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Preface

Fractional calculus of the origins can be traced back to the end of the seventeenth century, the time when Newton and Leibniz developed the foundations of differential and integral calculus. Nowadays, fractional differential equations have been proved to be an excellent tool in the modelling of many phenomena in various fields of engineering, physics, and economics. Many practical systems can be represented more accurately through fractional derivative formulation. However, time delay is a common phenomenon in the objective world and engineering fields. In order to describe the system more accurately in many practical systems, we need to take the influence of fractional-order derivative and delay into consideration altogether. Therefore, it has a practical significance to study the solution and its characteristics of fractional delayed differential systems.

In this book, existence and expression of solution, stability and control problem of fractional-order delayed differential systems are discussed. This book, composed of six chapters, mainly studies existence conditions of solution of fractional order functional differential equations, and general solution of fractional order linear differential difference equations, and stability and control problems of fractional-order delayed differential systems.

Firstly, the background and significance of the problems are given. We review the development history of fractional calculus. The main work done is also introduced in this book.

Secondly, we recall the existence and uniqueness of solutions to the Cauchy type problems for ordinary differential equations of fractional order on a finite interval of the real axis in spaces of continuous functions. Nonlinear and linear fractional differential equations in one-dimensional and vectorial cases are considered. The corresponding results of the Cauchy problem for ordinary differential equations are presented.

Thirdly, the existence results of solution of fractional order functional differential equation are derived based on Banach fixed point theorem, Schauder fixed point theorem and successive approximations technique, respectively. These results extend the corresponding ones of ordinary differential equations and functional differential equations of integer order.

Next, we investigate the general solution of fractional-order linear fractional neutral type differential-difference equations. The exponential estimates of the solutions and the expressions of general solution for linear fractional neutral differential difference equations are derived by using the Gronwall integral inequality and the Laplace transform method, respectively. At the same time, we also establish the variation of constant formulae for linear time varying Caputo fractional delay differential system.

Further, we apply the algebraic approach to discuss the asymptotic stability of fractional-order linear singular delay differential systems. The sufficient conditions are presented to ensure the asymptotic stability for any delay parameter. By applying the stability criteria, one can avoid solving the roots of transcendental equations. The results obtained are computationally flexible and efficient.

Finally, we discuss the controllability of linear fractional differential systems with delay in state and impulses. The expression of state response for such systems is derived, and the sufficient and necessary conditions of controllability criteria are established. Both the proposed criteria and illustrative examples show that the controllability property of the linear systems is independent on the order of fractional derivative, or on delay or impulses.

The purpose of this book focuses on the existence and expression of solution, stability and control problems of fractional-order delayed differential systems.

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Hai Zhang
October 2017

Contents

Chapter 1 Introduction

This chapter reviews the background of fractional calculus, and introduces the main work.

§1.1 Fractional Calculus

The subject of fractional calculus (that is, calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and other problems involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables.

The concept of fractional calculus is believed to have stemmed from a question asked in the year 1695 by Marquis de L'Hôpital (1661-1704) to Gottfried Wilhelm Leibniz (1646-1716) for an answer, which sought the meaning of Leibniz's (currently popular) notation $\frac{d^n y}{dx^n}$ for the derivative of order $n \in \mathbb{N} = \{0, 1, 2, \cdots\}$ when $n = \frac{1}{2}$ (What if $n = 1$?). In his reply, dated 30 September 1695, Leibniz wrote to L'Hôpital as follows: "... *This is an apparent paradox from which, one day, useful consequences will be drawn* ..."

Subsequent mention of fractional derivatives was made, in some context or the other, by (for example) Euler in 1730, Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Liouville in 1832, Riemann in 1847, Greer in 1859, Holmgren in 1865, Griinwald in 1867, Letnikov in 1868, Sonin in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, and Weyl in 1917. In fact, in his 700-page textbook, entitled "Traite du Calcul Differentiel et du Calcul Integral" (Second edition;

Courcier, Paris, 1819), S. F. Lacroix used two pages (pp. 409-410) on fractional calculus, showing eventually that

$$\frac{\mathrm{d}^{\frac{1}{2}}}{\mathrm{d}v^{\frac{1}{2}}}v = \frac{2\sqrt{v}}{\sqrt{\pi}}$$

In addition, of course, to the theories of differential, integral, and integro-differential equations, and special functions of mathematical physics as well as their extensions and generalizations in one and more variables, some of the areas of present-day applications of fractional calculus include Fluid Flow, Rheology, Dynamical Processes in Self-Similar and Porous Structures, Diffusive Transport Akin to Diffusion, Electrical Networks, Probability and Statistics, Control Theory of Dynamical Systems, Viscoelasticity, Electrochemistry of Corrosion, Chemical Physics, Optics and Signal Processing, and so on.

The first work, devoted exclusively to the subject of fractional calculus, is the book by Oldham and Spanier [1] published in 1974. One of the most recent works on the subject of fractional calculus is the book of Podlubny [2] published in 1999, which deals principally with fractional differential equations. Some of the latest (but certainly not the last) works especially on fractional models of anomalous kinetics of complex processes are the volumes edited by Carpinteri and Mainardi [3] in 1997 and by Hilfer [4] in 2000, and the book by Zaslavsky [5] published in 2005. Indeed, except those, numerous works (books, edited volumes, and conference proceedings) have also been published. These publications include (for example) the remarkably comprehensive encyclopedic-type monograph by Samko, Kilbas and Marichev [6], which was published in Russian in 1987 and in English in 1993, and the book devoted substantially to fractional differential equations by Miller and Ross [9], which was published in 1993. For more details on fractional calculus theory, one can see the monographs of Podlubny [2], Magin[7], Kilbas et al. [8], Miller and Ross [9] and Diethelm [10]. And today there exist at least two international journals which are devoted almost entirely to the subject of fractional calculus: (i)

Journal of Fractional Calculus and (ii) *Fractional Calculus and Applied Analysis*.

Next, the researchers recall the definitions of Riemann-Liouville fractional integrals and fractional derivatives, Caputo fractional derivatives. More detailed information can be found in the books by Podlubny [2], Kilbas et al. [8], Miller and Ross [9].

Definition 1.1 The Riemann-Liouville's fractional integral of order $\alpha > 0$ for a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ is defined as

$$D^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s, \quad t > 0, \tag{1.1}$$

where $\Gamma(\cdot)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} \mathrm{e}^{-t} \mathrm{d}t.$$

Definition 1.2 The Riemann-Liouville's fractional derivative of order α for a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ is defined as

$$D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{\mathrm{d}^m}{\mathrm{d}t^m} \int_0^t (t-s)^{m-\alpha-1} f(s) \mathrm{d}s, \quad t > 0, \tag{1.2}$$

where $0 \leqslant m-1 \leqslant \alpha < m, m \in \mathbb{Z}^+$.

Definition 1.3 The Caputo's fractional derivative of order α for a function $f : \mathbb{R}^+ \longrightarrow \mathbb{R}^n$ is defined as

$${}^{C}D^{\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s) \mathrm{d}s, \quad t > 0, \tag{1.3}$$

where $0 \leqslant m-1 \leqslant \alpha < m, m \in \mathbb{Z}^+$.

Definition 1.4 The Mittag-Leffler function in two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha > 0, \beta > 0$ and $z \in \mathbb{C}$, $\mathbb{C}$ denotes the complex plane. In particular, for $\beta = 1$, the Mittag-Leffler function in one parameter is defined as

$$E_\alpha(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1 + k\alpha)}, \quad \alpha > 0,\ z \in \mathbb{C}.$$

Definition 1.5 The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^{+\infty} \mathrm{e}^{-st} f(t)\mathrm{d}t, \quad s \in \mathbb{C},$$

where $f(t)$ is n-dimensional vector-valued function. For $m - 1 \leqslant \alpha < m$, it follows from [2, 7, 8] that

$$L[{}^C D^\alpha f(t)] = s^\alpha L[f(t)] - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0).$$

Fractional differential equations without delay involving the R-L fractional derivative or the Caputo fractional derivative have been paid more and more attention in [8-24] and references therein. Kilbas et al. [8] have discussed the explicit solutions of linear fractional ordinary differential equations based on the method of successive approximations. In [11-14], the theory of inequalities, local existence, extremal solutions, comparison results and global existence of the solutions of fractional differential equations have been established. Li et al. [16] have developed an operator theory to study fractional Cauchy problems with the Riemann-Liouville fractional derivatives in infinite-dimensional Banach spaces. Bonilla et al. [17] have considered the explicit solution for linear fractional ordinary differential equations employing the exponential matrix function and the fractional Green function. Odibat [18] has derived the exact solution for the initial value problems of linear fractional ordinary differential systems by analytical approaches.

Delayed differential equations (DDEs) are important in many areas of engineering and science [25]. The time delay is one of the inevitable problems in practical engineering applications, which has an important effect on the stability and performance of systems. While delayed differential systems with integer order have been thoroughly investigated during the past decades (see [25] and references therein), the research of fractional delay differential systems is still in the initial and developing stage [26-32].

This book focuses on existence and expression of solution, stability and control problems of fractional-order delayed differential systems.

§1.2 Outline of This Book

In this book, several problems of fractional delayed differential systems are discussed.

Chapter 2 recalls the existence and uniqueness of solutions to Cauchy type problems for ordinary differential equations of fractional order on a finite interval of the real axis in spaces of continuous functions. Nonlinear and linear fractional differential equations in one-dimensional and vectorial cases are considered. The corresponding results for the Cauchy problems for ordinary differential equations are presented.

Chapter 3 discusses the initial value problem for nonlinear fractional order retarded functional differential equations. Consider the existence of solutions of nonlinear fractional functional differential equations with Riemann-Liouville derivative

$$D^{\alpha}x(t) = f(t, x_t) \tag{1.4a}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [-\tau, 0], \tag{1.4b}$$

where $0 < \alpha < 1, f(t, \varphi) \in C([0, T] \times C, R)$, $C = C([-\tau, 0], R)$ denotes space of continuous functions mapping the interval $[-\tau, 0]$ into R. The different results of existence of solutions are derived based on Banach fixed point theorem, Schauder fixed point theorem and successive approximations technique, respectively.

Chapter 4 focuses on the general solution of linear fractional neutral differential difference equations. Consider the general solution to linear fractional neutral differential difference system with the form

$$\begin{cases} {}^{C}D^{\alpha}[x(t) - Cx(t-\tau)] = Ax(t) + Bx(t-\tau) + f(t), & t \geqslant 0, \\ x(t) = \varphi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.5}$$

where ${}^{c}D^{\alpha}x(t)$ denotes an α order Caputo fractional derivative of $x(t)$, $0 < \alpha \leqslant 1$, A, B are $n \times n$ constant matrices, τ is a constant with $\tau > 0$, $f(t)$ is a n-dimensional continuous vector-valued function, $\varphi \in C^1([-\tau, 0], \mathbb{R}^n)$, and $C^1([-\tau, 0], \mathbb{R}^n)$ denotes space of continuously differentiable functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$. The exponential estimate of the solutions and the variation of constant formula for linear fractional neutral differential difference equations are derived by using the Gronwall integral inequality and the Laplace transform method, respectively. The obtained results extend the corresponding ones of integer order linear ordinary differential equations and delayed differential equations. At the same time, we also establish the variation of constant formulae for linear time varying Caputo fractional delayed differential system with the form

$$\begin{cases} {}^{C}D^{\alpha}x(t) = A(t)x(t) + B(t)x(t-\tau) + f(t), & t \geqslant 0, \\ x(t) = \varphi(t), \quad t \in [-\tau, 0], \end{cases} \tag{1.6}$$

where ${}^{C}D^{\alpha}x(t)$ denotes an α order Caputo fractional derivative of $x(t)$, $0 < \alpha < 1$, A, B are $n \times n$ constant matrices, $A(t), B(t)$ are $n \times n$ continuous function matrices, τ is a constant with $\tau > 0$, and $f(t)$ is an n-dimensional continuous vector-valued function.

In Chapter 5, the algebraic approach is used to discuss the delay-independently asymptotic stability of fractional-order linear singular delayed differential systems. Consider the delay-independent stability of the Caputo fractional-order singular delayed differential system

$$\begin{cases} \overline{E}\,{}^{C}D^{\alpha}x(t) = \overline{A}x(t) + \overline{B}x(t-\tau), & t \geqslant 0, \\ x(t) = \overline{\varphi}(t), & -\tau \leqslant t \leqslant 0 \end{cases} \tag{1.7}$$

and the Caputo fractional-order singular neutral delayed differential system

$$\begin{cases} {}^{C}D^{\alpha}\left[Ex(t) - Cx(t-\tau)\right] = Ax(t) + Bx(t-\tau), \ t \geqslant 0, \\ x(t) = \varphi(t), \quad -\tau \leqslant t \leqslant 0, \end{cases} \tag{1.8}$$

where $0 < \alpha < 1; x(t) \in \mathbb{R}^n$ is the state vector; $D^{\alpha}x(t)$ denotes an α order Caputo fractional-order derivative of $x(t)$; matrices $\overline{A}, \overline{B}, A, B, C \in \mathbb{R}^{n\times n}$, and matrices $\overline{E}, E \in \mathbb{R}^{n\times n}$ are singular with $\text{rank}(\overline{E}) = r < n$, $\text{rank}(E) = r < n$; $\tau \in \mathbb{R}^{+}$ is the time delay, and $\overline{\varphi}, \varphi$ are both the consistent initial functions. Based on the algebraic approach, some sufficient conditions are presented to ensure the asymptotic stability for any delay parameter. By applying the stability criteria, one can avoid solving the roots of transcendental equations.

Chapter 6 concerns the controllability of linear fractional differential systems with delay in state and impulses with the form

$$\begin{cases} {}^{C}D^{\alpha}x(t) = Ax(t) + Bx(t-\tau) + Cu(t), & t \in [0,T]\backslash\{t_1, t_2, \cdots, t_k\}, \\ \Delta x(t_i) = x(t_i^{+}) - x(t_i^{-}) = I_i(x(t_i)), & i = 1, 2, \cdots, k, \\ x(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases} \tag{1.9}$$

The expression of state response for such systems is derived, and some sufficient and necessary conditions of controllability criteria are established. The proposed criteria and illustrative examples both show that the controllability property of the linear systems is independent on the order of fractional derivative, or on delay or on impulses. Furthermore, the researchers consider the reachability and controllability of the following fractional singular dynamical systems with control delay:

$$\begin{cases} E\,{}^{c}D^{\alpha}x(t) = Ax(t) + Bu(t) + Cu(t-\tau), \quad t \geqslant 0, \\ u(t) = \psi(t), \quad -\tau \leqslant t \leqslant 0, \\ x(0) = x_0, \end{cases} \tag{1.10}$$

where ${}^{c}D^{\alpha}x(t)$ denotes an α order Caputo fractional derivative of $x(t)$, and $0 < \alpha \leqslant 1$; E, A, B and C are the known constant matrices, $E, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{n\times m}$, and $\mathrm{rank}(E) < n$; $x \in \mathbb{R}^n$ is the state variable; $u \in \mathbb{R}^m$ is the control input; $\tau > 0$ is the time control delay; $\psi(t)$ is the initial control function. The state structure of fractional singular dynamical systems with control delay is characterized by analysing the state response and the reachable set. A set of sufficient and necessary conditions of controllability for such systems is established based on the algebraic approach.

Chapter 2 Overview of Solutions to Fractional ODEs

This chapter is devoted to proving the existence and uniqueness of solutions to the Cauchy problems for ordinary differential equations (ODEs) of fractional order on a finite interval of the real axis in spaces of summable functions and continuous functions. The nonlinear and linear fractional differential equations in one-dimensional and vectorial cases are considered. The corresponding results for the Cauchy problems for ordinary differential equations are presented.

§2.1 Existence Results of Fractional ODEs on a Finite Interval of the Real Axis

In this section, the researchers give a brief overview of the results of existence and uniqueness theorems for differential equations of fractional order on a finite interval of the real axis.

Most of the investigations in this field involve the existence and uniqueness of solutions to fractional differential equations with the Riemann-Liouville fractional derivative $(D_{a+}^{\alpha}y)(x)$ defined for $\Re(\alpha) > 0$ by (1.2). The nonlinear differential equation of fractional order α ($\Re(\alpha) > 0$) on a finite interval $[a, b]$ of the real axis $\mathbb{R} = (-\infty, +\infty)$ has the form

$$(D_{a+}^{\alpha}y)(x) = f[x, y(x)], \quad \Re(\alpha) > 0,\ x > a, \tag{2.1}$$

with the initial conditions

$$(D_{a+}^{\alpha-k}y)(a+) = b_k, \quad b_k \in \mathbb{C},\ k = 1, 2, \cdots, n, \tag{2.2}$$

where $n = [\Re(\alpha)] + 1$ for $\alpha \notin \mathbb{N}$ and $\alpha = n$ for $\alpha \in \mathbb{N}$. The notation $(D_{a+}^{\alpha-k}y)(a+)$ means that the limit is taken at almost all points of the right-sided neighborhood $(a, a+\varepsilon)(\varepsilon > 0)$ of a as follows:

$$(D_{a+}^{\alpha-k}y)(a+) = \lim_{n \longrightarrow a+} (D_{a+}^{\alpha-k}y)(x),\ 1 \leqslant k \leqslant n-1,$$

$$(D_{a+}^{\alpha-n}y)(a+) = \lim_{n\longrightarrow a+}(D_{a+}^{-(n-\alpha)}y)(x),\ \alpha \neq n,$$
$$(D_{a+}^{0}y)(a+) = y(a),\ \alpha = n,$$

where $D^{-(n-\alpha)}$ is the Riemann-Liouville fractional integral operator of order $n-\alpha$ defined by (1.1).

In particular, if $\alpha = n \in \mathbb{N}$, then the problem (2.1)-(2.2) is reduced to the classical Cauchy problems for the ordinary differential equation of order $n \in \mathbb{N}$:

$$y^{(n)}(x) = f[x, y(x)],\quad y^{n-k}(a) = b_k,\quad b_k \in \mathbb{C},\quad k = 1, 2, \cdots, n.$$

When $0 < \Re(\alpha) < 1$, the problem (2.1)-(2.2) takes the form

$$(D_{a+}^{\alpha}y)(x) = f[x, y(x)],\quad (D_{a+}^{-(1-\alpha)}y)(a+) = b,\quad b \in \mathbb{C}$$

and this problem can be rewritten as the weighted Cauchy type problem

$$(D_{a+}^{\alpha}y)(x) = f[x, y(x)],\quad \lim_{x\longrightarrow a+}((x-a)^{1-\alpha}y(x) = c,\quad c \in \mathbb{C}.$$

The investigations of the above problems followed their historical chronology. Essentially, they were based on reducing problem (2.1)-(2.2) to the following nonlinear Volterra integral equation of the second kind:

$$y(x) = \sum_{j=1}^{n}\frac{b_j}{\Gamma(\alpha-j+1)}(x-a)^{\alpha-j} + \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f[t, y(t)]\mathrm{d}t,\ x > a. \tag{2.3}$$

Pitcher and Sewell [33] in 1938, first considered the nonlinear fractional differential equation (2.1) with $0 < \alpha < 1$, provided that $f(x, y)$ is bounded in a special region G lying in $\mathbb{R} \times \mathbb{R}$ and satisfies the Lipschitz condition with respect to y:

$$\left|f(x, y_1) - f(x, y_2)\right| \leqslant L\left|y_1 - y_2\right|, \tag{2.4}$$

where the constant $L > 0$ does not depend on x. They proved the existence of the continuous solution $y(x)$ for the corresponding nonlinear integral