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Contests in Higher Mathematics: Miklós Schweitzer Competitions 1962-1991

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1962~1991年米洛克斯·史怀哲竞赛

[美] 伽伯·舍克里 著

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黑版贸审字 08-2017-047 号

图书在版编目(CIP)数据

高等数学竞赛:1962~1991 年米洛克斯·史怀哲竞赛:英文/(美)伽伯·舍克里著.—哈尔滨:哈尔滨工业大学出版社,2017.8
ISBN 978-7-5603-6841-2

Ⅰ.①高… Ⅱ.①伽… Ⅲ.①高等数学-高等学校-教学参考资料-英文 Ⅳ.①O13

中国版本图书馆 CIP 数据核字(2017)第 192343 号

策划编辑 刘培杰 张永芹
责任编辑 张永芹 聂兆慈
封面设计 孙茵艾
出版发行 哈尔滨工业大学出版社
社　　址 哈尔滨市南岗区复华四道街 10 号　邮编 150006
传　　真 0451-86414749
网　　址 http://hitpress.hit.edu.cn
印　　刷 哈尔滨市工大节能印刷厂
开　　本 787mm×1092mm　1/16　印张 37.5　字数 700 千字
版　　次 2017 年 8 月第 1 版　2017 年 8 月第 1 次印刷
书　　号 ISBN 978-7-5603-6841-2
定　　价 128.00 元

Preface

> *"I had the opportunity to speak with Leo Szilárd about the contests of the Mathematical and Physical Society, and about the fact that the winners of these contests turned out later to be almost identical with the set of mathematicians and physicists who became outstanding ..."*
>
> (J. Neumann, in a letter to L. Fejér, Berlin, Dec. 7, 1929)

The solutions to deep scientific problems rarely come to us easily. Thus, it is important to motivate students to begin efforts on these kinds of problems. Scientific competition has proved to be an effective stimulant toward intellectual efforts. Successful examples include the "Concours" for admission to the "Grandes Écoles" in France, and the "Mathematical Tripos" in Cambridge, England. At the turn of the century, mathematical contests helped Hungary become one of the strongholds of the mathematical world.

With the revolution in 1848 and the Compromise in 1867, Hungary broke free from many centuries of rule by the Turks and then the Hapsburgs, and became a nation on equal footing with her neighbor, Austria. By the end of the 19th century, Hungary entered a period of cultural and economic progress. In 1891, Baron Loránd Eötvös, an outstanding Hungarian physicist, founded the Mathematical and Physical Society. In turn, the Society founded two journals: the *Mathematical and Physical Journal* in 1892 and the *Mathematical Journal for Secondary Schools* in 1893. This latter journal offered a rich variety of elementary problems for high school students. One of the first editors of the journal, László Rátz, later became the teacher of John Neumann and Eugene Wigner (a Nobel prize winner in physics). In 1894, the Society introduced a mathematical competition for high school students. Among the winners there were Lipót Fejér, Alfréd Haar, Tódor Kármán, Marcel Riesz, Gábor Szegő, Tibor Radó, Ede Teller, and many others who became world-famous scientists.

The success of high school competitions led the Mathematical Society to found a college-level contest. The first contest of this kind was organized in 1949 and named after Miklós Schweitzer, a young mathematician who died in the Second World War. Schweitzer placed second in the High School Contest in 1941, but the statutes of the fascist regime of that time prevented his admission to college. Schweitzer Contest problems are proposed and selected by the most prominent Hungarian mathematicians. Thus, Schweitzer problems reflect the interest of these mathematicians and some aspects of the mainstream of Hungarian mathematics. The universities of Budapest, Debrecen, and Szeged have alternately been designated by the Society Presidium to conduct the Schweitzer Contests. The jury is chosen by the mathematics departments of the universities in question from among the mathematicians working in the host city. The jury sends out requests to leading Hungarian mathematicians to submit problems suitable for the contest. The list of problems selected by the jury is posted on the bulletin boards of mathematics departments and of local branches of the Mathematical Society (copies are available to anyone interested). Students may use any materials available in libraries or in their homes to solve the contest problems. In ten days the solutions are due, with the student's name, faculty, course, year, and university or high school recorded on the solution set.

The Schweitzer competition is one of the most unique in the world. Winners of the contests have gone on to become world-class scientists. Thus, the Schweitzer Contests are of interest to both math historians and mathematicians of all ages. They serve as reflections of Hungarian mathematical trends and as starting points for many interesting research problems in mathematics. The Schweitzer problems between 1949 and 1961 were previously published under the title *Contests in Higher Mathematics, 1949–1961* (Akadémiai Kiadó, Budapest, 1968; Chapter 4 of this book summarizes the mathematical work of M. Schweitzer). Our book is a continuation of that volume.

We hope that this collection of Schweitzer problems will serve as a guide for many young mathematicians and math majors. The large variety of research-level problems may spark the interest of seasoned mathematicians and historians of mathematics.

I wish to close by acknowledging the outstanding work of Dr. Marianna Bolla as Managing Editor. In addition, without the constant assistance of Dr. Dezső Miklós as Technical Editor, we could not have this book.

Bowling Green, OH *Gábor J. Székely*
August 26, 1995

Contents

1. Problems of the Contests

The letter in parentheses after the text of a problem refers to the section in Chapter 3 containing its solution. The topics include these areas of mathematics:

A: Algebra
C: Combinatorics
F: Theory of Functions
G: Geometry
M: Measure Theory
N: Number Theory
O: Operators
P: Probability Theory
S: Sequences and Series
T: Topology
ℵ: Set Theory

Thus, for example, **P.3** refers to problem in "Probability Theory" section.

When available, the names of proposers are in brackets at the very end of each problem.

1962

1. Let f and g be polynomials with rational coefficients, and let F and G denote the sets of values of f and g at rational numbers. Prove that $F = G$ holds if and only if $f(x) = g(ax + b)$ for some suitable rational numbers $a \neq 0$ and b. **(N.1)** [E. Fried]

2. Determine the roots of unity in the field of p-adic numbers. **(A.1)** [L. Fuchs]

3. Let A and B be two Abelian groups, and define the sum of two homomorphisms η and χ from A to B by

$$a(\eta + \chi) = a\eta + a\chi \quad \text{for all } a \in A.$$

With this addition, the set of homomorphisms from A to B forms an Abelian group H. Suppose now that A is a p-group (p a prime number). Prove that in this case H becomes a topological group under the topology defined by taking the subgroups $p^k H$ ($k = 1, 2, \dots$) as a neighborhood base of 0. Prove that H is complete in this topology and that every connected component of H consists of a single element. When is H compact in this topology? **(A.2)** [L. Fuchs]

4. Show that

$$\prod_{1 \leq x < y \leq \frac{p-1}{2}} (x^2 + y^2) \equiv (-1)^{\left[\frac{p+1}{8}\right]} \pmod p$$

for every prime $p \equiv 3 \pmod 4$. ([.] is integer part.) **(N.2)** [J. Surányi]

5. Let f be a finite real function of one variable. Let $\overline{D}f$ and $\underline{D}f$ be its upper and lower derivatives, respectively, that is,

$$\overline{D}f(x) = \limsup_{\substack{h,k \to 0 \\ h,k \geq 0 \\ h+k > 0}} \frac{f(x+h) - f(x-k)}{h+k}, \quad \underline{D}f(x) = \liminf_{\substack{h,k \to 0 \\ h,k \geq 0 \\ h+k > 0}} \frac{f(x+h) - f(x-k)}{h+k}.$$

Show that $\overline{D}f$ and $\underline{D}f$ are Borel-measurable functions. **(M.1)** [Á. Császár]

6. Let E be a bounded subset of the real line, and let Ω be a system of (nondegenerate) closed intervals such that for each $x \in E$ there exists an $I \in \Omega$ with left endpoint x. Show that for every $\varepsilon > 0$ there exist a finite number of pairwise nonoverlapping intervals belonging to Ω that cover E with the exception of a subset of outer measure less than ε. **(M.2)** [J. Czipszer]

7. Prove that the function

$$f(\vartheta) = \int_1^{\frac{1}{\vartheta}} \frac{dx}{\sqrt{(x^2 - 1)(1 - \vartheta^2 x^2)}}$$

(where the positive value of the square root is taken) is monotonically decreasing in the interval $0 < \vartheta < 1$. **(F.1)** [P. Turán]

8. Denote by $M(r, f)$ the maximum modulus on the circle $|z| = r$ of the transcendent entire function $f(z)$, and by $M_n(r, f)$ that of the nth partial sum of the power series of $f(z)$. Prove the existence of an entire function $f_0(z)$ and a corresponding sequence of positive numbers $r_1 < r_2 < \cdots \to +\infty$ such that

$$\limsup_{n\to\infty} \frac{M_n(r_n, f_0)}{M(r_n, f_0)} = +\infty.$$

(**F.2**) [P. Turán]

9. Find the minimum possible sum of lengths of edges of a prism all of whose edges are tangent to a unit sphere. (**G.1**) [Müller–Pfeiffer]

10. From a given triangle of unit area, we choose two points independently with uniform distribution. The straight line connecting these points divides the triangle, with probability one, into a triangle and a quadrilateral. Calculate the expected values of the areas of these two regions. (**P.1**) [A. Rényi]

1963

1. Show that the perimeter of an arbitrary planar section of a tetrahedron is less than the perimeter of one of the faces of the tetrahedron. (**G.2**) [Gy. Hajós]

2. Show that the center of gravity of a convex region in the plane halves at least three chords of the region. (**G.3**) [Gy. Hajós]

3. Let $R = R_1 \oplus R_2$ be the direct sum of the rings R_1 and R_2, and let N_2 be the annihilator ideal of R_2 (in R_2). Prove that R_1 will be an ideal in every ring $\tilde{R}$ containing R as an ideal if and only if the only homomorphism from R_1 to N_2 is the zero homomorphism. (**A.3**) [Gy. Pollák]

4. Call a polynomial positive reducible if it can be written as a product of two nonconstant polynomials with positive real coefficients. Let $f(x)$ be a polynomial with $f(0) \neq 0$ such that $f(x^n)$ is positive reducible for some natural number n. Prove that $f(x)$ itself is positive reducible. (**A.4**) [L. Rédei]

5. Let H be a set of real numbers that does not consist of 0 alone and is closed under addition. Further, let $f(x)$ be a real-valued function defined on H and satisfying the following conditions:

$$f(x) \le f(y) \text{ if } x \le y \text{ and } f(x+y) = f(x) + f(y) \quad (x, y \in H).$$

Prove that $f(x) = cx$ on H, where c is a nonnegative number. (**F.3**) [M. Hosszú, R. Borges]

6. Show that if $f(x)$ is a real-valued, continuous function on the half-line $0 \le x < \infty$, and

$$\int_0^\infty f^2(x)dx < \infty,$$

then the function

$$g(x) = f(x) - 2e^{-x} \int_0^x e^t f(t)dt$$

satisfies

$$\int_0^\infty g^2(x)dx = \int_0^\infty f^2(x)dx.$$

(F.4) [B. Szőkefalvi–Nagy]

7. Prove that for every convex function $f(x)$ defined on the interval $-1 \le x \le 1$ and having absolute value at most 1, there is a linear function $h(x)$ such that

$$\int_{-1}^{1} |f(x) - h(x)|dx \le 4 - \sqrt{8}.$$

(F.5) [L. Fejes–Tóth]

8. Let the Fourier series

$$\frac{a_0}{2} + \sum_{k\ge 1} (a_k \cos kx + b_k \sin kx)$$

of a function $f(x)$ be absolutely convergent, and let

$$a_k^2 + b_k^2 \ge a_{k+1}^2 + b_{k+1}^2 \quad (k = 1, 2, \dots).$$

Show that

$$\frac{1}{h}\int_0^{2\pi} (f(x+h) - f(x-h))^2\, dx \qquad (h > 0)$$

is uniformly bounded in h. **(S.1)** [K. Tandori]

9. Let $f(t)$ be a continuous function on the interval $0 \le t \le 1$, and define the two sets of points

$$A_t = \{(t,0) : t \in [0,1]\}, \quad B_t = \{(f(t),1) : t \in [0,1]\}.$$

Show that the union of all segments $\overline{A_t B_t}$ is Lebesgue-measurable, and find the minimum of its measure with respect to all functions f. **(M.3)** [Á. Császár]

10. Select n points on a circle independently with uniform distribution. Let P_n be the probability that the center of the circle is in the interior of the convex hull of these n points. Calculate the probabilities P_3 and P_4. **(P.2)** [A. Rényi]

1964

1. Among all possible representations of the positive integer n as $n = \sum_{i=1}^k a_i$ with positive integers k, $a_1 < a_2 < \cdots < a_k$, when will the product $\prod_{i=1}^k a_i$ be maximum? **(C.1)**

2. Let p be a prime and let

$$l_k(x,y) = a_k x + b_k y \quad (k = 1, \dots, p^2),$$

be homogeneous linear polynomials with integral coefficients. Suppose that for every pair (ξ, η) of integers, not both divisible by p, the values $l_k(\xi, \eta)$, $1 \le k \le p^2$, represent every residue class mod p exactly p times. Prove that the set of pairs $\{(a_k, b_k) : 1 \le k \le p^2\}$ is identical mod p with the set $\{(m, n) : 0 \le m, n \le p-1\}$. **(N.3)**

3. Prove that the intersection of all maximal left ideals of a ring is a (two-sided) ideal. **(A.5)**

4. Let $A_1, A_2, \ldots, A_n$ be the vertices of a closed convex n-gon K numbered consecutively. Show that at least $n-3$ vertices A_i have the property that the reflection of A_i with respect to the midpoint of $\overline{A_{i-1}A_{i+1}}$ is contained in K. (Indices are meant mod n.) **(G.4)**

5. Is it true that on any surface homeomorphic to an open disc there exist two congruent curves homeomorphic to a circle? **(G.5)**

6. Let $y_1(x)$ be an arbitrary, continuous, positive function on $[0, A]$, where A is an arbitrary positive number. Let

$$y_{n+1}(x) = 2\int_0^x \sqrt{y_n(t)}\,dt \quad (n = 1, 2, \ldots).$$

Prove that the functions $y_n(x)$ converge to the function $y = x^2$ uniformly on $[0, A]$. **(S.2)**

7. Find all linear homogeneous differential equations with continuous coefficients (on the whole real line) such that for any solution $f(t)$ and any real number c, $f(t+c)$ is also a solution. **(F.6)**

8. Let F be a closed set in the n-dimensional Euclidean space. Construct a function that is 0 on F, positive outside F, and whose partial derivatives all exist. **(F.7)**

9. Let E be the set of all real functions on $I = [0, 1]$. Prove that one cannot define a topology on E in which $f_n \to f$ holds if and only if f_n converges to f almost everywhere. **(S.3)**

10. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{2n}$ be independent random variables such that $P(\varepsilon_i = 1) = P(\varepsilon_i = -1) = 1/2$ for all i, and define $S_k = \sum_{i=1}^k \varepsilon_i$, $1 \le k \le 2n$. Let N_{2n} denote the number of integers $k \in [2, 2n]$ such that either $S_k > 0$, or $S_k = 0$ and $S_{k-1} > 0$. Compute the variance of N_{2n}. **(P.3)**

1965

1. Let p be a prime, n a natural number, and S a set of cardinality p^n. Let $\mathbf{P}$ be a family of partitions of S into nonempty parts of sizes divisible by p such that the intersection of any two parts that occur in any of the partitions has at most one element. How large can $|\mathbf{P}|$ be? **(N.4)**

2. Let R be a finite commutative ring. Prove that R has a multiplicative identity element (1) if and only if the annihilator of R is 0 (that is, $aR = 0$, $a \in R$ imply $a = 0$). **(A.6)**

3. Let $a, b_0, b_1, \ldots, b_{n-1}$ be complex numbers, A a complex square matrix of order p, and E the unit matrix of order p. Assuming that the eigenvalues of A are given, determine the eigenvalues of the matrix

$$B = \begin{pmatrix} b_0E & b_1A & b_2A^2 & \cdots & b_{n-1}A^{n-1} \\ ab_{n-1}A^{n-1} & b_0E & b_1A & \cdots & b_{n-2}A^{n-2} \\ ab_{n-2}A^{n-2} & ab_{n-1}A^{n-1} & b_0E & \cdots & b_{n-3}A^{n-3} \\ & & \ddots & & \\ ab_1A & ab_2A^2 & ab_3A^3 & \cdots & b_0E \end{pmatrix}.$$

(O.1)

4. The plane is divided into domains by n straight lines in general position, where $n \geq 3$. Determine the maximum and minimum possible number of angular domains among them. (We say that n lines are in general position if no two are parallel and no three are concurrent.) **(G.6)**

5. Let $A = A_1A_2A_3A_4$ be a tetrahedron, and suppose that for each $j \neq k$, $[A_j, A_{jk}]$ is a segment of length ρ extending from A_j in the direction of A_k. Let p_j be the intersection line of the planes $[A_{jk}A_{jl}A_{jm}]$ and $[A_kA_lA_m]$. Show that there are infinitely many straight lines that intersect the straight lines p_1, p_2, p_3, p_4 simultaneously. **(G.7)**

6. Consider the radii of normal curvature of a surface at one of its points P_0 in two conjugate directions (with respect to the Dupin indicatrix). Show that their sum does not depend on the choice of the conjugate directions. (We exclude the choice of asymptotic directions in the case of a hyperbolic point.) **(G.8)**

7. Prove that any uncountable subset of the Euclidean n-space contains an uncountable subset with the property that the distances between different pairs of points are different (that is, for any points $P_1 \neq P_2$ and $Q_1 \neq Q_2$ of this subset, $\overline{P_1P_2} = \overline{Q_1Q_2}$ implies either $P_1 = Q_1$ and $P_2 = Q_2$, or $P_1 = Q_2$ and $P_2 = Q_1$). Show that a similar statement is not valid if the Euclidean n-space is replaced with a (separable) Hilbert space. **(T.1)**

8. Let the continuous functions $f_n(x)$, $n = 1, 2, 3, \ldots$, be defined on the interval $[a, b]$ such that every point of $[a, b]$ is a root of $f_n(x) = f_m(x)$ for some $n \neq m$. Prove that there exists a subinterval of $[a, b]$ on which two of the functions are equal. **(S.4)**

9. Let f be a continuous, nonconstant, real function, and assume the existence of an F such that $f(x + y) = F[f(x), f(y)]$ for all real x and y. Prove that f is strictly monotone. **(F.8)**

10. A gambler plays the following coin-tossing game. He can bet an arbitrary positive amount of money. Then a fair coin is tossed, and the gambler wins or loses the amount he bet depending on the outcome. Our gambler, who starts playing with x forints, where $0 < x < 2C$, uses the following strategy: if at a given time his capital is $y < C$, he risks all of it; and if he has $y > C$, he only bets $2C - y$. If he has exactly $2C$ forints, he stops playing. Let $f(x)$ be the probability that he reaches $2C$ (before going bankrupt). Determine the value of $f(x)$. (**P.4**)

1966

1. Show that a segment of length h can go through or be tangent to at most $2[h/\sqrt{2}] + 2$ nonoverlapping unit spheres. ([.] is integer part.) (**G.9**) [L. Fejes–Tóth, A. Heppes]
2. Characterize those configurations of n coplanar straight lines for which the sum of angles between all pairs of lines is maximum. (**G.10**) [L. Fejes–Tóth, A. Heppes]
3. Let $f(n)$ denote the maximum possible number of right triangles determined by n coplanar points. Show that

$$\lim_{n\to\infty} \frac{f(n)}{n^2} = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{f(n)}{n^3} = 0.$$

(**G.11**) [P. Erdős]

4. Let I be an ideal of the ring of all polynomials with integer coefficients such that
   (a) the elements of I do not have a common divisor of degree greater than 0, and
   (b) I contains a polynomial with constant term 1.
   Prove that I contains the polynomial $1 + x + x^2 + \cdots + x^{r-1}$ for some natural number r. (**A.7**) [Gy. Szekeres]
5. A "letter T" erected at point A of the x-axis in the xy-plane is the union of a segment AB in the upper half-plane perpendicular to the x-axis and a segment CD containing B in its interior and parallel to the x-axis. Show that it is impossible to erect a letter T at every point of the x-axis so that the union of those erected at rational points is disjoint from the union of those erected at irrational points. (**M.4**) [Á. Császár]
6. A sentence of the following type is often heard in Hungarian weather reports: "Last night's minimum temperatures took all values between -3 degrees and $+5$ degrees." Show that it would suffice to say, "Both -3 degrees and $+5$ degrees occurred among last night's minimum temperatures." (Assume that temperature as a two-variable function of place and time is continuous.) (**T.2**) [Á. Császár]
7. Does there exist a function $f(x, y)$ of two real variables that takes natural numbers as its values and for which $f(x, y) = f(y, z)$ implies $x = y = z$? (**ℵ.1**) [A. Hajnal]
8. Prove that in a Euclidean ring R the quotient and remainder are always uniquely determined if and only if R is a polynomial ring over some field and the value of the norm is a strictly monotone function of the degree of the polynomial. (To be precise, there are two more trivial cases: R can also be a field or the null ring.) (**A.8**) [E. Fried]
9. If $\sum_{m=-\infty}^{+\infty} |a_m| < \infty$, then what can be said about the following expression?

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{m=-\infty}^{+\infty} |a_{m-n} + a_{m-n+1} + \cdots + a_{m+n}|$$

(**S.5**) [P. Turán]

10. For a real number x in the interval $(0,1)$ with decimal representation

$$0.a_1(x)a_2(x)\dots a_n(x)\dots,$$

denote by $n(x)$ the smallest nonnegative integer such that

$$\overline{a_{n(x)+1}a_{n(x)+2}a_{n(x)+3}a_{n(x)+4}} = 1966.$$

Determine $\int_0^1 n(x)dx$. ($\overline{abcd}$ denotes the decimal number with digits a, b, c, d.) (**P.5**) [A. Rényi]

1967

1. Let

$$f(x) = a_0 + a_1x + a_2x^2 + a_{10}x^{10} + a_{11}x^{11} + a_{12}x^{12} + a_{13}x^{13} \qquad (a_{13} \neq 0)$$

and

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_{11}x^{11} + b_{12}x^{12} + b_{13}x^{13} \qquad (b_3 \neq 0)$$

be polynomials over the same field. Prove that the degree of their greatest common divisor is at most 6. (**A.9**) [L. Rédei]

2. Let K be a subset of a group G that is not a union of left cosets of a proper subgroup. Prove that if G is a torsion group or if K is a finite set, then the subset

$$\bigcap_{k \in K} k^{-1}K$$

consists of the identity alone. (**A.10**) [L. Rédei]

3. Prove that if an infinite, noncommutative group G contains a proper normal subgroup with a commutative factor group, then G also contains an infinite proper normal subgroup. (**A.11**) [B. Csákány]

4. Let $a_1, a_2, \dots, a_N$ be positive real numbers whose sum equals 1. For a natural number i, let n_i denote the number of a_k for which $2^{1-i} \geq a_k > 2^{-i}$ holds. Prove that

$$\sum_{i=1}^{\infty} \sqrt{n_i 2^{-i}} \leq 4 + \sqrt{\log_2 N}.$$

(**A.12**) [L. Leindler]

5. Let f be a continuous function on the unit interval $[0,1]$. Show that

$$\lim_{n\to\infty} \int_0^1 \dots \int_0^1 f\left(\frac{x_1 + \dots + x_n}{n}\right) dx_1 \dots dx_n = f\left(\frac{1}{2}\right)$$

and

$$\lim_{n\to\infty} \int_0^1 \dots \int_0^1 f(\sqrt[n]{x_1 \dots x_n}) dx_1 \dots dx_n = f\left(\frac{1}{e}\right).$$

(**P.6**)

6. Let A be a family of proper closed subspaces of the Hilbert space $H = l^2$ totally ordered with respect to inclusion (that is, if $L_1, L_2 \in A$, then either $L_1 \subset L_2$ or $L_2 \subset L_1$). Prove that there exists a vector $x \in H$ not contained in any of the subspaces L belonging to A. (**T.3**) [B. Szőkefalvi–Nagy]
7. Let U be an $n \times n$ orthogonal matrix. Prove that for any $n \times n$ matrix A, the matrices

$$A_m = \frac{1}{m+1} \sum_{j=0}^{m} U^{-j} A U^j$$

converge entrywise as $m \to \infty$. (**O.2**) [I. Kovács]
8. Suppose that a bounded subset S of the plane is a union of congruent, homothetic, closed triangles. Show that the boundary of S can be covered by a finite number of rectifiable arcs. (**G.12**) [L. Gehér]
9. Let F be a surface of nonzero curvature that can be represented around one of its points P by a power series and is symmetric around the normal planes parallel to the principal directions at P. Show that the derivative with respect to the arc length of the curvature of an arbitrary normal section at P vanishes at P. Is it possible to replace the above symmetry condition by a weaker one? (**G.13**) [A. Moór]
10. Let $\sigma(S_n, k)$ denote the sum of the kth powers of the lengths of the sides of the convex n-gon S_n inscribed in a unit circle. Show that for any natural number greater than 2 there exists a real number k_0 between 1 and 2 such that $\sigma(S_n, k_0)$ attains its maximum for the regular n-gon. (**G.14**) [L. Fejes–Tóth]

1968

1. Consider the endomorphism ring of an Abelian torsion-free (resp. torsion) group G. Prove that this ring is Neumann-regular if and only if G is a discrete direct sum of groups isomorphic to the additive group of the rationals (resp., a discrete direct sum of cyclic groups of prime order). (A ring R is called *Neumann-regular* if for every $\alpha \in R$ there exists a $\beta \in R$ such that $\alpha\beta\alpha = \alpha$.) (**A.13**) [E. Fried]
2. Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers. Prove that

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} a_i^{n-1}\right) \le n \prod_{i=1}^{n} a_i + (n-1) \sum_{i=1}^{n} a_i^n .$$

(**S.6**) [J. Surányi]
3. Let K be a compact topological group, and let F be a set of continuous functions defined on K that has cardinality greater than continuum. Prove that there exist $x_0 \in K$ and $f \neq g \in F$ such that

$$f(x_0) = g(x_0) = \max_{x \in K} f(x) = \max_{x \in K} g(x).$$

(**T.4**) [I. Juhász]

4. Let f be a complex-valued, completely multiplicative, arithmetical function. Assume that there exists an infinite increasing sequence N_k of natural numbers such that

$$f(n) = A_k \neq 0 \quad \text{provided} \quad N_k \leq n \leq N_k + 4\sqrt{N_k}.$$

Prove that f is identically 1. **(N.5)** [I. Kátai]

5. Let k be a positive integer, z a complex number, and $\varepsilon < 1/2$ a positive number. Prove that the following inequality holds for infinitely many positive integers n:

$$\left| \sum_{0 \leq \ell \leq \frac{n}{k+1}} \binom{n - k\ell}{\ell} z^\ell \right| \geq (\tfrac{1}{2} - \varepsilon)^n.$$

(F.9) [P. Turán]

6. Let $\mathfrak{A} = \langle A; \dots \rangle$ be an arbitrary, countable algebraic structure (that is, $\mathfrak{A}$ can have an arbitrary number of finitary operations and relations). Prove that $\mathfrak{A}$ has as many as continuum automorphisms if and only if for any finite subset A' of A there is an automorphism $\pi_{A'}$ of $\mathfrak{A}$ different from the identity automorphism and such that

$$(x)\pi_{A'} = x$$

for every $x \in A'$. **(A.14)** [M. Makkai]

7. For every natural number r, the set of r-tuples of natural numbers is partitioned into finitely many classes. Show that if $f(r)$ is a function such that $f(r) \geq 1$ and $\lim_{r\to\infty} f(r) = +\infty$, then there exists an infinite set of natural numbers that, for all r, contains r-tuples from at most $f(r)$ classes. Show that if $f(r) \not\to +\infty$, then there is a family of partitions such that no such infinite set exists. **(C.2)** [P. Erdős, A. Hajnal]

8. Let n and k be given natural numbers, and let A be a set such that

$$|A| \leq \frac{n(n+1)}{k+1}.$$

For $i = 1, 2, \dots, n+1$, let A_i be sets of size n such that

$$|A_i \cap A_j| \leq k \qquad (i \neq j),$$

$$A = \bigcup_{i=1}^{n+1} A_i.$$

Determine the cardinality of A. **(C.3)** [K. Corrádi]

9. Let $f(x)$ be a real function such that

$$\lim_{x\to+\infty} \frac{f(x)}{e^x} = 1$$

and $|f''(x)| < c|f'(x)|$ for all sufficiently large x. Prove that

$$\lim_{x\to+\infty} \frac{f'(x)}{e^x} = 1.$$

(F.10) [P. Erdős]

10. Let h be a triangle of perimeter 1, and let H be a triangle of perimeter λ homothetic to h. Let $h_1, h_2, \ldots$ be translates of h such that, for all i, h_i is different from h_{i+2} and touches H and h_{i+1} (that is, intersects without overlapping). For which values of λ can these triangles be chosen so that the sequence $h_1, h_2, \ldots$ is periodic? If $\lambda \geq 1$ is such a value, then determine the number of different triangles in a periodic chain $h_1, h_2, \ldots$ and also the number of times such a chain goes around the triangle H. **(G.15)** [L. Fejes–Tóth]

11. Let $A_1, \ldots, A_n$ be arbitrary events in a probability field. Denote by C_k the event that at least k of $A_1, \ldots, A_n$ occur. Prove that

$$\prod_{k=1}^{n} P(C_k) \leq \prod_{k=1}^{n} P(A_k).$$

(P.7) [A. Rényi]

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1. Let G be an infinite group generated by nilpotent normal subgroups. Prove that every maximal Abelian normal subgroup of G is infinite. (We call an Abelian normal subgroup maximal if it is not contained in another Abelian normal subgroup.) **(A.15)** [J. Erdős]

2. Let $p \geq 7$ be a prime number, ζ a primitive pth root of unity, c a rational number. Prove that in the additive group generated by the numbers $1, \zeta, \zeta^2, \zeta^3 + \zeta^{-3}$ there are only finitely many elements whose norm is equal to c. (The norm is in the pth cyclotomic field.) **(A.16)** [K. Győry]

3. Let $f(x) \geq 0$ be a nonzero, bounded, real function on an Abelian group G, $g_1, \ldots, g_k$ are given elements of G and $\lambda_1, \ldots, \lambda_k$ are real numbers. Prove that if

$$\sum_{i=1}^{k} \lambda_i f(g_i x) \geq 0$$

holds for all $x \in G$, then

$$\sum_{i=1}^{k} \lambda_i \geq 0.$$

(S.7) [A. Máté]

4. Show that the following inequality holds for all $k \geq 1$, real numbers $a_1, a_2, \ldots, a_k$, and positive numbers $x_1, x_2, \ldots, x_k$.

$$\ln \frac{\sum_{i=1}^{k} x_i}{\sum_{i=1}^{k} x_i^{1-a_i}} \leq \frac{\sum_{i=1}^{k} a_i x_i \ln x_i}{\sum_{i=1}^{k} x_i}$$