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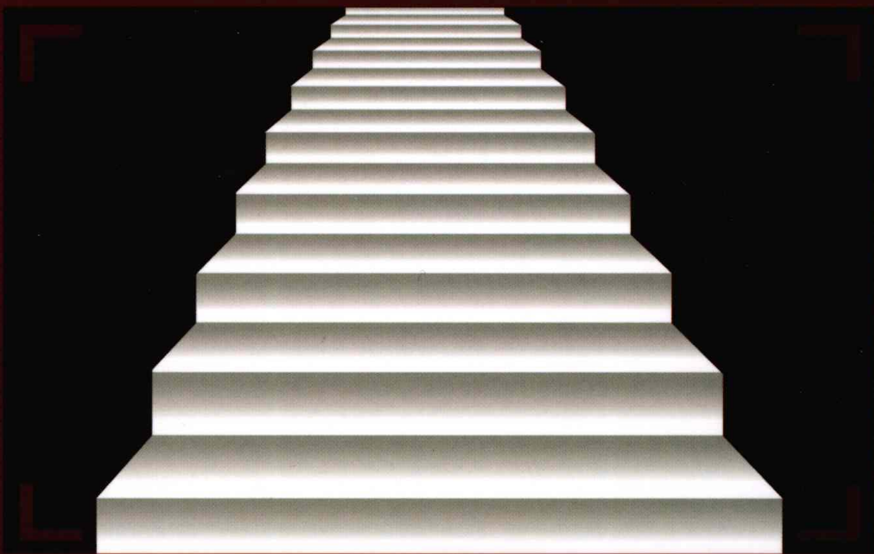
重大装备中问题驱动的应用数学理论和方法

李开泰 著



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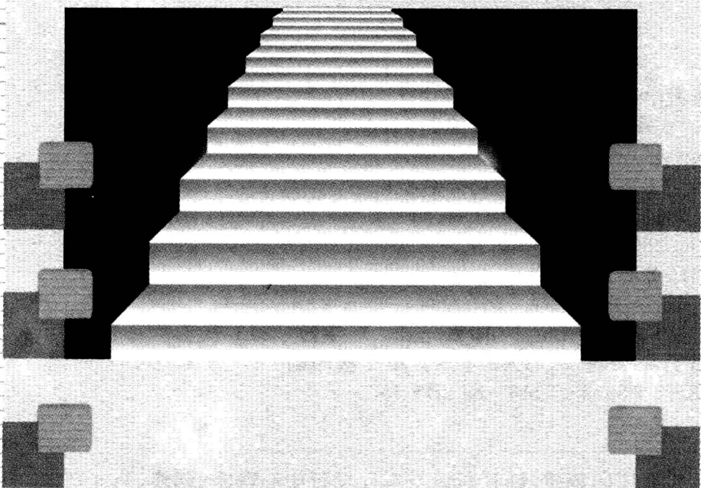

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内容提要

本书内容涉及叶轮机械内部流动、涡轮增压、柴油发动机热力气动循环、薄壁构造和壳体理论、轴承润滑、核反应堆和核电站运行、Navier-Stokes方程理论和建立在近似惯性流形基础上的算法和维数分裂算法、非线性问题分歧理论和算法等等，这些问题与工程问题有紧密联系，可以为从事相关技术问题研究的读者或在校的研究生提供参考。

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序 言

重大装备中问题驱动的应用数学研究很受国家的重视和提倡，也引起数学界和工程界极大的关注。重大装备中问题驱动的应用数学研究，是一个多学科交叉的研究领域，需要有自然科学、计算机科学和数学知识，同时还要对实际问题的背景有所了解。美国科学院院士，A. Friedmann教授撰写《工业应用数学》丛书时，走遍美国、日本和欧洲100多个工厂和科研单位。日本岩波出版社20世纪60年代出版的《应用数学》丛书的作者们，也是如此。

这本文集，收集了作者从70年代到现在有关此方面的研究成果和学术论文。其中包括叶轮机械内部流动及叶片设计、柴油机热力气动循环、薄壁构件及壳体理论方法、轴承润滑、核反应堆物理及核电站、重型机械等技术课题中提出的数学问题。反应了作者研究问题的过程是：建立数学模型，探讨数学方法和进行数学分析，最后到研制相关软件。在80年代·钱令希院士，钱伟长院士在给作者的信中就十分欣赏这种研究方法和风格，并且主张应该加以提倡。本书可对从事这方面研究或有兴趣的读者提供参考。

著者

2008. 2. 8

目　录

A. 叶轮机械、柴油发动机、壳体、轴承润滑和地球物理流动

B. 近似惯性流形和 Navier-Stokes 方法的惯性算法

C. 外部问题的边界积分方程和有限元耦合方法

D. 非线性方程分歧问题的算法

A

叶轮机械、柴油发动机、壳体、轴承润滑和地球物理流动

A Geometrical Design Method for Blade's Surface Shape and General Minimal Surface

1 Introduction

The applications of optimal shape design are uncountable. For systems governed by partial differential equations, they range from structure mechanics to electromagnetism and fluid mechanics and, more recently, to a combination of the three. Among the applications to fluids are (a) weight reduction and aeroacoustic design of engines, cars, airplanes, and even music instruments; (b) electromagnctically optimal shapcs, such as in stcalth objects with acrodynamics constrains; (c) wave cancelling in boat design; and (d) drag reduction in air and water by-static or active mechanics. In industry, optimum design is not a once and for all solution tool because engineering design is made of compromises owing to the multidisciplinary aspect of the problems and the necessity of doing multi-point constrained design.

Optimal shape design is a branch of differentiable optimization and more precisely of optimal control distributed systems, for example, blade design is a optimal shape control. As well known that classical minimal surface is to find a surface spanning on a closed Jordanian curvilinear C such that

$$J(\mathcal{S}) = \text{Aug} \inf_{S \subset F} J(S)$$

where $J(S) = \iint_S dS$ is the area of S. Turbomachinery design is driven by the need for improving performance and reliability. So far we have not found a geometric design entirely from mathematical point of view. In this paper we try to propose a principle for a fully mathematical design of the surface of the blade in a impeller.

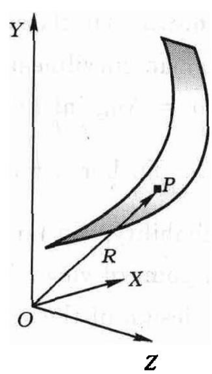
* Li Kaitai, Su Jian and Mei Liquan

This principle models a general minimal surface by minimizing a functional proposed by us. A key point in this modeling process is theoretical rationality and the realization of our design procedure. Using a tensor analysis technique we realize this procedure and obtain Eular-Lagrange equation for blade's surface which is an elliptic boundary value problem for the bland 's surface, and prove the existence of solution of minimal problem.

The content of paper organize as following. In section second we give main results. In third section, we give rotating Navier-Stokes equations in the channel in the impeller with mixed boundary condition, discuss this problem of weak and strong solution, indicate a open problem on dependence of solution upon geometry of blade's surface; In third section we establish a new coordinate system and transform the flow domain into a fixed domain in new coordinate system; present formulation of Navier-Stokes equations in term of blade's surface in the new coordinate system; In section forth we establish a new principle of geometric design of shape of blade, prove existence of solution of corresponding optimal control problem. In section fifth we present formulation of object functional and its gradient under new coordinate system, hence Eular-Lagrange equation on surface is derived. In last section, we proposal a algorithm for minimum problem and operator splinting method for rotating Navier-Stokes equations with mixed boundary.

2 Main Results

Supposc $(x^1, x^2) \in D \subset \mathscr{R}^2$ (2D-Euclidian Space). A smooth mapping $\Theta(x^1, x^2)$ is image a surface. On the other hand, suppose the (r, θ, z) is a polar cylindrical coordinate system rotating with impeller's angular velocity ω.



$(\vec{e}_r, \vec{e}_0, \vec{k})$ are the corresponding base veetors. z-axis is in the rotaing axis of the *impeller*. N is the number of blade and $\varepsilon = \pi/N$. The angle between successively two blades is $\frac{2\pi}{N}$. The flow passage of the impeller is bounded by $\partial\Omega_\varepsilon = \Gamma_{in} \cup \Gamma_{out} \cup \Gamma_t \cup \Gamma_b \cup S_+ \cup S_-$. The middle surface S of the blade is defined as the image $\vec{\mathscr{R}}$ of the closure of a domain $D \subset R^2$ where $\vec{\mathscr{R}}$: $D \to \mathscr{R}^3$ is a smooth injective mapping which can be expressed by that for any point $\vec{\mathscr{R}}(C) \in S$

$$\vec{\mathscr{R}}(x) = x^2\vec{e}_r + x^2\Theta(x^1, x^2)\vec{e}_0 + x^1\vec{k}, \ \forall x = (x^1, x^2) \in \bar{D} \qquad (2.1)$$

where $\Theta \in C^2(D, R)$ is a smooth function. $x = (x^1, x^2)$ is called a Gaussian coordinate system on S. It is easy to prove that there exists a family S_ε of surfaces with a single parameter to cover the domain Ω_ε defined by the mapping $D \to S_\varepsilon = \{\vec{R}(x^1, x^2; \xi) : \forall (x^1, x^2) \in D\}$:

$$\vec{R}(x^1, x^2; \xi) = x^2\vec{e}_r + x^2(\varepsilon\xi + \Theta(x^1, x^2))\vec{e}_\Theta + x^1\vec{k} \qquad (2.2)$$

It is clear the metric tensor $a_{\alpha\beta}$ of S_ε is homogenous and nonsingular independent of ξ and is given as follows

$$a_{\alpha\beta} = \frac{\partial\vec{R}}{\partial x^\alpha}\frac{\partial\vec{R}}{\partial x^\beta} = \delta_{\alpha\beta} + r^2\Theta_\alpha\Theta_\beta, \quad a = \det(a_{\alpha\beta}) = 1 + r^2(\Theta_1^2 + \Theta_2^2) > 0 \qquad (2.3)$$

From this we establish a curvilinear coordinate system (x^1, x^2, ξ) in $\mathscr{R}^3$.

$$(r, \theta, z) \to (x^1, x^2, \xi): x^1 = z, \quad x^2 = r, \ \xi = \varepsilon^{-1}(\theta - \Theta(x^1, x^2)) \qquad (2.4)$$

that maps the flow passage domian

$$\Omega_\varepsilon = \{\vec{R}(x^1, x^2, \xi) = x^2\vec{e}_r + x^2(\varepsilon\xi + \Theta(x^1, x^2))\vec{e}_\theta + x^1\vec{k}, \ \forall (x^1, x^2, \xi) \in \Omega\}$$

into a fixed domain in E^3:

$$\Omega = \{(x^1, x^2) \in D, -1 \leqslant \xi \leqslant 1\} \quad \text{in} \quad \mathscr{R}^3$$

which is independent of Surface S of the blade, and Jiacobian

$$J\left(\frac{\partial(r, \theta, z)}{\partial(x^1, x^2, \xi)}\right) = \varepsilon$$

The transformation is nonsingular.

Assume that $(x^{1'}, x^{2'}, x^{3'}) = (r, \theta, z)$, as well known that corresponding metric tensor of $\mathscr{R}^3$ is $(g_{1'1'} = 1, \ g_{2'2'} = r^2, \ g_{3'3'} = 1, g_{i'j'} = 0 \ \forall i' \neq j')$. According to rule of tensor transformation under coordinate transformation we have following claculation formulae

$$g_{ij} = g_{i'j'}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^{j'}}{\partial x^j}$$

Substituting(2.4) into above formulae the metric tensor of E^3 in new curvilinear coordinate system can be obtain

$$g_{\alpha\beta} = a_{\alpha\beta}, \quad g_{3\beta} = g_{\beta 3} = r^3\varepsilon\Theta_\beta, \quad g_{33} = \varepsilon^2 r^2, \quad g = \det(g_{ij}) = \varepsilon^2 r^2 \quad (2.5)$$

Through this paper we denote $\Theta_\alpha = \frac{\partial\Theta}{\partial x^\alpha}$. Its contravariant components are given by

$$g^{\alpha\beta}=\delta^{\alpha\beta},\quad g^{3\beta}=g^{\beta 3}=-\varepsilon^{-1}\Theta_{\beta},\quad g^{33}=\varepsilon^{-2}r^{-2}(1+r^{2}\mid\nabla\Theta\mid^{2})\qquad(2.6)$$

where $|\nabla\Theta|^2=\Theta_1^2+\Theta_2^2$ and $\Theta_\alpha=\dfrac{\partial\Theta}{\partial x^\alpha}$.

Model First

Theorem 1 Suppose the Θ is a blade's surface defined by (2.1). Then Θ is proposed as a solution of following elliptic boundary value problem:

$$\begin{cases}\dfrac{\partial^2}{\partial x^\lambda\partial x^\sigma}\left(K^{\alpha\beta\lambda\sigma}(w)\dfrac{\partial^2\Theta}{\partial x^\beta\partial x^\alpha}\right)+\dfrac{\partial^2}{\partial x^\lambda\partial x^\sigma}(r\hat{\Phi}^{\lambda\sigma}(w,\Theta))\\ \quad-\dfrac{\partial}{\partial x^\lambda}(r\hat{\Phi}^{\lambda}(w,\Theta))+\hat{\Phi}^0(w,\hat{w})r=0,\quad\forall(x^1,x^2)\in D\subset\mathscr{R}^2\\ \Theta=\Theta_0,\quad\dfrac{\partial\Theta}{\partial n}=\Theta_*,\quad \text{On }\partial D\end{cases}\qquad(2.7)$$

combing Navier-Stokes equations and linearized Navier-Stokes equations, where (w,p) and $(\hat{w},\hat{p})$ are solutions of compressible or incompressible rotating Navier-Stokes equations (3.1) and linearized Navier-Stokes equations (3.7) or (3.24) respectively and

$$K^{\alpha\beta\lambda\sigma}(w,\Theta)=2\mu r^3W^{\alpha\sigma}\delta^{\beta\lambda},\quad W^{\alpha\beta}=\int_{-1}^{1}w^\alpha w^\beta\mathrm{d}\xi\qquad(2.8)$$

$\tilde{\Phi}^0,\tilde{\Phi}^\lambda,\tilde{\Phi}^{\lambda\sigma}$ are defined by (4.17)(4.18) respectively.

Variational formulation associated with (2.7) is given by

$$\begin{cases}\text{Fin}\Theta\in V_\Gamma(D)=\{q\mid q\in H^2(D),q\mid_{\Gamma_0}=\Theta_0,\left.\dfrac{\partial q}{\partial n}\right|_{\Gamma_0}=0\}\text{ such that}\\ \displaystyle\iint_D[(K^{\lambda\sigma\nu\mu}(w)\Theta_{\nu\mu}+r\hat{\Phi}^{\lambda\sigma}(w,\Theta)]\eta_{\lambda\sigma}\mathrm{d}x\\ \quad+\displaystyle\iint_D[r\hat{\Phi}^\lambda(w,\Theta)\eta_\lambda+r\hat{\Phi}^0(w,\hat{w},\Theta)\eta]\mathrm{d}x=0,\quad\forall\eta\in H_0^2(D)\end{cases}\qquad(2.9)$$

Model Second

Theorem 2 Suppose the Θ is a blade's surface defined by (2.1). Then Θ is proposed as a solution of following elliptic boundary value problem:

$$\begin{cases}-(K_0(w)\tilde{\Delta}\Theta+K^{\lambda\nu}(w,\Theta)\Theta_{\nu\lambda})+F^{\nu\mu}(w)\Theta_\nu\Theta_\mu+F^\lambda(w)\Theta_\lambda+F_0(w,\Theta)=0\\ \Theta\mid_\gamma=\Theta_0\end{cases}\qquad(2.10)$$

where $K_0(w),K^{\lambda\nu}(w),F^{\nu\mu}(w),F^\lambda(w),F_0(w,\Theta)$ are defined by (5.17).

The variational formulation associated with (2.10) is given by

$$\begin{cases} \text{Find } \Theta \in H^1_\gamma(D) = \{v \mid v \in H^1(D), v = \Theta^* \text{ on} \gamma = \partial D\} \quad \text{such that} \\ \int_D \{[\Psi_0(w,p,\Theta)\eta + \Psi^\lambda(w,p,\Theta)\eta_\lambda - \mu r^2 W^\sigma \frac{\partial(\Theta_\lambda \eta_\lambda)}{\partial x^\sigma}]\varepsilon\omega r^2\} \mathrm{d}x = 0, \\ \quad \forall \eta \in H^1_0(D) \end{cases} \tag{2.11}$$

where

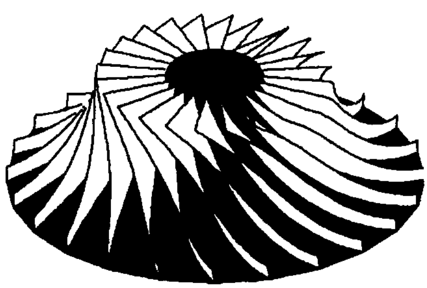
$$\Psi^\lambda(w,p,\Theta) = \Psi^\lambda_0(w,p) + \Psi^\lambda_\nu(w,p)\Theta_\nu + \Psi^\lambda_{\nu\mu}(w,p)\Theta_\nu\Theta_\mu \tag{2.12}$$

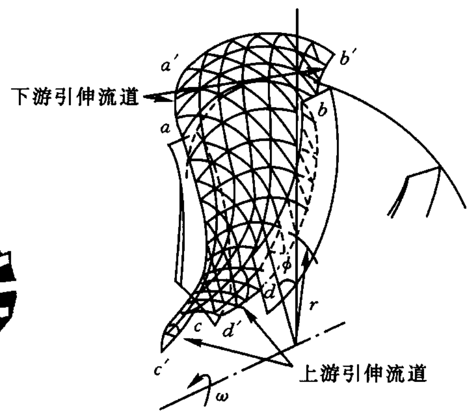
where $\Psi_0(w,p,\Theta)$, $\Psi^\lambda_0(w,p)$, $\Psi^\lambda_\nu(w,p)$, $+\Psi^\lambda_{\nu\mu}(w,p)$, are defined by (5.10) (5.11).

3 Rotating Navier-Stokes Equations With Mixed Boundary Conditions

At first, we consider the three-dimensional rotating Navier-Stokes equations in a frame rotating around the axis of a rotating imperller with an angular velociy ω:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho w) = 0 \\ \rho a = \nabla \sigma + f \\ \rho c_v(\frac{\partial T}{\partial t} + w^j \nabla_j T) - \operatorname{div}(\kappa \operatorname{grad} T) + p \operatorname{div} w - \Phi = h \\ p = p(\rho, T) \end{cases} \tag{3.1}$$





where ρ is the density of the fluid, w the velocity of the fluid, h the heat source, T the temperature, k the coefficient of heat conductivity, C_v specific heat at constant volume, and μ viscosity. Furthermore, the deformation rate tensor, stress tensor, dissipative function and viscous tensor are given by respectively:

$$
\begin{aligned}
&e_{ij}(w) = \frac{1}{2}(\nabla_i w_j + \nabla_j w_i); \quad i,j = 1,2,3 \\
&e^{ij}(w) = g^{ik}g^{jm}e_{km}(w) = \frac{1}{2}(\nabla^i w^j + \nabla^j w^i) \\
&\sigma^{ij}(w,p) = A^{ijkm}e_{km}(w), \quad \Phi = A^{ijkm}e_{ij}(w)e_{ij}(w) \\
&A^{ijkm} = \lambda g^{ij}g^{km} + \mu(g^{ik}g^{jm} + g^{im}g^{jk}), \quad \lambda = -\frac{2}{3}\mu
\end{aligned} \tag{3.2}
$$

where g_{ij} and g^{ij} are the covarant and contravariant components of the metric tensor of dimensional three Euclidian space, respectively

$$
\begin{aligned}
&\nabla_i w^j = \frac{\partial w^j}{\partial x^i} + \Gamma^j_{ik}w^k; \quad \nabla_i w_j = \frac{\partial w_j}{\partial x^i} - \Gamma^k_{ij}w_k \\
&\Gamma^i_{jk} = g^{il}\left(\frac{\partial g_{kl}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l}\right)
\end{aligned} \tag{3.3}
$$

The absolute acceleration of the fluid is given by

$$
\begin{aligned}
&a^i = \frac{\partial w^i}{\partial t} + w^j\nabla_j w^i + 2\varepsilon^{ijk}\omega_j w_k - \omega^2 r^i \\
&a = \frac{\partial w}{\partial t} + (w\nabla)w + 2\vec{\omega}\times\vec{w} + \vec{\omega}\times(\vec{\omega}\times\vec{R})
\end{aligned} \tag{3.4}
$$

where $\vec{\omega}=\omega\vec{k}$ is the vector of angular velocity, $\vec{k}$ the unite vector along axis, and $\vec{R}$ the radium vector of the fluid particle. The flow domain Ω_ε occupied by the fluids in the channel in the impleller. The boundary $\partial\Omega_\varepsilon$ of flow domain Ω_ε consists of inflow boundary Γ_{in}, out flow boundary Γ_{out}, positive blade' surface S_+, negative blade's surface S_- and top wall Γ_t and Bottom wall Γ_b:

$$
\partial\Omega_\varepsilon = \Gamma = \Gamma_{in} \cup T_{out} \cup S_- \cup S_+ \cup \Gamma_t \cup \Gamma_b \tag{3.5}
$$

Boundary conditions are given by

$$
\begin{cases}
w|_{S_-\cup S_+} = 0, \quad w|_{\Gamma_b} = 0, \quad w|_{\Gamma_t} = 0 \\
\sigma^{ij}(w,p)n_j|_{\Gamma_{in}} = g^i_{in}, \quad \sigma^{ij}(w,p)n_j|_{\Gamma_{out}} = g^i_{out}, \quad \text{Natural conditions} \\
\frac{\partial T}{\partial n} + \lambda(T - T_0) = 0 \quad \text{when } \lambda \geqslant 0 \text{ is constant}
\end{cases} \tag{3.6}
$$

If the fluid is incompressible and flow is stationary then

$$
\begin{cases}
\operatorname{div} w = 0 \\
(w\nabla)w + 2\vec{\omega}\times\vec{w} + \nabla p - \nu\operatorname{div}(e(w)) = -\vec{\omega}\times(\vec{\omega}\times\vec{R}) + f \\
w|_{\Gamma_0} = 0, \quad \Gamma_0 = S_+ \cup S_- \cup \Gamma_t \cup \Gamma_b \\
(-pn + 2\nu e(w))|_{\Gamma_{in}} = g_{in}, \quad \Gamma_1 = \Gamma_{in} \cup \Gamma_{out} \\
(-pn + 2\nu e(w))|_{\Gamma_{out}} = g_{out} \\
w|_{t=0} = w_0(x), \quad \Omega_\varepsilon
\end{cases} \tag{3.7}
$$

For the polytropic ideal gas and flow is stationary, system (3.1) turns to the conservation form

$$\begin{cases}\operatorname{div}(\rho w) = 0 \\ \operatorname{div}(\rho w \otimes w) + 2\rho\omega \times w + R\nabla(\rho T) = \mu\Delta w + (\lambda+\mu)\nabla \operatorname{div} w - \rho\omega \times (\omega \times \vec{R}) \\ \operatorname{div}[\rho(\frac{|\omega|^2}{2} + c_v T + RT) w] \\ \quad = \kappa\Delta T + \lambda \operatorname{div}(w \operatorname{div} w) + \mu \operatorname{div}[w \nabla w] + \frac{\mu}{2}\Delta |w|^2 \end{cases} \tag{3.8}$$

while for isentropic ideas gases, it turns

$$\begin{cases}\operatorname{div}(\rho w) = 0 \\ \operatorname{div}(\rho w \otimes w) + 2\rho\omega \times w + \alpha\nabla(\rho^\gamma) = 2\mu \operatorname{div}(e) + \lambda\nabla \operatorname{div} w - \rho\omega \times (\omega \times \vec{R}) \end{cases} \tag{3.9}$$

where $\gamma > 1$ is the specific heat radio and α a positive constant.

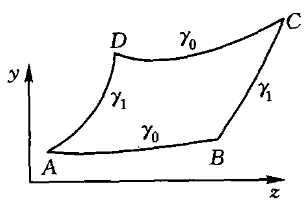
The rate of work done by the impeller and dissipative energy are given by

$$I(S, w(S)) = \iint_{S_- \cup S_+} \sigma \cdot n \cdot e_\theta \omega r \, dS, \quad J(S, w(S)) = \iiint_{\Omega_\epsilon} \Phi(w) dV \tag{3.10}$$

where e_θ is base vector along the angular direction in a polar cylindrical coordinate system.

Let us employ new coordinate system. Flow's domain Ω_ϵ is mapped into $\Omega = D \times [-1,1]$. where D is a domain in $(x^1, x^2) \in \mathcal{R}^2$ surround by four are $\widehat{AB}$, $\widehat{CD}$, $\widehat{CB}$, $\widehat{DA}$ such that

$$\partial D = \gamma_0 \cup \gamma_1, \ \gamma_0 = \widehat{AB} \cup \widehat{CD}, \ \gamma_1 = \widehat{CB} \cup \widehat{DA}$$



and there exist four positive functions $\gamma_0(z)$, $\tilde{\gamma}_0(z)$, $\gamma_1(z)$, $\tilde{\gamma}_1(z)$ such that

$$\begin{aligned} &r := x^2 = \gamma_0(x^1) = \gamma_0(z) \quad \text{on } \widehat{AB}, \quad x^2 = \tilde{\gamma}_0(x^1) \quad \text{on } \widehat{CD} \\ &r := x^2 = \gamma_1(x^1) = \gamma_1(z) \quad \text{on } \widehat{DA}, \quad x^2 = \tilde{\gamma}_1(x^1) \quad \text{on } \widehat{BC} \\ &r_0 \leqslant \gamma_0(z) \leqslant r_1 \quad \text{on } \widehat{AB}, \ r_0 \leqslant \tilde{\gamma}_0(z) \leqslant r_1 \quad \text{on } \widehat{CD} \\ &r_0 \leqslant \gamma_1(z) \leqslant r_1 \quad \text{on } \widehat{DA}, \ r_0 \leqslant \tilde{\gamma}_1(z) \leqslant r_1 \quad \text{on } \widehat{BC} \end{aligned} \tag{3.11}$$

$$\tilde{\Gamma}_{in}^{Let} = \mathcal{R}(\Gamma_{in}), \ \tilde{\Gamma}_{out} = \mathcal{R}(\Gamma_{out}), \ \tilde{\Gamma}_b = \mathcal{R}(\tilde{\Gamma}_b), \ \tilde{\Gamma}_l = \mathcal{R}(\tilde{\Gamma}_l)$$

$$\tilde{\Gamma}_1 = \tilde{\Gamma}_{out} \cup \tilde{\Gamma}_{in}, \ \tilde{\Gamma}_0 = \tilde{\Gamma}_b \cup \tilde{\Gamma}_t \cup \{\zeta = 1\} \cup \{\xi = -1\} \tag{3.12}$$

$$\partial D = \gamma_0 \cup \gamma_1, \quad \partial\Omega = \tilde{\Gamma}_0 \cup \tilde{\Gamma}_1$$
$$\gamma_0 = (D \cap \tilde{\Gamma}_b) \cup (D \cap \tilde{\Gamma}_t), \quad \gamma_1 = (D \cup \tilde{\Gamma}_{out}) \cup (D \cup \tilde{\Gamma}_{in}) \tag{3.13}$$

Where $\mathscr{R}$ Is defined by (2.1).

Let denote

$$V(\Omega) := \{v|, v \in H^1(\Omega)^3, v|_{\Gamma_0} = 0\}, H^1_\Gamma(\Omega) = \{q|, q \in H^1(\Omega) \; q|_{\Gamma_0} = 0\} \tag{3.14}$$

The variational formulations for Navier-Stokes problem (3.7) and (3.9) are respectively given by

$$\begin{cases} \text{Find}(w,p), w \in V(\Omega), p \in L^2(\Omega), \text{such that} \\ a(w,v) + 2(\omega \times w, v) + b(w,w,v) - (p, \text{div} v) = \langle F, v\rangle, \quad \forall v \in V(\Omega) \\ (q, \text{div}\omega) = 0, \quad \forall q \in L^2(\Omega) \end{cases} \tag{3.15}$$

and

$$\begin{cases} \text{Find}(w,\rho), w \in V(\Omega), \rho \in L^\gamma(\Omega), \text{such that} \\ a(w,v) + 2(\omega \times w, v) + b(\rho w, w, v) \\ \quad + (-\alpha p + \lambda \text{div} w, \text{div} v) = \langle F, v\rangle, \quad \forall v \in V(\Omega) \\ (\nabla q, \rho w)) = \langle \rho w n, q\rangle|_{\Gamma_1}, \quad \forall q \in H^1_\Gamma(\Omega) \end{cases} \tag{3.16}$$

where

$$\langle F, v\rangle := \langle f, v\rangle + \langle \tilde{g}, v\rangle_{\Gamma_1}, \quad \langle \tilde{g}, v\rangle = \langle g_{\text{in}}, v\rangle|_{\Gamma_{\text{in}}} + \langle g_{\text{out}}, v\rangle >| \Gamma_{\text{out}}$$
$$a(w,v) = \int_\Omega A^{ijkm} e_{ij}(w) e_{km}(v)\sqrt{g}\,dx d\xi \tag{3.17}$$
$$b(w,w,v) = \int_\Omega g_{km} w^j \nabla_j w^k v^m \sqrt{g}\,dx d\xi$$

Next we rewrite (3.7)(3.9) in new coordinate system. Because second kind of Christoffel symbols in new coordinate system are

$$\begin{cases} \Gamma^\alpha_{\beta\gamma} = -\gamma\delta_{2\alpha}\Theta_\beta\Theta_\gamma, \quad \Gamma^\alpha_{3\beta} = -\varepsilon r \delta_{2\alpha}\Theta_\beta \\ \Gamma^3_{\alpha\beta} = \varepsilon^{-1} r^{-1}(\delta_{2\alpha}\delta^\lambda_\beta + \delta_{2\beta}\delta^\lambda_\alpha)\Theta_\lambda + \varepsilon^{-1}\Theta_{\alpha\beta} + \varepsilon^{-1} r\Theta_2\Theta_\alpha\Theta_\beta \\ \Gamma^3_{3\alpha} = \Gamma^3_{\alpha 3} = r^{-1}\delta_{2\alpha} + r\Theta_2\Theta_\alpha, \quad \Gamma^\alpha_{33} = -\varepsilon^2 r\delta_{2\alpha}, \quad \Gamma^3_{33} = \varepsilon r\Theta_2 \end{cases} \tag{3.18}$$

the covariant derivatives of the velocity field $\nabla_i w^j = \frac{\partial w^j}{\partial w^i} + \Gamma^j_{ik} w^k$ can be expressed as

Lemma 1 Under the curvilinear coordinate system (x^1, x^2, ξ) defined by (2.4), the covariant derivatives of the velocity field can be expressed as