现代无穷小分析导引

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序　　言

1960年，美国数理逻辑学家A. Robinson以其卓越的洞察力看出，现代数理逻辑的分支——模型论可以为实无限小演算奠定严格的逻辑基础．他的工作结果，创始了一种新的数学方法——非标准分析（Nonstandard Analysis）．

古典的分析学（后文有时相应地称之为标准分析）建立在否认无限小存在的数域上．而非标准分析则是在确立了包含有无限小元素的某种数域的前提下进行严谨论述的．它已成功地解决了300年前涉及无限小的古老悖论．但是，近20余年来的历史表明，非标准分析的价值已远远超出重建微积分的意义．应用这一方法业已在数学的各个分支取得的丰硕成果显示，非标准分析以其直观而又简捷的特点和更合于发明的创造技巧，已成为强有力的数学研究新工具．可以肯定地说，非标准分析之所以如此富有生命力，并激起不少数学家的研究热情，正在于它所具有的和谐、朴素而又简易的美学特征，在于其丰富的表现力以及其深广的应用．Robinson的成就无疑是本世纪的主要数学进展之一．

然而不幸的是，由于Robinson在其著作中使用了相当多的数理逻辑知识，而后者是多数数学家所不熟悉的，因而限制了其进一步的普及和应用．此后，为了更好地理解并掌握这一新工具，不少数学家做了很大的努力．如Luxemburg[17]，Stroyan/Luxemburg[5]，Osdol[2]，Davis[16]， Nelson[10], Hrbacek[11], Lutz/Goze[20], Hurd/Loeb[7]等等都为解释、阐述非标准分析做出了贡献. H. J. Keisler的著作[3]是一本出色的用非标准观点写的微积分教材. 它为初学者（主要是大学低年级学生和高中学生）掌握非标准微积分的基本方法，作了可贵的尝试. 情况报道和评论见[8]、[9]. 不过，它终究是一本浅近的初等微积分教本，未能涉及非标准分析的关键部分，因而在[3]的基础上进入更高水平的研究和应用领域，看来是困难的. 此外，Keisler所引进的6条公理，虽颇具匠心，然而往往使初学者难以领会，也不易据此看清非标准模型的结构本质.

这样，编写一本国内使用的入门性的现代无穷小分析教材，就是很必要的了. 八、九年前我们就想写这样的一本书. 正如上文多数作者那样，本书依照Robinson在其原始论文[14]中的建议，采用超幂的思想，构造了非标准数域 $^*R$，在此基础上，为了讲述主要定理即一般框架内的转换原理，我们介绍了超结构的基本概念. 然后在其余各章简要论述了非标准微积分理论.

本书的部分题材曾由作者们分别在吉林大学的讨论班上和吉林工业大学举办的讲座中报告过. 近6年来，我们对全部书稿几经重写、刷新、修改和扩充.

本书取材基本上自给自足，自成体系. 书中注重思想方法的阐明. 例如，在阐述一系列基本概念时，为便于使读者洞察非标准分析与标准分析方法之间的内在联系，我们总是采取对比讲法. 这样做，相信能帮助读者更加深刻地领会非标准方法的实质，并了解其直接简易的特点. 在不妨碍精确性的前提下，本书陈述方式力求简明扼要、直观易懂，故篇

幅虽不大，但介绍了现代无穷小分析最基本的概念、原理和方法，而且全书所讲述的定义、命题和定理，基本上包括了初等微积分的重要内容，并涉及部分高等微积分知识．书中还有一些多样性例题和少量习题，这对初学者掌握非标准微积分方法想来是会有帮助的．

在题材处理手法上，也有一些不同于其它书籍之处．例如，我们采用“等价过程量”的表述方式叙述了$^*R$的构造方法，这使初学者更容易从直观上看清非标准数域$^*R$的结构．我们还将经典Duhamel定理推广到$^*R$，建立了“广义Duhamel原理”或“等和原理”．借助此原理，可以用*-有限结构较真切地逼近标准无限结构，并统一了涉及各种积分过程的存在唯一性的论证手续．此外，还引用了“相对微商”的概念，使Riemann-Stieltjes积分的讨论趋于简化．

现代无穷小分析方面属于教程一类的书籍，国内尚未出版过，我们编写这本教程只是一个尝试．假若此书能对讲授这类课程的教师以及自学青年有所帮助，我们将感到莫大欣慰．同时，也热忱希望读者能对书中的缺点或错误不吝指正．

编著者

1989年1月

PREFACE

The history formed a ring being full of comedy. The infinitesimals have appeared on the mathematical stage again as both rigorous and fruitful role which has considerable potentialities.

In 1960, Abraham Robinson gave a rogorous logical foundation for the infinitesimal analysis, thus recalling the infinitesimals to life. Robinson's infinitesimal method, nowadays called nonstandard analysis or modern infinitesimal analysis, has become a powerful and living new method for research in many branches of pure and applied mathematics and in some fields outside of mathematics, which has made remarkable achievements. The reason why Robinson's method has aroused the enthusiasm of research mathematicians is that it possesses of mathematical simplicity, expressiveness, intuition, inherent beauty and the far-reaching application. Robinson's work may be regarded as one of the most significant mathematical achievements of the twentieth century.

In 1670's W. G. Leibniz created a general framework in which he treated infinitesimals as "ideal" numbers which were smaller in absolute value than any ordinary strictly positive real number and possessed of all the usual properties of arithmetic. Since Leibniz's view necessarily led to the paradox of the infinitesimals, it was replaced by the $\varepsilon-\delta$ method of Weierstrass, which has provided a strict logical foundation for calculus, but the vitality of the infinitessimals has all gone, thus the indirect inference has not given full play to the wisdom and creativeness of mathematicians.

In 1960 Robinson used methods from the branch of mathematical logic known as Model Theory (the compactness theorem and the completeness theorem) to solve completely Leibniz's problem of the infinitesimal and showed that the real number field can be regarded as a subset of the hyperreal number field which contains infinitesimals as well as infinitely large numbers and which in a well-defined sense satisfies all of the arithmetic rules obeyed by ordinary real numbers.

But Robinson's proof of the existence of hyperreal numbers was based on mathematical logic which was unfamiliar to most mathemalicians and made it difficult to study the method. In the past twenty years, many books have been written for use in presenting such method. Stroyan and Luxemburg[5], Keisler[4], Davis[6], Hurd and Loeb[7], etc, have made brilliant contributions to interpret and expound the nonstandard methods, which have propelled nonstandard analysis forward into the far-reaching application.

In China, such books have seldom been published so that the condition prevants the method from spreading among mathematical research. The present book is a popular literature, whose purpose is to make the modern infinitesimal method available to students having only a knowledge of calculus and to make the newly acquired tool fruitful for the classical fields of interest to mathematicians with few basic facts of nonstandard analysis. To achieve this purpose we treat infinitesimal analysis at the elementary level, in which we have tried to give as clear and simple an explanation as possible instead of proof of the Transfer Principle to do our best to lighten the burden on mathematical logic of the readers. This seems especially important in a popular text because it serves to emplasize the fact that the Transfer Principle is the most basic one, but the logic used in proving it is unavailable for nonstandard analysis. Here care is taken to select the necessary material for describing the Transfer Principle. We also have tried to enhance the intuitive understanding of the readers and to vary, by proving some classical results, their old train of thought, which occurs so easily when studying the new methods.

The main differences between this book and the previous textbooks on nonstandard analysis is, first to introduce (Chapter Ⅱ) the concept of "procedure quantities" into the ultrapower construction, which eases perhape the readers of understanding the precession of the sequences of real number; secondly in view of the present situation of Chinese readers, to make (Chapter Ⅲ) a clear explanation of significance instead of proof of the transfer princple; thirdly to present (Chapters Ⅳ—Ⅸ) the contrasts between the nonstandard and standard expressions in order to demonstrate the power of expressions of the former in the most of the book; fourthly to establish quasi-Duhamel Principle, which simplifies many proofs in Chapter XII and parts of the rest of the book such as the existence of the various integrals and the interchangeability of the integration and the infinite summation; fifthly to study nonstandardly (Chapter XI) the Riemann-Stieltjes integral, in which we have the aid of relative derivate and relative primitive function of a function with respect to another function to inspect some properties of the Riemann-Stieltjes integral. Sixthly, to make improvements on what has been achieved in the first eleven chapters, we summarize some fundamental concepts found in the research literature. This is a bridge between the elementary treatment in chaplers I—XI and the needs of the far-reaching application of nonstandard analysis.

This book is divided into five parts. In the first part (Chapter I: Introdnction) we give a brief account of how things are in the rapid development of nonstandard analysis (section 1), and so far as we superficially know, the significance of the method is appraised from the historical developments in section 2. Section 3 summarizes the achievements, the excellent character and the philosophy of A. Robinson to cherich the memeory of him, whose untimely death has been a great loss to scientific world.

In the second part [Chapter II: Hyperreal Fields (H-fields)] we use "ultrapower" construction instead of the compactness theorem and the completeness theorem to establish the existence of nonstandard models of analysis, which affords us an insight into the structure of nonstandard models of analysis, and which may be helpful to the beginners unfamiliar to model theory.

In the third part (Chapter III: Transfer Principle), in order to give main principle (Transfer Principle), which is not only suitable for the properties expressed in a first-order language but also appropriate for various structures under consideration(real and complex analysis, algebra, topology, and the like), this chapter begins by constructing a superstructure $V(X)$ which contains all of the entities involved in the mathematics of a basic set X. Second, we introduce the formal language for superstructures and, with the aid of the interpretation map, the formal language, especially the bounded sentences, are endowed with mathematical meaning. Third, we define a superstructure embedding map $*: V(X) \to V(^*X)$ and show clearly the method forming $*$-transform. Finally we give the Transfer Principle. In the remaining parts of the chapter, the notions of internal and external entities and sentences or formulae are defined because of which the readers will comprehend profoundly the implication of the Transfer Principle, and which are the most important ones of modern infinitesmal analysis.

In the fourth part (Chapters Ⅳ—Ⅸ: Nonstandard Mathematical Analysis) we present the nonstandard alternatives to some classical basic concepts, essential statements and main methods of elementary and advanced calculus, and some typical elementary applications of infinitesimals to classical analysis. The students who are the users of this book will learn how to use "downward or upward transfer principle" to yield the desired results, in which they will have an intimate knowledge of internal and external entities playing a key role in our methods as mentioned above. They can see that classical results very familiar to them are expressed in a way clearly distinct from the classical ones and may compare the effect of both methods. It will leave a deeper impression on the readers that expressiveness of the way of thinking of infinitesimal analysis is directer and more conformable to the art of invention than the classical one as Leibniz has realized in 1701. This part contains Robinson's Sequential Lemma and its applications as to show the conservation theorem of the continuity, which reminds the readers in case that one may use the Transfer Principle wrongly to the external sets. In Chapter Ⅶ, in particular, we generalize a conclusion from many similar facts involved with two sums of n infinitesimals, in which the classical Duhamel theorem is extended to H-field $^*R$.

Duhamel theorem is as follows.

If the sum of n infinitesimals(each being a function of n) approaches a limit as n increases(becomes infinite), then the same limit is approached by the sum of the infinitesimals formed by adding to each of these infinitesimals other infinitesimals which are uniformly of highter order than the ones to which they are added.

We have obtained quasi-Duhamel Principle as follows. Let $\{s_j\}=\{s_1, s_2, \cdots, s_\omega)$ and $\{s_j'\}=\{s_1', s_2', \cdots, s_\omega'\}$ be internal $*$-finite sequences in H-field $^*R$ so that $s_j'/s_j \simeq 1$ $(j=1,2,\cdots,\omega)$ and $\sum_{j=1}^{\omega}|s_j|=l$ is finite, where $\omega \in {}^*N-N$; and let $\{r_j\}=\{r_1, r_2, \cdots, r_\omega\}$ and $\{r_j'\}=\{r_1', r_2', \cdots, r_\omega'\}$ be internal $*$-finite sequences so that $r_j \simeq r_j'$ $(j=1,2,\cdots,\omega)$, where $r_j, r_j' \in {}^*R$ is finite. Then

$$\sum_{j=1}^{\omega} r_j s_j \simeq \sum_{j=1}^{\omega} r_j' s_j'.$$

This result is elementary, but as stated before, has widespread applications.It will be expected that this result is very useful for asymptotic approximation of sums and integrals. It has become a strong and convenient way by which standard infinite structures can be approximated by $*$-finite structures, e.g., we can use it to justify easily Infinite Sum Theorem due to Keisler[4] (Chapter X) and to show many classical theorems (ChapterX).

A few material found in literature is selected in the appendix. It contains enlargments, concurrent relations, exhaustiveness, comprehensiveness, κ-saturated superstructures, and the more sophisticated

Ultrapower construction of superstructures as well. It is aimed at the advanced learners who wish to take some fundamental notions as an elective course.

It is still our belief that teaching content should be fewer but better. For this reason, in chapters Ⅳ—Ⅸ we have chosen to include only a brief sketch, rather than a datailed treatment, of the classical calculus, which seems sufficient for the needs learning new methods.

Most chapters contain introductory remarks and the gist of their coetents. We offer a small amount of exercises, of which most are trivial and others give additional theoretical developments. The book is indexed in detail.

On the whole, while paying attention to the elementary facts, we haven't gone more deeply into the subject in the book. It seems desirable to present only the simplest and most fundamental results, so that we left this rather sketchy book here especially for Chinese beginners to read.

We hope that our book might serve not only as a text but also as an introduction to infinitesimal analysis for some of experts who want to understand the methods at an advanced level.

Vigour revived in infinitesimal, whose applications are now in the ascendant. we dedicate the booklet to a new era of revival of infinitesimals.

We are indebted to Prof. Li Bang-He for his help in writing this book and to Prof. W.A.J. Luxemburg for informing the recent developments being made in the applications of nonstandard analysis and presenting his excellent works to us. Our thanks are due to Prof. Edwin Hewitt who gave us friendly support. We also wish to acknowledge Professors H. J. Keisler, A. E. Hurd, P. A. Loeb, W. A. J. Luxemburg, E. A. Zakon and many other eminent savants whose valuable works are the source of our writing.

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目　　录