郭永怀文集

GUO YONGHUAI WENJI

郭永怀 著

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内 容 简 介

郭永怀是著名的力学家、应用数学家，我国近代力学事业奠基人之一。为纪念郭永怀诞辰一百周年，我们再次编辑出版《郭永怀文集》。

本文集共收录了作者 24 篇期刊论文和研究报告。其中，1943~1955 年期间发表的 14 篇论文内容涉及直管中的可压缩流动、有限振幅柱面和球面波的传播、可压缩流体二维无旋亚声速、超声速混合型流动和上临界马赫数、光滑跨声速绕流及其稳定性、斜激波从平板边界层的反射、中等雷诺数下绕平板的流动等。这些论文反映了作者在跨声速流动和奇异摄动理论两个领域中为国际公认的学术成就，后者被命名为 PLK 方法。1956~1957 年期间发表的 5 篇论文：绕平板和楔的高超声速流动、普朗特数和解离对高超声速流动的影响以及增补的 5 篇文章是作者在高超声速流动领域的研究成果以及他在回国后的学术报告和发表的文章，涉及现代空气动力学的发展方向、发射卫星和返地回收的科学和技术问题，体现了他在参与“两弹一星”技术领导工作中的学术思想。

本书可供流体力学、空气动力学、应用数学专业、航空航天工程科技人员以及高等院校有关专业教师、研究生和高年级学生参考。

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为什么回到祖国？——我作为一个中国人，有责任回到祖国，和人民一道，共同建设我们的美丽的山河。

对于所有的问题，我们都是空怀若谷，不经过讨论，就没有定论。对每一方面，我们需要权威，但是权威绝不能专有真理。

郭永怀

代　序*

现在已是八十年代的第一春．还要倒数到第十一个冬天，郭永怀同志因公乘飞机，在着陆事故中牺牲了．是的，就那么十秒钟吧，一个有生命、有智慧的人，一位全世界知名的优秀应用力学家就离开了人世；生和死，就那么十秒钟！

十秒钟是短暂的．但回顾往事，郭永怀同志和我相知却跨越了近三十个年头，而这是世界风云多变的三十个年头呵．我第一次与他相识是在 1941 年底，在美国加州理工学院．当时在航空系的有林家翘先生、有钱伟长同志，还有郭永怀同志和我．在地球物理系的有傅承义同志．林先生是一位应用数学家．傅承义同志专的是另外一行．钱伟长同志是个多才多艺的人．所以，虽然我们经常在一起讨论问题，但和我最相知的只有郭永怀一人．他具备应用力学工作所要求的严谨与胆识．当时航空技术的大问题是突破“声障”进入超声速飞行，所以研究跨声速流场是个重要课题，但描述运动的偏微分方程是非线性的，数学问题难度很大．永怀同志因问题对技术发展有重大意义，故知难而进，下决心攻关，终于发现对某一给定外形，在均匀的可压缩理想气体来流中，当来流马赫数达到一定值，物体附近的最大流速达到局部声速，即来流马赫数为下临界马赫数；来流马赫数再高，物体附近出现超声速流场，但数学解仍然存在；来流马赫数再增加，数学解会突然不可能，即没有连续解，这就是上临界马赫数．所以真正有实际意义的是上临界马赫数而不是以前大家所注意的下临界马赫数，这是一个重大发现．

1946 年秋，郭永怀同志任教于由 W. R. Sears 主持的美国康奈尔大学航空学院，我也去美国麻省理工学院，两校都在美国东部，而加州理工学院在西部，相隔近三千公里，他和我就驾车旅行．有这样知己的同游，是难得的，所以当他到了康奈尔而留下来，而我还要一个人驾车继续东行到麻省理工学院时，我感到有点孤单．

1949 年我再次搬家，又到美国加州理工学院任教，所以再一次开车西去，中途到康奈尔．这次我们都结了婚，是家人相聚了，蒋英也再次见到我常称道的郭永怀和李佩同志．这次聚会还有 Sears 夫妇，都是我们在加州理工学院的熟朋友．我们都是我们的老师 Theodore von Kármán 的学生，学术见解很一致，谈起来逸趣横生．这时郭永怀同志已对跨声速气动力学提出了一个新课题：既然超出上临界马赫数不可能有连续解，在流场的超声速区就要出现激波，而激波的位置和形状是受附面层影响的，因此必须研究激波与附面层的相互作用．这个问题比上临界马赫数问题更难，连数学方法都得另辟新途径．这就是 PLK 方法中 Kuo（郭）的来源，现在我们称奇异摄动法．这项工作是郭永怀同志的又一重大贡献．

郭永怀同志之所以能取得这两项重大成果，是因为他治学严谨而遇事看得准，有见识；而一旦看准，有胆量去攻关．当然这是我们从旁见到的，我们也许见不到的是他刻苦的功夫，呕心沥血的劳动．

我以后再见到永怀同志是 1953 年冬，他和李佩同志到加州理工学院．他讲学；我也有机会向他学习奇异摄动法．我当时的心情是很坏的，美国政府因不许我归回祖国而限制我的人身自由，我满腔怒火，向我多年的知己倾诉．他的心情其实也是一样的，但他

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* 摘自《郭永怀文集》，北京：科学出版社, 1982. 原题为“写在《郭永怀文集》的后面”．

克制地劝我说，不能性急，也许要到 1960 年美国总统选举后，形势才能转化，我们才能回国．所幸的是：在中国共产党领导下，新中国有亿万人民的团结，迅速强大起来了，我们都比这个日程早得多回到祖国，我在 1955 年，他在 1956 年．

郭永怀同志归国后，奋力工作，是中国科学院力学研究所的主要学术领导人；他做的比我要多得多．但这还不是他的全部工作，1957 年初，有关方面问我谁是承担核武器爆炸力学工作最合适的人，我毫无迟疑地推荐了郭永怀同志．郭永怀同志对发展我国核武器是有很大的贡献的．

所以我认为郭永怀同志是一位优秀的应用力学家，他把力学理论和火热的改造客观世界的革命运动结合起来了．其实这也不只是应用力学的特点，也是一切技术科学所共有的，一方面是精深的理论，一方面是火样的斗争，是冷与热的结合，是理论与实践的结合．这里没有胆小鬼的藏身处，也没有私心重的活动地；这里需要的是真才实学和献身精神．郭永怀同志的崇高品德就在这里！

由于郭永怀同志的这些贡献，我想人民是感谢他的．周恩来总理代表党和全国人民对郭永怀同志无微不至的关怀就是证据．大家辛勤工作，为翻译、编辑和出版这本文集付出了劳动，也是个证据．是的，人民感谢郭永怀同志！作为我们国家的一个科学技术工作者，作为一个共产党员，活着的目的就是为人民服务，而人民的感谢就是一生最好的评价！

我们忘不了郭永怀同志，这本文集是一件很好的纪念品，一本很好的学习材料．

钱学森

1980 年 1 月 16 日

再版前言

在郭永怀先生诞辰一百周年之际，我们再次编辑出版了《郭永怀文集》(简称《文集》)。与 30 年前的中译本不同，郭永怀先生在国外发表的 19 篇期刊论文和研究报告全部以原文刊出。同时，文集中还增加了他回国初期的 5 篇重要文章。这对于了解他的科学成就，领悟他的学术思想，保存他的历史文献具有重要意义。

他在国外期间发表的学术论文，集中展现了他在空气动力学和应用数学领域为世界公认的科学成就：

在跨声速流动领域的 8 篇文章中，他和钱学森一起提出了“上临界马赫数”的概念，研究了激波与边界层的相互作用，发现光滑的跨声速混合流动可以维持到上下临界马赫数间的来流速度，激波的出现将导致波后压力陡增，产生漩涡，乃至发生分离，并可通过黏性层影响上游。从而回答了当时航空工程界聚焦的问题：当飞行速度接近声速时，在绕翼型和飞行器的流场中什么时候会实际出现激波？激波的出现对其气动性能可能产生什么样的影响以及如何减轻和避免气动失速影响的措施。这是 20 世纪 40 年代人类为突破“声障”最有影响的研究成果，并为跨声速飞机设计奠定理论基础。

与奇异摄动理论领域相关的 3 篇论文，研究了中等雷诺数和高速边界层问题。由于黏性扩散或压缩性效应使边界层厚度增加，流动偏折明显，需要计算高阶近似。这时由于没有现成的数学方法可以依循，郭永怀提出了边界层问题中消除高阶奇性并使解一致有效的方法。他不仅给出平板边界层二阶阻力系数和激波边界层黏性干扰的结果，同时也发展了 H. Poincaré 的参数摄动法和 J. Lighthill 的坐标摄动法。1956 年，钱学森在《应用力学评论》(*Applied Mechanics Reviews*)撰文，将该方法命名为“PLK 方法”。

20 世纪 50 年代中，郭永怀在解决了航空领域的难题后，敏锐地意识到人类即将进入空间时代，在 1956~1957 发表的 5 篇文章中，开始关注高超声速流动问题，并深入分析了 Prandtl 数和气体解离对流动，摩阻和传热的可能影响。1963 年，他还指导研究人员完成了高速湍流边界层传热的分析。

郭永怀先生在 1957 年中国力学学术大会和 60 年代星际航行座谈会上的报告，精辟地指明了现代空气动力学的发展方向，深刻分析我国发射卫星和返地回收需要解决的若干科学和技术问题。在我们已经实现了载人航天的今天，重新阅读这些文献，就会更加赞叹郭永怀先生的远见卓识。

郭永怀先生在国外学习和研究的 16 年都是在做准备，都是为了“和人民一道共同建设美丽的山河!”。在他回国后有限的时间里，精力全部集中在领导我国力学学科规划工作，倡导高超声速空气动力学、磁流体力学和爆炸力学等新兴学科，支持研制系列激波管和激波风洞，规划我国的空气动力学试验基地等方面。他高瞻远瞩确定的方向，迄今仍是航空、航天和能源等工程中富有生命力和挑战性的课题。这一时期郭永怀先生的科学成就主要体现在他为我国近代力学事业奠基和“两弹一星”的科学实践活动中。

在我国科学技术界，郭永怀先生是将国家需求和学科前沿结合的楷模，是研究工作

和技术工作衔接的典范，是理论和实验研究并重的表率，郭永怀先生的学术思想是留给我们的宝贵财富。我们一定要继承和发扬郭永怀先生的科学精神和高尚品格，为把我国建设成创新型的现代化国家而不懈努力！

在编辑《文集》的过程中，力学所图书信息中心的朱涛、张凌晨同志做了细致工作，也得到了科学出版社的全力支持，谨此致谢！

中国科学院力学研究所

中国力学学会

2009 年 3 月

目　　录

ON THE FORCE AND MOMENT ACTING ON A BODY IN SHEAR FLOW* 1)

BY YUNG-HUAI KUO (*California Institute of Technology*)

Recently, H. S. Tsien solved the problem[1] of a Joukowsky airfoil in a steady, two-dimensional flow of constant vorticity distribution. It is interesting to note that the hydrodynamical forces can be expressed in a form similar to the well known Blasius' theorem, involving contour integration of the complex potential function. The following derivation of the formulae is believed to be simpler than that of Tsien.

1. **Equations of motion.** Let u and v be the velocity components parallel to the x- and y-axis, respectively. In the case of two-dimensional steady motion, the Eulerian dynamical equations are:

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{1.1}$$

$$u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+u\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{1.2}$$

where p is the pressure and ρ, the density of the fluid. The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{1.3}$$

For the type of shear flow considered by Tsien,[1] the vorticity is constant everywhere in the field and equal to $-k$. Thus

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=-k, \qquad k>0. \tag{1.4}$$

At the first sight, it seems that the problem might not be definite as one has four equations for three variables. By eliminating p between Eqs. (1.1) and (1.2), however, the result can be reduced to Eq. (1.3) by means of Eq. (1.4). This shows that any solution which satisfies Eqs. (1.3) and (1.4) is consistent with Eqs. (1.1) and (1.2).

To simplify the problem, the solution is written in the following form:

$$u=ky+u', \tag{1.5}$$

$$v=v'. \tag{1.6}$$

Then Eqs. (1.3) and (1.4) reduce to

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0, \tag{1.7}$$

$$\frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}=0. \tag{1.8}$$

* Received June 21, 1943.

[1] H. S. Tsien, *Symmetrical Joukowski airfoils in shear flow*, Quarterly Appl. Math., **1**, 129 (1943).

1) Published in *Quarterly of Applied Mathematics*, 1943, 1(1): 273~275

These equations are satisfied by

$$u' = \frac{\partial \psi}{\partial y}, \qquad v' = -\frac{\partial \psi}{\partial x}; \tag{1.9}$$

or

$$u' = \frac{\partial \varphi}{\partial x}, \qquad v' = \frac{\partial \varphi}{\partial y}; \tag{1.10}$$

where ψ and φ are the imaginary and real parts of the complex potential $F(z)$; namely,

$$\varphi + i\psi = F(z), \qquad z = x + iy; \tag{1.11}$$

and

$$u' - iv' = w'(z). \tag{1.12}$$

For a given problem the function $F(z)$ is so determined that the velocity component normal to the contour of the body is zero.

By virtue of Eqs. (1.4), (1.5), and (1.6), Eqs. (1.1) and (1.2) give

$$p = -\frac{\rho}{2} q'^2 - \rho k u' y + \rho k \psi, \tag{1.13}$$

where $q'^2 = u'^2 + v'^2$, and the constant of integration is absorbed in ψ.

2. Force and moment. If the motion is two-dimensional and steady, the components of the hydrodynamical force and moment[2] acting on the body are given by

$$X = -\oint p\,dy - \rho \oint u(u\,dy - v\,dx), \tag{2.1}$$

$$Y = \oint p\,dx + \rho \oint v(v\,dx - u\,dy), \tag{2.2}$$

$$M = \oint p(x\,dx + y\,dy) - \rho \oint (-v^2 x\,dx - u^2 y\,dy + uvy\,dx + uvx\,dy), \tag{2.3}$$

where the contour integrals are taken along a closed curve containing the body. Using Eqs. (1.5), (1.6) and (1.13), the above equations can be written as:

$$X = -\frac{\rho}{2}\oint [(u'^2 - v'^2)dy - 2u'v'dx] - \rho k \oint [(\psi + u'y)dy - v'y\,dx], \tag{2.4}$$

$$Y = -\frac{\rho}{2}\oint [(u'^2 - v'^2)dx + 2u'v'dy] + \rho k \oint [(\psi - u'y)dx - v'y\,dy], \tag{2.5}$$

$$M = -\operatorname{Re}\left[\frac{\rho}{2}\oint z w'^2 dz\right] + \rho k \oint [(\psi - u'y)(x\,dx + y\,dy) - (v'yx - 2u'y^2)dy + v'y^2 dx]. \tag{2.6}$$

[2] W. F. Durand, *Aerodynamic theory*, vol. 2, Springer, Berlin, 1935, pp. 31–33.

If only bodies with closed boundary are considered, no sources can exist within the field of flow. Then the stream function ψ is single-valued, and

$$\oint \psi dx = \oint x(v'dx - u'dy),$$

$$\oint \psi dy = \oint y(v'dx - u'dy).$$

From these relations, it is not difficult to deduce

$$X = -\frac{\rho}{2}\oint [(u'^2 - v'^2)dy - 2u'v'dx], \tag{2.7}$$

$$Y = -\frac{\rho}{2}\oint [(u'^2 - v'^2)dx + 2u'v'dy] + \rho k\oint [v'(xdx - ydy) - u'(ydx + xdy)], \tag{2.8}$$

$$M = -\operatorname{Re}\left[\frac{\rho}{2}\oint zw'^2dz\right] + \frac{\rho k}{2}\oint [-u'\{(x^2 - y^2)dy + 2xydx\} + v'\{(x^2 - y^2)dx - 2xydy\}]. \tag{2.9}$$

These at once suggest the following alternative expressions:

$$X - iY = \frac{i\rho}{2}\oint w'^2dz + i\operatorname{Im}\left[\rho k\oint w'zdz\right], \tag{2.10}$$

and

$$M = -\operatorname{Re}\left[\frac{\rho}{2}\oint z\left(w' - \frac{ikz}{2}\right)^2 dz\right]. \tag{2.11}$$

Eqs. (2.10) and (2.11) may be regarded as an extension of Blasius' theorem. They can be easily identified with the expressions given by Tsien.[1] The calculation of force and moment, however, can be simplified to a certain extent by using these new expressions.

The writer wishes to thank Dr. H. S. Tsien for the use of his paper before publication and for his helpful discussions.

THE FLOW OF A COMPRESSIBLE VISCOUS FLUID THROUGH A STRAIGHT PIPE 1)

BY Y. H. KUO

Introduction

The problem of determining a steady flow of incompressible viscous fluid through a straight pipe of any section is easily reduced to a problem of Dirichlet. This is the familiar Hagen-Poiseuille[1] flow, and a pipe of circular section is particularly well known. If, however, the fluid is compressible, the problem of steady flow becomes much more difficult. Although the problem has been treated experimentally[2] and by the rather crude method of hydraulics,[3] there does not appear to have been any previous mathematical treatment.

The method of the present paper is that of development in series in powers of the Mach number, which is supposed to be small. The operation with series is formal, no attempt being made to discuss the convergence.

The boundary condition of zero velocity on the wall of the tube, together with the information regarding the pressure drop, is not sufficient to make the mathematical problem definite. It is, in fact, a question of finding *a* solution rather than *the* solution; the simplest solution is taken, as in the classical case of the incompressible fluid, where the same partial indeterminacy also occurs.

Part I deals with a pipe of general section. The equations of motion expressed in terms of momentum vector and specific volume are given and the process of power series development is explained. This may, for small values of Mach number, be regarded as a process of successive approximations.

The zero approximation is the Hagen-Poiseuille flow.

In the first approximation the flow remains parallel to the walls of the pipe, and the determination of the velocity again reduces to a Dirichlet problem (5.3).

In the second approximation the velocity is no longer parallel to the walls. The component of momentum parallel to the walls is a linear function of the distance along the pipe (6.3), the determination of the coefficient of this function again depending on Dirichlet problems (6.8), (6.9). The transverse momentum component is proportional to $f_\alpha^{(2)}$ in (6.13), and this is independent of distance along the pipe; its determination is reduced to a Dirichlet problem (6.14) and a biharmonic problem (6.15).

In the zero approximation the pressure is a linear function of distance along the pipe (4.7), in the first approximation a quadratic function (5.8), and in the second approximation a cubic function (6.27).

The drag and flux are considered in §7.

In Part II the pipe is of circular section and explicit solutions are given.

[1] Hagen, Pogg. Ann., **46**, 423 (1839).
[2] Poiseuille, Comp. Rend. **11** and **12**, (1840–1).
[3] Reynolds, Phil. Trans. Roy. Soc. London, **174**, 935 (1883).

1) Published in *Journal of Mathematics and Physics*, 1943, 22: 13~30

Part I. Pipe of a general section

1. The equations of motion

We shall consider a compressible viscous fluid flowing steadily through a pipe of any constant section, under the influence of the difference between the pressures applied at the ends of the pipe. It is assumed that the pipe has a well-rounded mouth-piece so that it introduces no initial disturbances. The domain of the mathematical theory is the interior of the pipe between two normal sections at a distance l' apart (the "length" of the pipe); these sections are called the "entrance" and "outlet".

Let x'_i be Cartesian coordinates with the origin at the mean centre of the entrance, the axis of x'_0 lying along the axis of the pipe; let u'_i be the components of the velocity. Then in the pipe itself the motion satisfies the general equations[4,5] of steady motion in the absence of body force

$$\rho' u'_j u'_{i,j} = -p'_{,i} + \mu \Delta' u'_i + \frac{\mu}{3} \theta'_{,i}, \tag{1.1}$$

where $\theta' = \partial u'_j / \partial x'_j$, $\Delta'(= \partial^2/\partial x'_j \partial x'_j)$ is the Laplacian operator, ρ' the density, p' the pressure and μ the viscosity, which is a function of temperature, but within wide limits, independent of pressure. With these is associated the equation of continuity

$$(\rho' u'_j)_{,j} = 0. \tag{1.2}$$

Here the equations are expressed in accordance with the indicial notation; the Latin suffixes have the range 0, 1, 2, the comma indicates partial differentiation and summation is understood for repeated suffixes. We shall assume the motion to be either isothermal or adiabatic, so that a definite pressure density relation exists. We shall in both cases treat μ as a constant, its variations under change of pressure being neglected in both cases and its variation under change of temperature being neglected in the adiabatic case.

We now find it convenient to define a few constants which are useful in the later calculation. Let $\bar{p}'$ be the mean pressure over the cross-section at the outlet, defined as

$$\bar{p}' = \frac{1}{A} \int\int p' \, dx'_1 \, dx'_2 \qquad \text{for} \quad x'_0 = l', \tag{1.3}$$

where A is the area of the cross-section. Let $\bar{\rho}'$ be the density corresponding to $\bar{p}'$ according to the pressure-density relation of the fluid. Finally, let $\bar{u}'$ be the mean velocity over a cross-section, weighted by the density so that it is independent of the particular section chosen, namely,

$$\bar{u}' = \frac{1}{\bar{\rho}' A} \int\int \rho' u'_0 \, dx'_1 \, dx'_2 \qquad \text{for any} \quad x'_0. \tag{1.4}$$

[4] W. Müller, Einführung in die Theorie der zähen Flüssigkeiten, (Leipzig, 1932) p. 13.

[5] H. Lamb, Hydrodynamics, (Cambridge, 1932) p. 577.

We shall now be able to express the equations in dimensionless form by the following transformation:

$$x_i = \frac{x_i'}{2m}, \qquad u_i = \frac{u_i'}{\bar{u}'}, \qquad p = \frac{p' - \bar{p}'}{\bar{\rho}'\bar{u}'^2}, \qquad \rho = \frac{\rho'}{\bar{\rho}'}, \tag{1.5}$$

where m is the hydraulic mean radius of the section, i.e. $m = A/P$ where P is the perimeter of the bounding curve C. Substituting (1.5) in (1.1) and (1.2), we have

$$\rho u_\beta u_{\alpha,\beta} + \rho u_0 u_{\alpha,0} = -p_{,\alpha} + \frac{2}{R}\Delta u_\alpha + \frac{2}{3R}\theta_{,\alpha}, \tag{1.6}$$

$$\rho u_\beta u_{0,\beta} + \rho u_0 u_{0,0} = -p_{,0} + \frac{2}{R}\Delta u_0 + \frac{2}{3R}\theta_{,0}, \tag{1.7}$$

$$(\rho u_\beta)_{,\beta} + (\rho u_0)_{,0} = 0, \tag{1.8}$$

the Greek suffixes having the range 1, 2; here $\theta = \partial u_j/\partial x_j$, R is the Reynolds' number, i.e., $R = 4m\bar{u}'/\nu$ where $\nu\left(= \frac{\mu}{\bar{\rho}'}\right)$ is the mean kinematic viscosity.

The boundary conditions to be satisfied by u_i are

$$u_i = 0 \quad \text{on} \quad C. \tag{1.9}$$

These seem to be the only essential boundary conditions. As we shall see later that, they are generally not sufficient to ensure a definite solution. We shall make use of this partial indeterminacy to obtain analytically simple solutions, hoping that the physical validity of such solutions may be justified in the same way that the solutions of Saint Venant in elasticity are justified.

It is convenient to have the constancy of the flux of mass across any section expressed in integral form; in dimensionless variables, it reads

$$\iint \rho u_0 \, dx_1 \, dx_2 = \frac{P^2}{4A}. \tag{1.10}$$

Once the velocity u_i is known we can solve for p from (1.6) and (1.7). As the differential equations involving p are of the first order there will be one arbitrary constant at our disposal, which will be determined by the condition at the outlet. This is

$$\iint p \, dx_1 \, dx_2 = 0 \qquad \text{for} \quad x_0 = l = \frac{l'}{2m}. \tag{1.11}$$

2. The pressure-density relation

The equations (1.6), (1.7) and (1.8) which involve five unknowns cannot be solved without using a relation connecting p and ρ. As remarked above, we shall suppose the change of state to be either isothermal or adiabatic. It would appear necessary to treat the case of a gas and that of a liquid separately. However, as we shall see below, it is possible to justify a common treatment as approximately valid from a physical point of view.

Let M denote the Mach number; it is defined as the ratio of the mean velocity $\bar{u}'$ of the fluid to that of sound in the fluid, namely,

$$M = \frac{\bar{u}'}{c_0}, \tag{2.1}$$

where c_0 is the velocity of sound in the fluid at the pressure $\bar{p}'$, i.e. $c_0^2 = (dp'/d\rho')_{\rho=\bar{\rho}'}$.

Suppose we are dealing with the perfect gas under the isothermal condition, the pressure-density relationship is Boyle's law

$$\rho' = kp',$$

where k is a constant. With this law the value of $\bar{\rho}'$ corresponding to $\bar{p}'$ is

$$\bar{\rho}' = k\bar{p}'.$$

Taking the difference of these two equations and converting to dimensionless variables, we have

$$\rho = 1 + M^2 p. \tag{2.2}$$

Similarly, under the adiabatic condition the pressure-density relationship takes the form

$$\rho^\gamma = 1 + \gamma M^2 p, \tag{2.3}$$

where γ is the ratio of the specific heats.

For mathematical reasons, we consider only those cases in which the viscosity μ may be regarded as a constant. This is certainly the case for the isothermal motion of a gas. It will not be accurate when the change is adiabatic, for the temperature no longer remains constant. However in gases, air for instance, the viscosity depends only slightly on temperature; for an increase of temperature from 0° to 20°C, the increase[6] in μ is about 6 percent, so we can take it as constant for small change in temperature.

In the case of a liquid, the general thermodynamic properties are rather vague and no definite laws have ever been established. However, it is an experimental fact that when[7] water flows through a pipe, the temperature everywhere is sensibly the same so long as the flow is laminar; and that[8] under a constant temperature, the pressure density-relation is approximately linear for a moderate range of pressures. Consequently, the isothermal law (2.2) will be assumed to be valid also for liquid.

As a mathematical convenience, we may regard the equation (2.2) as a special case of the equation (2.3), although they represent entirely different physical processes. Hence we shall use (2.3), reducing to (2.2) when required by putting $\gamma = 1$.

[6] H. Lamb, Loc. cit.[5] p. 576.
[7] Barnes and Coker, Proc. Roy. Soc. A 74, 341 (1904).
[8] P. W. Bridgman, Proc. Roy. Soc. A 48, 309 (1912); A 49, 1 (1913); A 66, 185 (1931).