Moreover, if V is a $\mathfrak{g}$-module, then there exists a Lie superalgebraic structure on the direct sum $\mathfrak{g} \oplus V$ of $\mathbb{Z}_2$-graded vector spaces defined by

$$[(x,u),(y,v)] = ([x,y], \rho(x)v - (-1)^{|u||y|}\rho(y)u)$$

for all homogenous elements $x, y \in \mathfrak{g}$ and $u, v \in V$. In this case, $\mathfrak{g} \oplus V$ is called the *semidirect sum* of $\mathfrak{g}$ and V and denoted by $\mathfrak{g} \ltimes_\rho V$.

Let $m \geq n \geq 1$ be two positive integers and $\mathfrak{gl}(m|n)$ be the space of $(m+n) \times (m+n)$ matrices. If $M \in \mathfrak{gl}(m|n)$, we divide it into four blocks: $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A is an $(m \times m)$-, D is an $(n \times n)$-, B is an $(m \times n)$-, C is an $(n \times m)$-matrix. We say that M has degree 0 (resp. -1, 1) if the nonzero coefficients of M are in A or D (resp. C, B). If M and N have degrees γ and γ', we set $[M,N] = MN - (-1)^{\gamma\gamma'} NM$. This makes $\mathfrak{gl}(m|n)$ into a $\mathbb{Z}$-graded Lie superalgebra with the even part $\mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$. The supertrace of M is $\mathrm{str}(M) = \mathrm{tr}(A) - \mathrm{tr}(D)$.

Let $\mathfrak{h}_{\mathfrak{gl}}$ be the space of diagonal matrices in $\mathfrak{gl}(m|n)$. We set $\varepsilon_i(h) = h_i$ and $\delta_j(h) = h'_j$ for $h = \mathrm{diag}(h_1, \cdots, h_m, h'_1, \cdots, h'_n) \in \mathfrak{h}_{\mathfrak{gl}}$, This defines a basis of $\mathfrak{h}^*_{\mathfrak{gl}}$. We define a bilinear form on $\mathfrak{h}_{\mathfrak{gl}}^*$ by $(\varepsilon_i, \varepsilon_k) = \delta_{ik}, (\delta_j, \delta_l) = -\delta_{jl}$ and $(\varepsilon_i, \delta_j) = 0$ for $i, k = 1, \cdots, m$ and $j, l = 1, \cdots, n$.

We define the *special linear superalgebra* $\mathfrak{sl}(m|n) = \mathrm{Ker}(\mathrm{str})$ which inherits a structure of $\mathbb{Z}$-graded superalgebra from that of $\mathfrak{gl}(m|n)$. Set $\mathfrak{h}_{\mathfrak{sl}} = \mathfrak{h}_{\mathfrak{gl}} \cap \mathfrak{sl}(m|n)$. We identify $\mathfrak{h}_{\mathfrak{sl}}^* \cong \mathfrak{h}_{\mathfrak{gl}}^*/\mathbb{C}\mathrm{str}$, where $\mathrm{str} = \sum_{i=1}^m \varepsilon_i - \sum_{j=1}^n \delta_j$. The set of positive roots is $\Delta^+ = \Delta_{\bar{0}}^+ \cup \Delta_{\bar{1}}^+$, with the set $\Delta_{\bar{0}}^+$ of even positive roots and the set $\Delta_{\bar{1}}^+$ of odd positive roots given respectively by

$$\Delta_{\bar{0}}^+ = \{\varepsilon_i - \varepsilon_k | 1 \le i < k \le m\} \cup \{\delta_j - \delta_l | 1 \le j < l \le n\},$$

$$\Delta_{\bar{1}}^+ = \{\varepsilon_i - \delta_j | 1 \le i \le m, 1 \le j \le n\}.$$

Let ρ_0 (resp. ρ_1) be the half sum of positive even (resp. odd) roots, and let $\rho = \rho_0 - \rho_1$. We have

$$\rho_0 = \frac{1}{2}\sum_{i=1}^m (m - 2i + 1)\varepsilon_i + \frac{1}{2}\sum_{j=1}^n (n - 2j + 1)\delta_j, \tag{1.7}$$

$$\rho_1 = \frac{n}{2}\sum_{i=1}^m \varepsilon_i - \frac{m}{2}\sum_{j=1}^n \delta_j. \tag{1.8}$$

A weight $\Lambda \in \mathfrak{h}_{\mathfrak{sl}}^*$ is called *dominant* if $2(\Lambda, \alpha)/(\alpha, \alpha) \ge 0$ for all $\alpha \in \Delta_{\bar{0}}^+$. We express a given weight

$$\Lambda = \sum_{i=1}^m a_i \varepsilon_i + \sum_{j=1}^n b_j \delta_j,$$

where $\sum_{i=1}^m a_i + \sum_{j=1}^n b_j = 0$. The above weight Λ is dominant if and only if all the numbers $a_i - a_{i+1} (i = 1, \cdots, m-1)$ and $b_j - b_{j+1} (j = 1, \cdots, n-1)$ are nonnegative integers ([28, Section 3.1]).

A dominant weight Λ is called *singular* with respect to $\alpha \in \Delta_{\bar{1}}^+$ if $(\Lambda + \rho, \alpha) = 0$. The degree of atypicality of Λ is the number $\mathrm{atp}\Lambda$ of odd positive roots with respect to which Λ is singular. A dominant weight Λ is called *typical* if $\mathrm{atp}\Lambda = 0$, *atypical* if $\mathrm{atp}\Lambda > 0$ and *singly atypical* if $\mathrm{atp}\Lambda = 1$.

We denote by $\mathfrak{g}$ the special linear superalgebra $\mathfrak{sl}(m|n)$. Let $\Lambda \in \mathfrak{h}_{\mathfrak{sl}}^*$ be a dominant weight and $L(\Lambda)$ be the irreducible $\mathfrak{g}_0$-module with highest weight Λ. Set $\mathfrak{p}_\pm = \mathfrak{g}_0 \oplus \mathfrak{g}_{\pm 1}$. We consider $L(\Lambda)$ as a $\mathfrak{p}_+$-module (resp. a $\mathfrak{p}_-$-module) in which $\mathfrak{g}_{+1}$ (resp. $\mathfrak{g}_{-1}$) acts trivially. We define two $\mathfrak{g}$-modules $K(\Lambda)$ and $\check{K}(\Lambda)$, called, respectively, a *Kac module* and an *opposite Kac module*, by

$$K(\Lambda) = \mathrm{Ind}_{\mathfrak{p}_+}^{\mathfrak{g}} L(\Lambda) \cong U(\mathfrak{g}_{-1}) \otimes_{\mathbb{C}} L(\Lambda), \check{K}(\Lambda) = \mathrm{Ind}_{\mathfrak{p}_-}^{\mathfrak{g}} L(\Lambda) \cong U(\mathfrak{g}_1) \otimes_{\mathbb{C}} L(\Lambda). \tag{1.9}$$

The Kac module $K(\Lambda)$ is an irreducible $\mathfrak{g}$-module if and only if the highest weight Λ is typical ([38, Theorem 1]). In case the Kac module is not irreducible, it contains a maximal submodule $I(\Lambda)$ and the quotient module $V(\Lambda) = K(\Lambda)/I(\Lambda)$ is an irreducible module. The fundamental results concerning the representations of $\mathfrak{g}$ are the following [38, Proposition 2.2]: 1) let Λ be a dominant weight, then any finite dimensional irreducible representation of $\mathfrak{g}$ is of the form $V(\Lambda) = K(\Lambda)/I(\Lambda)$; 2) two $\mathfrak{g}$-modules $V(\Lambda_1)$ and $V(\Lambda_2)$ are equivalent if and only if $\Lambda_1 = \Lambda_2$.

Any irreducible module $V(\Lambda)$ is the unique irreducible quotient of a unique opposite Kac module $\check{K}(\Lambda')$. We denote by $T^-\Lambda$ the highest weight of $\mathrm{soc}K(\Lambda)$ and $T^+\Lambda$ the highest weight of $\mathrm{soc}\check{K}(\Lambda')$. The following proposition is taken from [28, Proposition 6.1.2].

Proposition 1.1. *Let $\Lambda, \Phi \in \mathfrak{h}_{\mathfrak{sl}}^*$ be dominant weights.*

(1) *If Λ is typical, then*

$$\mathrm{Ext}^1(V(\Lambda), V(\Phi)) = \begin{cases} \mathbb{C} & \textit{if } \Lambda = \Phi, \\ 0 & \textit{otherwise}. \end{cases} \tag{1.10}$$

(2) *If Λ is singly atypical, then*

$$\mathrm{Ext}^1(V(\Lambda), V(\Phi)) = \begin{cases} \mathbb{C} & \textit{if } \Phi \in \{T^+\Lambda, T^-\Lambda\}, \\ 0 & \textit{otherwise}. \end{cases} \tag{1.11}$$

The classical result due to Kac [37] states that all finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero that are not Lie algebras consist of the following classical and Cartan Lie superalgebras:

classical : $A(m,n), n > m \geq 0; A(n,n), n \geq 1; B(m,n), m \geq 0, n \geq 1; C(n), n \geq 3;$

$D(m,n), m \geq 2, n \geq 1; D(2,1;\alpha), \alpha \neq 0, -1; F(4); G(3); P(n), n \geq 2;$

$Q(n), n \geq 2;$

Cartan : $W(n), n \geq 3; S(n), n \geq 4; \tilde{S}(n), n \geq 4, n$ even; $H(n), n \geq 5.$

Note that $D(2,1;\alpha)$ and $D(2,1;\beta)$ are isomorphic if and only if α and β lie in the same orbit of the group V of order 6 generated by $\alpha \mapsto -1-\alpha, \alpha \mapsto 1/\alpha$.

Chapter 2
Left-symmetric Superalgebras

2.1 Background

Left-symmetric algebras (or under other names like pre-Lie algebras, Koszul-Vinberg algebras, quasi-associative algebras) are a class of Lie-admissible algebras whose commutators are Lie algebras. They are not associative in general. They appeared in the work of Cayley as a kind of rooted tree algebras in 1896 ([14]). They arose again from the study of convex homogenous cones, affine manifolds and affine structures on Lie groups and the cohomology theory of associative algebras in 1960s ([54, 39, 29]). Moreover, left-symmetric algebras appeared in many fields of mathematics and mathematical physics. They are also closely related to rooted trees ([18]), certain integrable systems ([9, 53]), classical and quantum Yang-Baxter equation ([25, 30, 24, 3]). The recent survey paper [11] discussed the origin and applications of left-symmetric algebras in geometry and physics in detail and the algebra theory of left-symmetric algebras such as structure theory, cohomology theory and the classification of some simple left-symmetric algebras.

Since left-symmetric algebras are Lie-admissible algebras, a fundamental problem is to decide whether a given Lie algebra admits a left-symmetric algebra and to give a classification of such products. This problem is important in geometry. In fact, if G is a connected and simply connected Lie group over the field of real numbers whose Lie algebra is $\mathfrak{g}$, then there is a left-invariant flat and torsion free connection, that is, an affine structure on G if and only if $\mathfrak{g}$ admits a left-symmetric algebra ([39, 46]).