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Nail H. Ibragimov · Vladimir F. Kovalev

Approximate and Renormgroup Symmetries

逼近与重整化群对称

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Preface

This is an introduction to a new field in applied group analysis. Namely, the book deals with the so-called *renormalization group* (briefly *renormgroup*) symmetries considered in the framework of approximate transformation groups. The notion of the renormalization group and the renormalization group method were introduced in theoretical physics by N. N. Bogoliubov and D. V. Shirkov in 1950s. Renormgroup symmetries provide a basis for the *renormgroup algorithm* for improving solutions to boundary value problems by converting "less applicable solutions" into "more applicable solutions". The algorithm is particularly useful for improving approximate solutions given by the perturbation theory.

We present in a concise form the essence of the mathematical apparatus for computing approximate and renormgroup symmetries using the infinitesimal techniques of the modern group analysis. In order to make the book self-contained, we provide in Chapter 1 an outline of basic notions from the classical Lie group analysis of differential equations. Chapters 2 and 3 reflect new trends in the modern group analysis. Chapter 2 contains a brief discussion of approximate transformation groups. In Chapter 3 we discuss methods for calculating symmetries of integro-differential equations. Renormgroup symmetries are introduced and illustrated by several examples in Chapter 4. The renormgroup algorithm is applied to various nonlinear problems in mathematical physics in Chapter 5.

The authors wish to express their gratitude to Professor Dmitry V. Shirkov, a world leader in the study of renormalization groups in quantum field theory. Our collaboration with him over many years plays a decisive role in preparing the "physical part" (Chapters 3, 4 and 5) of the monograph. We also would like to say a word of genuine appreciation in memory of late Dr. Veniamin V. Pustovalov who made our collaboration possible and who inspired many ideas that form a ground of this book.

Nail H. Ibragimov and Vladimir F. Kovalev

Contents

Chapter 1
Lie Group Analysis in Outline

The mathematical discipline known today as the *Lie group analysis*, was originated in 1870s by an outstanding mathematician of the 19th century, Sophus Lie (1842–1899).

One of the most remarkable achievements of Lie was the discovery that the majority of known methods of integration of ordinary differential equations, which until then had seemed artificial and not intrinsically related to one another, could be derived in a unified manner using his theory. Moreover, Lie provided a classification of all ordinary differential equations in terms of their symmetry groups, and thus described the whole set of equations integrable by group-theoretical methods. These results are presented, e.g. in his textbook [10].

This chapter is aimed at discussing basic concepts from the Lie group analysis: continuous transformation groups and their generators, definition and calculation of symmetry groups of differential equations, simplest methods of integration of nonlinear equations using their symmetries. It contains also an introduction to the theory of Lie-Bäcklund transformation groups and approximate groups. The reader interested in studying more about the Lie group methods of integration of differential equations is referred to [7] and to the recent textbook [8].

1.1 Continuous point transformation groups

1.1.1 One-parameter groups

We will consider here only one-parameter groups. Let T_a be an invertible transformation depending on a real parameter a and acting in the (x,y)-plane:

$$\bar{x} = f(x,y,a), \quad \bar{y} = g(x,y,a), \tag{1.1}$$

where the functions f and g satisfy the conditions

$$f\big|_{a=0} = x, \quad g\big|_{a=0} = y. \tag{1.2}$$

The invertibility is guaranteed if one requires that the Jacobian of f, g with respect to x, y is not zero in a neighborhood of $a = 0$. Further, it is assumed that the functions f and g as well as their derivatives that appear in the subsequent discussion are continuous in x, y, a.

Definition 1.1.1. A set G of transformations (1.1) is a *one-parameter transformation group* if it contains the identical transformation $I = T_0$ and includes the inverse T_a^{-1} as well as the composition $T_b T_a$ of all its elements $T_a, T_b \in G$. By a suitable choice of the group parameter a, the main group property $T_b T_a \in G$ can be written

$$T_b T_a = T_{a+b},$$

that is

$$\begin{aligned} f\big(f(x,y,a), g(x,y,a), b\big) &= f(x,y,a+b), \\ g\big(f(x,y,a), g(x,y,a), b\big) &= g(x,y,a+b). \end{aligned} \tag{1.3}$$

In practical applications, the conditions (1.3) hold only for sufficiently small values of a and b. Then one arrives at what is called a *local one-parameter group G*. For brevity, local groups are also termed groups.

1.1.2 Infinitesimal transformations

The expansion of the functions f, g into the Taylor series in a near $a = 0$, taking into account the initial condition (1.2), yields the *infinitesimal transformation* of the group G (1.1):

$$\bar{x} \approx x + \xi(x,y)a, \quad \bar{y} \approx y + \eta(x,y)a, \tag{1.4}$$

where

$$\xi(x,y) = \frac{\partial f(x,y,a)}{\partial a}\Big|_{a=0}, \quad \eta(x,y) = \frac{\partial g(x,y,a)}{\partial a}\Big|_{a=0}. \tag{1.5}$$

The vector (ξ, η) with components (1.5) is the tangent vector (at the point (x,y)) to the curve described by the transformed points $(\bar{x}, \bar{y})$, and is therefore called the *tangent vector field* of the group G.

Example 1.1.1. The group of rotations

$$\bar{x} = x\cos a + y\sin a, \quad \bar{y} = y\cos a - x\sin a$$

has the following infinitesimal transformation:

$$\bar{x} \approx x + ya, \quad \bar{y} \approx y - xa.$$

The tangent vector field (1.5) is sometimes also written as a first-order differential operator

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{1.6}$$

which behaves as a *scalar* under an arbitrary change of variables, unlike the *vector* (ξ, η). Lie called the operator (1.6) the *symbol* of the infinitesimal transformation (1.4) or of the corresponding group G. In the current literature, the operator X (1.6) is called the *generator* of the group G of transformations (1.1).

Example 1.1.2. The generator of the group of rotations from Example 1.1.1 has the form

$$X = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{1.7}$$

1.1.3 Lie equations

Given an infinitesimal transformation (1.4), or the generator (1.6), the transformations (1.1) of the corresponding one-parameter group G are defined by solving the following equations known as the *Lie equations*:

$$\frac{\mathrm{d}f}{\mathrm{d}a} = \xi(f,g), \quad f\big|_{a=0} = x,$$
$$\frac{\mathrm{d}g}{\mathrm{d}a} = \eta(f,g), \quad g\big|_{a=0} = y. \tag{1.8}$$

We will write Eq. (1.8) also in the following equivalent form:

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}a} = \xi(\bar{x},\bar{y}), \quad \bar{x}\big|_{a=0} = x,$$
$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}a} = \eta(\bar{x},\bar{y}), \quad \bar{y}\big|_{a=0} = y. \tag{1.9}$$

Example 1.1.3. Consider the infinitesimal transformation

$$\bar{x} \approx x + ax^2, \ \bar{y} \approx y + axy.$$

The corresponding generator has the form

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \tag{1.10}$$

The Lie equations (1.9) are written as follows:

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}a} = \bar{x}^2, \qquad \bar{x}\big|_{a=0} = x,$$
$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}a} = \bar{x}\bar{y}, \qquad \bar{y}\big|_{a=0} = y.$$

The differential equations of this system are easily solved and yield

$$\bar{x} = -\frac{1}{a+C_1}, \quad \bar{y} = \frac{C_2}{a+C_1}.$$

The initial conditions imply that $C_1 = -1/x$, $C_2 = -y/x$. Consequently we arrive at the following one-parameter group of *projective transformations*:

$$\bar{x} = \frac{x}{1-ax}, \quad \bar{y} = \frac{y}{1-ax}. \tag{1.11}$$

1.1.4 Exponential map

One can represent the solution to the Lie equations (1.9) by means of infinite power series (Taylor series). Then the group transformation (1.1) for a generator X (1.6) is given by the so-called *exponential map*:

$$\bar{x} = \mathrm{e}^{aX}(x), \quad \bar{y} = \mathrm{e}^{aX}(y), \tag{1.12}$$

where

$$\mathrm{e}^{aX} = 1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots. \tag{1.13}$$

Example 1.1.4. Consider again the generator (1.10) discussed in Example 1.1.3:

$$X = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}.$$

According to (1.12)–(1.13), one has to find $X^s(x)$ and $X^s(y)$ for all $s = 1, 2, \ldots$. We calculate several terms, e.g.

$$X(x) = x^2, \ X^2(x) = X(X(x)) = X(x^2) = 2!x^3, \ X^3(x) = X(2!x^3) = 3!x^4,$$

and then make a guess:

$$X^s(x) = s!x^{s+1}.$$

The proof of the latter equation is given by induction:

$$X^{s+1}(x) = X(s!x^{s+1}) = (s+1)!x^2x^s = (s+1)!x^{s+2}.$$

Furthermore, one obtains

$$X(y) = xy, \ X^2(y) = X(xy) = yX(x) + xX(y) = yx^2 + xxy = 2!yx^2,$$
$$X^3(y) = 2![yX(x^2) + x^2X(y)] = 2![y(2x^3) + x^2xy] = 3!yx^3,$$

then makes a guess

$$X^s(y) = s!yx^s$$

and proves it by induction:

$$X^{s+1}(y) = s!X(yx^s) = s![syx^{s+1} + x^s(xy)] = (s+1)!yx^{s+1}.$$

Substitution of the above expressions in the exponential map yields

$$e^{aX}(x) = x + ax^2 + \cdots + a^s x^{s+1} + \cdots.$$

One can rewrite the right-hand side as $x(1 + ax + \cdots + a^s x^s + \cdots)$. The series in brackets is manifestly the Taylor expansion of the function $1/(1-ax)$ provided that $|ax| < 1$. Consequently,

$$\bar{x} = e^{aX}(x) = \frac{x}{1-ax}\cdot$$

Likewise, one obtains

$$e^{aX}(y) = y + ayx + a^2yx^2 + \cdots + a^s yx^s + \cdots$$
$$= y(1 + ax + \cdots + a^s x^s + \cdots).$$

Hence,

$$\bar{y} = e^{aX}(y) = \frac{y}{1-ax}\cdot$$

Thus, we have arrived at the transformations (1.11):

$$\bar{x} = \frac{x}{1-ax}, \quad \bar{y} = \frac{y}{1-ax}\cdot$$

1.1.5 Canonical variables

Theorem 1.1.1. *Every one-parameter group of transformations* (1.1) *reduces to the group of translations* $\bar{t} = t + a$, $\bar{u} = u$ *with the generator*

$$X = \frac{\partial}{\partial t}$$

by a suitable change of variables

$$t = t(x,y), \quad u = u(x,y).$$

The variables t, u are called canonical variables.

Proof. Under a change of variables the differential operator (1.6) transforms according to the formula

$$X = X(t)\frac{\partial}{\partial t} + X(u)\frac{\partial}{\partial u}\cdot \tag{1.14}$$

Therefore, canonical variables are found from the linear partial differential equations of the first order:

$$
\begin{aligned}
X(t) &\equiv \xi(x,y)\frac{\partial t(x,y)}{\partial x} + \eta(x,y)\frac{\partial t(x,y)}{\partial y} = 1,\\
X(u) &\equiv \xi(x,y)\frac{\partial u(x,y)}{\partial x} + \eta(x,y)\frac{\partial u(x,y)}{\partial y} = 0.
\end{aligned}
\tag{1.15}
$$

1.1.6 Invariants and invariant equations

Definition 1.1.2. A function $F(x,y)$ is an invariant of the group G of transformations (1.1) if $F(\bar{x},\bar{y}) = F(x,y)$, i.e.,

$$F\big(f(x,y,a), g(x,y,a)\big) = F(x,y) \tag{1.16}$$

identically in the variables x,y and the group parameter a.

Theorem 1.1.2. *A function $F(x,y)$ is an invariant of the group G if and only if it solves the following first-order linear partial differential equation:*

$$XF \equiv \xi(x,y)\frac{\partial F}{\partial x} + \eta(x,y)\frac{\partial F}{\partial y} = 0. \tag{1.17}$$

Proof. Let $F(x,y)$ be an invariant. Let us take the Taylor expansion of $F\big(f(x,y,a), g(x,y,a)\big)$ with respect to a:

$$F\big(f(x,y,a), g(x,y,a)\big) \approx F(x+a\xi, y+a\eta) \approx F(x,y) + a\Big(\xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y}\Big),$$

or

$$F(\bar{x},\bar{y}) = F(x,y) + aX(F) + o(a),$$

and substitute it in to Eq. (1.16):

$$F(x,y) + aX(F) + o(a) = F(x,y).$$

It follows that $aX(F) + o(a) = 0$, whence $X(F) = 0$, i.e., Eq. (1.17).

Conversely, let $F(x,y)$ be a solution of Eq. (1.17). Assuming that the function $F(x,y)$ is analytic and using its Taylor expansion, one can extend the exponential map (1.12) to the function $F(x,y)$ as follows:

$$F(\bar{x},\bar{y}) = \mathrm{e}^{aX}F(x,y) \overset{\text{def}}{=} \Big(1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots\Big)F(x,y).$$

Since $XF(x,y) = 0$, one has $X^2F = X(XF) = 0, \ldots, X^sF = 0$. We conclude that $F(\bar{x},\bar{y}) = F(x,y)$, i.e., Eq. (1.16) thus proving the theorem.

It follows from Theorem 1.1.2 that every one-parameter group of transformations in the plane has one independent invariant, which can be taken to be the left-hand side of any first integral $\psi(x,y) = C$ of the characteristic equation for (1.17):

$$\frac{\mathrm{d}x}{\xi(x,y)} = \frac{\mathrm{d}y}{\eta(x,y)}\cdot \tag{1.18}$$

Any other invariant F is then a function of ψ, i.e., $F(x,y) = \Phi\big(\psi(x,y)\big)$.

Example 1.1.5. Consider the group with the generator (see Exercise 1.1)

$$X = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}\cdot$$

The characteristic equation (1.18) is written as

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{2y}$$

and yields the first integral $\psi = y/x^2$. Hence, the general invariant is given by $F(x,y) = \Phi(y/x^2)$ with an arbitrary function Φ of one variable.

The concepts introduced above can be generalized in an obvious way to the multi-dimensional case by considering groups of transformations

$$\bar{x}^i = f^i(x,a), \quad i = 1,\dots,n, \tag{1.19}$$

in the n-dimensional space of points $x = (x^1,\dots,x^n)$ instead of transformations (1.1) in the (x,y)-plane. The generator of the group of transformations (1.19) is written as

$$X = \xi^i(x)\frac{\partial}{\partial x^i}, \tag{1.20}$$

where

$$\xi^i(x) = \frac{\partial f^i(x,a)}{\partial a}\Big|_{a=0}.$$

The Lie equations (1.9) become

$$\frac{\mathrm{d}\bar{x}^i}{\mathrm{d}a} = \xi^i(\bar{x}), \quad \bar{x}^i\big|_{a=0} = x^i. \tag{1.21}$$

The exponential map (1.12) is written:

$$\bar{x}^i = \mathrm{e}^{aX}(x^i), \quad i = 1,\dots,n, \tag{1.22}$$

where

$$\mathrm{e}^{aX} = 1 + \frac{a}{1!}X + \frac{a^2}{2!}X^2 + \cdots + \frac{a^s}{s!}X^s + \cdots. \tag{1.23}$$

Definition 1.1.2 of invariant functions of several variables remains the same, namely an invariant is defined by the equation $F(\bar{x}) = F(x)$. The invariant test given by Theorem 1.1.2 has the same formulation with the evident replacement of Eq. (1.17) by its n-dimensional version:

$$\sum_{i=1}^{n} \xi^i(x) \frac{\partial F}{\partial x^i} = 0. \tag{1.24}$$

Then $n-1$ functionally independent first integrals $\psi_1(x), \ldots, \psi_{n-1}(x)$ of the characteristic system for Eq. (1.24)

$$\frac{dx^1}{\xi^1(x)} = \frac{dx^2}{\xi^2(x)} = \cdots = \frac{dx^n}{\xi^n(x)} \tag{1.25}$$

provides a basis of invariants. Namely, any invariant $F(x)$ is given by

$$F(x) = \Phi\big(\psi_1(x), \ldots, \psi_{n-1}(x)\big). \tag{1.26}$$

Let us dwell on this higher-dimensional case and consider a system of equations

$$F_1(x) = 0, \ldots, F_s(x) = 0, \quad s < n. \tag{1.27}$$

We shall assume that the rank of the matrix $\|\partial F_k/\partial x^i\|$ is equal to s at all points x satisfying the system of Eqs. (1.27). The system of equations (1.27) then defines an $(n-s)$-dimensional surface M.

Definition 1.1.3. The system of Eqs. (1.27) is said to be invariant with respect to the group G of transformations (1.19) if each point x on the surface M is moved by G along the surface M, i.e., $x \in M$ implies $\bar{x} \in M$.

Theorem 1.1.3. *The system of Eqs.* (1.27) *is invariant with respect to the group G of transformations* (1.19) *with the generator X* (1.20) *if and only if*

$$XF_k\Big|_M = 0, \quad k = 1, \ldots, s. \tag{1.28}$$

1.2 Symmetries of ordinary differential equations

1.2.1 Frame of differential equations

Any differential equation has two components, namely, the *frame* and the *class of solutions* (see [7]). For example, the frame of a first-order ordinary differential equation

$$F(x, y, y') = 0$$

is the surface $F(x,y,p) = 0$ in the space of three *independent variables* x,y,p. It is obtained by replacing the first derivative y' in the differential equation $F(x,y,y') = 0$ by the variable p.

The class of solutions is defined in accordance with certain "natural" mathematical assumptions or from a physical significance of the differential equations under discussion.

The crucial step in integrating differential equations is a "simplification" of the frame by a suitable change of the variables x,y. The Lie group analysis suggests methods for simplification of the frame by using *symmetry groups* (or *admissible groups*) of differential equations.

Consider, as an example, the following Riccati equation:

$$y' + y^2 - \frac{2}{x^2} = 0. \tag{1.29}$$

Its frame is defined by the algebraic equation

$$p + y^2 - \frac{2}{x^2} = 0 \tag{1.30}$$

and is a "hyperbolic paraboloid". For the Riccati equation (1.29), a one-parameter symmetry group is provided by the following scaling transformations (non-homogeneous dilations) obtained in Sect. 1.3.1:

$$\bar{x} = xe^a, \quad \bar{y} = ye^{-a}. \tag{1.31}$$

Indeed, transformations (1.31) after the extension to the first derivative y' and the substitution $y' = p$ are written as

$$\begin{aligned} &\bar{x} = xe^a, \quad \bar{y} = ye^{-a}, \\ &\bar{p} = pe^{-2a}. \end{aligned} \tag{1.32}$$

One can readily verify that the frame of Eq. (1.30) is invariant with respect to the transformations (1.32). Let us check the infinitesimal invariance condition (1.28). The generator (1.20) of the group of transformations (1.32) has the form

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - 2p\frac{\partial}{\partial p}.$$

One can readily verify that the invariance condition is satisfied. Indeed,

$$X\left(p + y^2 - \frac{2}{x^2}\right) = -2p - 2y^2 + \frac{4}{x^2} = -2\left(p + y^2 - \frac{2}{x^2}\right),$$

and hence $X\left(p + y^2 - \frac{2}{x^2}\right)\Big|_{(1.30)} = 0$. For the transformations (1.31), the canonical variables are (Exercise 1.3)

$$t = \ln x, \quad u = xy. \tag{1.33}$$

In the canonical variables (1.33), the Riccati equation (1.29) becomes

$$u' + u^2 - u - 2 = 0 \quad (u' = \mathrm{d}u/\mathrm{d}t). \tag{1.34}$$

Its frame is obtained by substituting $u' = q$ in (1.34) and is given by the following algebraic equation:

$$q + u^2 - u - 2 = 0. \tag{1.35}$$

The left-hand side of Eq. (1.35) does not involve the variable t. Thus the curved frame (1.30) has been reduced to a cylindrical surface protracted along the t-axis. Namely it is a "parabolic cylinder". We see that, in integrating differential equations, the decisive step is that of simplifying the frame by converting it into a cylinder. For such purpose, it is sufficient to simplify the symmetry group by introducing canonical variables. In consequence, e.g. the Riccati equation (1.29) takes the integrable form (1.34).

1.2.2 Extension of group actions to derivatives

The transformation of derivatives $y', y'', \ldots$ under the action of the point transformations (1.1), regarded as a change of variables, is well-known from Calculus. It is convenient to write these transformation formulae by using the operator of *total differentiation*:

$$D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \cdots.$$

Then the transformation formulae, e.g. for the first and second derivatives are written as

$$\bar{y}' \equiv \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} = \frac{Dg}{Df} = \frac{g_x + y'g_y}{f_x + y'f_y} \equiv P(x,y,y',a), \tag{1.36}$$

$$\bar{y}'' \equiv \frac{\mathrm{d}\bar{y}'}{\mathrm{d}\bar{x}} = \frac{DP}{Df} = \frac{P_x + y'P_y + y''P_{y'}}{f_x + y'f_y}\cdot \tag{1.37}$$

Starting from the group G of point transformations (1.1) and then adding the transformation (1.36), one obtains the group $G_{(1)}$, which acts in the space of the three variables (x,y,y'). Further, by adding the transformation (1.37) one obtains the group $G_{(2)}$ acting in the space (x,y,y',y'').

Definition 1.2.1. The groups $G_{(1)}$ and $G_{(2)}$ are termed the first and second *prolongations* of G, respectively. The higher prolongations are determined similarly.

1.2.3 Generators of prolonged groups

Substituting into (1.36), (1.37) the infinitesimal transformation (1.4),