Graduate Texts in Mathematics **1**

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Harmonic Analysis and Approximation on the Unit Sphere

(球面上的调和分析与逼近)

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Preface

In recent years the research in Fourier Analysis and Approximation Theory has been extended from the classical setting, i.e., from the investigation on $\mathbb{R}^n$ and $\mathbb{T}^n$, respectively, to the investigation on manifolds. Nikol'skii has published a series of papers in this respect. The unit sphere

$$\Omega_n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}, \qquad n \geq 2,$$

is a typical manifold in $\mathbb{R}^n$. Fourier analysis on Ω_n is also called Fourier-Laplace analysis since the Laplace-Beltrami operator takes an important role in almost all problems. When $n = 2$, we have $\Omega_2 = \mathbb{T}$ and get back to the classical case.

The investigation of Fourier-Laplace analysis has already a long history. The earliest research papers may have been published at the beginning of the 20th century. A basic lecture note is the one by C. Müller. There were also important papers we would like to point out. In 1968 Berens, Butzer and Pawelke studied basic approximation and saturation problems on the sphere. In 1973 Bonami and Clerc established important theorems on the convergence of Cesàro means of Fourier-Laplace series. Since 1980 the approximation of functions on the sphere has become an even more active field with the publication of the papers by Nikol'skii, Lizorkin, Kamzolov and others.

Why do analysts pay more attention to Ω_n recently? Here are two reasons. Firstly, although classical Fourier analysis on Ω_2 is getting more and more complete, quite a few problems on Ω_n for $n \geq 3$ are still left open. Secondly, the research on the sphere is in demand by practical problems in physics, geography, seismology, etc.

Since 1990 Professor Sun Yongsheng has urged his group to do research on the sphere, and under his guidance and encouragement his students started the research on Fourier-Laplace analysis and related problems on approximation. Since then the author's research has been supported by the NSF of China for the three periods, 1992~1994 (No. 19171008), 1995~1997 (19471007), and 1998~2000 (No.19771009).

The monograph is a summary of our research until 1998.

In order to make the book self- contained we wrote a preliminary first chapter. The second chapter provides basic knowledge on Fourier-Laplace series and some early research results.

In the third chapter, we present a kind of operators which are equiconvergent with Cesàro means with the same orders. These operators are simply convolutions with Jacobi polynomials as kernels. So it is convenient to investigate convergence problems with help of these operators. Also these operators can be applied to investigate general linear summability and strong summability of Fourier-Laplace series.

In the fourth chapter, we discuss how to describe the constructive properties of functions defined on the sphere. Then in Chapter 5 we give a detailed proof of Jackson inequality. This inequality says that for any function defined on the sphere, the best approximation by polynomials is dominated by its modulus of continuity. In L^p metric this inequality was established in 1987, but for L^1 and uniform metric this problem has been keeping open until 1994 the first author of the book and Riemenschneider found a constructive proof jointly.

In the last chapter, we discuss the problem of approximation by linear means such as Riesz means, Cesàro means and de la Vallée Poussin means.

The first five chapters are written originally by the first author and have been served as the material for his seminars. The last chapter (Chapter 6) is written by the second author.

We are sincerely grateful to our supervisor Professor Sun Yong Sheng. We are also grateful to Professor H.Berens, Professor G.Brown, Professor S.Riemenschneider and Professor Z.Ditzian. With them we have been collaborating since long and our research gets their kind concern and encouragement.

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To our supervisor

Professor Sun Yongsheng

Contents

Chapter 1
Preliminaries

The materials in the first two sections of this chapter are mostly taken from the books [SW], [M] and [Sz]. As for the concept of convolution of functions defined on the sphere we follow the argument in [D]. In section 1.2 we give a very basic introduction on Jacobi polynomials and particularly on Gegenbauer polynomials. We also introduce the normalized Gegenbauer polynomials from the point of view of the harmonic analysis on the sphere. This follows the argument in [M]. For a complete discussion on Jacobi polynomials we refer the reader to Szegö's famous book [Sz].

In the section 1.3 we will extend the Jacobi polynomials to the case of complex indices and give for such polynomials some asymptotic estimates. The results of this section have been given in [BW].

1.1 Basic concepts

1.1.1 Definition of $\mathcal{P}_k^n$, $\mathcal{A}_k^n$ and $\mathcal{H}_k^n$

Let $\mathbb{N}$ denote the set of natural numbers and let $\mathbb{R}^n$ $(n \in \mathbb{N})$ be the n-dimensional Euclidean space with norm $|x| := \sqrt{x_1^2 + x_2^2 \cdots + x_n^2}$ for $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$. What we will treat with is complex valued functions defined on $\mathbb{R}^n$. As usual we denote by $\Delta = \Delta_n$ the Laplace operator, i.e.,

$$\Delta \equiv \Delta_n := \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Suppose f is a function defined on $\mathbb{R}^n$. If f satisfies the Laplace equation, i.e., $\Delta_n f = 0$, we say that f is harmonical. If for all $\alpha \in \mathbb{C}$ and all $x \in \mathbb{R}^n$, $f(\alpha x) = \alpha^k f(x)$ with $k \in \mathbb{Z}_+ = \{0\} \cup \mathbb{N}$, then we say that f is homogeneous of degree k.

Definition 1.1.1 The set of all homogeneous polynomials of n variables of degree k is written as $\mathcal{P}_k^n$. The subset of all harmonic functions in $\mathcal{P}_k^n$ is denoted by $\mathcal{A}_k^n$. An element of $\mathcal{A}_k^n$ is called *solid spherical harmonic* (see [SW, p.141]).

When $n \geq 2$ we denote by Ω_n the unit spherical surface of $\mathbb{R}^n$, i.e.,

$$\Omega_n := \{\xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n : \ \xi_1^2 + \cdots + \xi_n^2 = 1\}.$$

Definition 1.1.2 Suppose $f \in \mathcal{A}_k^n$ $(n \geq 2)$. The restriction of f on Ω_n is called spherical harmonic of n variables of degree k. The set of all such functions is denoted by $\mathcal{H}_k^n$.

It is obvious that $\mathcal{A}_k^n$ and $\mathcal{H}_k^n$ are all linear spaces over complex scalar field $\mathbb{C}$ and they have the same dimension which is denoted by a_k^n. In order to find the value of a_k^n we first consider the linear space $\mathcal{P}_k^n$. We denote the dimension of $\mathcal{P}_k^n$ by d_k^n. Let $P \in \mathcal{P}_k^n$. Then P has the following representation:

$$P(x) = \sum_{\langle\alpha\rangle = k} c_\alpha x^\alpha, \quad \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n, \ c_\alpha \in \mathbb{C}.$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\langle\alpha\rangle = \alpha_1 + \cdots + \alpha_n$. From this we can deduce by induction on the dimensions that

$$d_k^n = C_{n-1+k}^{n-1} = \frac{(n+k-1)!}{k!(n-1)!}. \tag{1.1.1}$$

We now prove the following lemma from which the relation of d_k^n and a_j^n will be derived.

Lemma 1.1.1 For any $P \in \mathcal{P}_k^n$ $(n \geq 2)$ there is a unique decomposition:

$$P(x) = \sum_{j=0}^{\ell} |x|^{2j} P_{k-2j}(x), \quad \ell = \left[\frac{k}{2}\right], \tag{1.1.2}$$

where $P_{k-2j} \in \mathcal{A}_{k-2j}^n$ $(j = 0, 1, \cdots, \ell)$, and

$$d_k^n = \sum_{j=0}^{\ell} a_{k-2j}^n. \tag{1.1.3}$$

Proof We introduce an inner product on $\mathcal{P}_k^n$ by defining

$$< P, Q >_k := P(D)\overline{Q}, \quad \forall P,\ Q \in \mathcal{P}_k^n,$$

where $P(D)$ denotes the differential operator determined by P, while D^α is defined for any $\alpha \in \mathbb{Z}_+^n$ by

$$D^\alpha := \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

It is easy to verify that $(\mathcal{P}_k^n,\ < \cdot, \cdot >_k)$ is a complete inner product space. And the convergence in $(\mathcal{P}_k^n, < \cdot, \cdot >_k)$ is just equivalent to the convergence of the coefficients of polynomials.

The cases $k = 0$ and $k = 1$ are trivial. We now assume $k \geq 2$ and define for $j \geq 2$

$$\mathcal{B}_j^n := |x|^2 \mathcal{P}_{j-2}^n = \{P(x) = |x|^2 Q(x) \ :\ Q \in \mathcal{P}_{j-2}^n\}.$$

Obviously, $\mathcal{B}_j^n$ is a closed subspace of $\mathcal{P}_j^n$ with inner product $< \cdot, \cdot >_j$. We write $\mathcal{B}_j^n$ as M temporarily. We are going to prove $\mathcal{A}_j^n$ is just the orthogonal complement of M in $\mathcal{P}_j^n$.

Suppose $P \in \mathcal{P}_j^n$ and $R(x) = |x|^2 Q(x) \in M$. If $< P, R >_j = 0$ then

$$< R, P >_j = \Delta Q(D)\overline{P} = Q(D)\overline{\Delta P} = < Q, \Delta P >_{j-2} = 0,$$

where $\Delta = \Delta_n$. If $< Q, \Delta P >_{j-2} = 0$ holds for all $Q \in \mathcal{P}_{j-2}^n$ $(j \geq 2)$ then $\Delta P = 0$. Hence we know $P \in \mathcal{A}_j^n$. This means $M^\perp \subset \mathcal{A}_j^n$.

Now let $P \in \mathcal{A}_j^n$ and $R(x) = |x|^2 Q(x) \in M$. Then

$$< P, R >_j = < Q, \Delta P >_{j-2} = 0.$$

So $M^\perp \supset \mathcal{A}_j^n$.

We have proved $M^\perp = \mathcal{A}_j^n$. Therefore by the orthogonal decomposition theorem we deduce that for any $P \in \mathcal{P}_k^n$ $(k \geq 2)$ there are unique $P_0 \in \mathcal{A}_k^n$ and unique $Q \in \mathcal{P}_{k-2}^n$ such that

$$P(x) = P_0(x) + |x|^2 Q(x) \tag{1.1.4}$$

and obviously (for $k \geq 2$)

$$d_k^n = \dim \mathcal{A}_k^n + \dim \mathcal{B}_j^n = a_k^n + d_{k-2}^n. \tag{1.1.5}$$

By using (1.1.4) and (1.1.5) repeatedly for ℓ times we get (1.1.2) and (1.1.3) and complete the proof. □

Observing that

$$d_0^n = a_0^n = 1 \quad \text{and} \quad d_1^n = a_1^n = n$$

and noticing (1.1.1) we get the following

Corollary 1.1.2 Let $n \geq 2$. Then

$$a_k^n = \begin{cases} 1, & \text{if} \quad k = 0, \\ (n+2k-2)\dfrac{(n+k-3)!}{k!(n-2)!}, & \text{if} \quad k \in \mathbb{N}. \end{cases} \tag{1.1.6}$$

Also we deduce from Lemma 1.1.1 the following useful

Corollary 1.1.3 The restriction on Ω_n of any polynomial is just a sum of some spherical harmonics.

1.1.2 $L^2(\Omega_n)$ $(n \geq 2)$

The space $L^2(\Omega_n)$ has naturally an inner product defined by

$$< f, g >:= \int_{\Omega_n} f(\xi)\overline{g(\xi)}\, d\omega_n(\xi), \tag{1.1.7}$$

where $d\omega_n$ denotes the surface element of Ω_n and the letter ξ in connection with $d\omega_n$ means that the integration is carried out with respect to ξ. If no confusion between $d\omega(\xi)$ and the Lebesgue measure on $\mathbb{R}^n$ is possible, we will write $d\omega(\xi)$ as $d\xi$ simply. With this inner product $L^2(\Omega_n)$ is a Hilbert space. And every $\mathcal{H}_k^n$ is its closed subspace. Now we are facing to prove the following

Lemma 1.1.4 If $k, \ell \in \mathbb{Z}_+$ and $k \neq \ell$ then $\mathcal{H}_k^n \perp \mathcal{H}_\ell^n$.

Proof Suppose $f = u|_{\Omega_n}$ with $u \in \mathcal{A}_k^n$ and $g = v|_{\Omega_n}$ with $v \in \mathcal{A}_\ell^n$. By Green's formula

$$0 = \int_{|x|\leq 1} (u\Delta v - v\Delta u)\, dx = \int_{\Omega_n} \left(u\frac{\partial v}{\partial r} - v\frac{\partial u}{\partial r} \right) d\omega_n$$

for a monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, we have

$$\frac{\partial x^\alpha}{\partial r} = \sum_{k=1}^{n} x_k \frac{\partial x^\alpha}{\partial x_k} =< \alpha > x^\alpha.$$

So

$$\frac{\partial v(x)}{\partial r}|_{x=\xi} = \ell g(\xi), \quad \frac{\partial u(x)}{\partial r}|_{x=\xi} = kf(\xi), \quad \xi \in \Omega_n.$$

Substituting these into the above equation we get

$$\int_{\Omega_n} (\ell - k) f(\xi) g(\xi) d\omega_n(\xi) = 0.$$

Since $\ell \neq k$ we get $< f, \overline{g} >= 0$. Noticing that $\mathcal{H}_k^n$ (and $\mathcal{A}_k^n$ also) is unchanged under scalar conjugating operation we get $< f, g >= 0$. Hence the proof is complete. □

Lemma 1.1.5 Span$(\bigcup_{k=0}^{\infty} \mathcal{H}_k^n)$ is dense in $L^2(\Omega_n)$.

Proof According to Weierstrass theorem the set of the restrictions on Ω_n of polynomials is dense in $C(\Omega_n)$. But by Corollary 1.1.3 the restriction of any polynomial on Ω_n is just a function in span$(\bigcup_{k=0}^{\infty} \mathcal{H}_k^n)$. Hence we complete the proof. □

Now we obtain the following

Theorem 1.1.6 Let $n \geq 2$. $L^2(\Omega_n) = \oplus \sum_{k=0}^{\infty} \mathcal{H}_k^n$ where "$\oplus \Sigma$" denotes orthogonal direct sum. □

By Theorem 1.1.6, for every $f \in L^2(\Omega_n)$ there holds the following expansion in L^2 metric:

$$f = \sum_{k=0}^{\infty} Y_k(f),$$

where $Y_k(f)$ denotes the projection of f onto $\mathcal{H}_k^n$.

1.1.3 The case $n = 2$

By (1.1.6) we know

$$a_0^2 = 1 \quad \text{and} \quad a_k^2 = 2 \quad \text{for} \quad k \in \mathbb{N}.$$

Assume $k \geq 1$. Let $f^k(x_1, x_2) = (x_1 + ix_2)^k$, $(x_1, x_2) \in \mathbb{R}^2$. It is obvious that $f^k \in \mathcal{A}_k^2$ and also $\Re f^k$, $\mathrm{Im}\, f^k$ are all in $\mathcal{A}_k^2$. We denote their restrictions on Ω_2 by y_1^k and y_2^k respectively. Then y_1^k, $y_2^k \in \mathcal{H}_k^2$ and by usual polar coordinate we have for $(\xi_1, \xi_2) = (\cos\theta, \sin\theta) \in \Omega_2$

$$y_1^k(\xi_1, \xi_2) = \cos k\theta, \quad y_2^k(\xi_1, \xi_2) = \sin k\theta.$$

And we have

$$\begin{aligned} < y_1^k, y_2^k > &= \int_{\Omega_2} y_1^k(\xi) y_2^k(\xi)\, d\omega_2(\xi) \\ &= \int_0^{2\pi} \cos k\theta \sin k\theta \, d\theta = 0. \end{aligned}$$

So $\frac{1}{\sqrt{\pi}} y_1^k$ and $\frac{1}{\sqrt{\pi}} y_2^k$ consist an orthonormal base of $\mathcal{H}_k^2$ $(k \in \mathbb{N})$.

Therefore the system

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos k\theta, \frac{1}{\sqrt{\pi}} \sin k\theta : k \in \mathbb{N} \right\}$$

is a polar coordinate form of an orthonormal base of $L^2(\Omega_2)$. Of course we are very familiar with this well-known trigonometric system. So the investigation of $L^2(\Omega_2)$ coincides just with $L^2(0, 2\pi)$.

1.1.4 Zonal harmonics

Let $n \geq 2$. We denote $\mathcal{H}_k^n$ by $\mathcal{H}_k$, $\mathcal{A}_k^n$ by $\mathcal{A}_k$ and a_k^n by a_k simply. We know $\mathcal{H}_k$ is a subspace of $L^2(\Omega_n)$. Let $\mathcal{H}_k^*$ be the conjugate space of $\mathcal{H}_k$. Then by Riesz representation theorem, we know, for any $L \in \mathcal{H}_k^*$, there is a unique $g \in \mathcal{H}_k$ such that

$$L(f) = < f, g >$$

holds for all $f \in \mathcal{H}_k$.

Given $\xi \in \Omega_n$, define a functional L on $\mathcal{H}_k$ by

$$L(f) := f(\xi), \quad \forall f \in \mathcal{H}_k. \tag{1.1.8}$$

Then $L \in \mathcal{H}_k^*$. Hence we get the following

Definition 1.1.3 Let $\xi \in \Omega_n$ and let L be defined by (1.1.8). The unique $z \in \mathcal{H}_k$ which satisfies

$$L(f) = < f, z > = f(\xi), \quad \forall f \in \mathcal{H}_k$$

is called *Zonal harmonic* with pole ξ of n variables of degree k and is written as $z = z_\xi^{n,k}$ or z_ξ^k when it is not necessary to indicate n.

The following characterizations of the zonal harmonics are important.

Lemma 1.1.7 Let z_ξ^k be zonal harmonic. Then
(i) for any orthonormal base $(y_1, \cdots, y_{a_k})$ of $\mathcal{H}_k$

$$z_\xi^k(\eta) = \sum_{j=1}^{a_k} \overline{y_j(\xi)} y_j(\eta), \quad \forall \eta \in \Omega_n; \tag{1.1.9}$$

(ii) for any ξ and η in Ω_n

$$\overline{z_\xi^k(\eta)} = z_\xi^k(\eta) = z_\eta^k(\xi); \tag{1.1.10}$$

(iii) for any rotation ρ of $\mathbb{R}^n$ and for any ξ, η in Ω_n

$$z_{\rho\xi}^k(\rho\eta) = z_\xi^k(\eta). \tag{1.1.11}$$

Proof By the definition of zonal harmonic we have

$$\overline{< y_j, z_\xi^k >} = \overline{y_j(\xi)} = < z_\xi^k, y_j >, \quad j = 1, 2, \cdots, a_k.$$

Hence

$$z_\xi^k = \sum_{j=1}^{a_k} < z_\xi^k, y_j > y_j = \sum_{j=1}^{a_k} \overline{y_j(\xi)} y_j.$$

That is conclusion (i).

Since the representation (1.1.9) is independent of the choice of the base $\{y_1, \cdots, y_{a_k}\}$ we get the conclusion (ii) by choosing a special base consisting of real functions.

Now suppose $f \in \mathcal{H}_k$. Define g by $g(\xi) = f(\rho^{-1}\xi)$ for all $\xi \in \Omega_n$. It is easy to verify that $g \in \mathcal{H}_k$ and we have

$$\begin{aligned} \int_{\Omega_n} f(\eta) z_{\rho\xi}^k(\rho\eta) d\omega_n(\eta) &= \int_{\Omega_n} f(\rho^{-1}\eta) z_{\rho\xi}^k(\eta) d\omega_n(\eta) \\ &= < g, z_{\rho\xi}^k > \\ &= g(\rho\xi) = f(\xi) \\ &= \int_{\Omega_n} f(\eta) z_\xi^k(\eta) d\omega_n(\eta). \end{aligned}$$

So by the arbitrariness of f in $\mathcal{H}_k$ we get the conclusion (iii). The proof is complete. □

Since we frequently concern rotation, we make use of the notation SO(n) , as usual, to denote the group of rotations on $\mathbb{R}^n$, i.e., the special orthogonal group. The action of any $\rho \in \mathrm{SO}(n)$ on a point $x \in \mathbb{R}^n$ is denoted by ρx as usual.

We now introduce the concept of parallel of latitude which is related to zonal harmonics closely.

Definition 1.1.4 Let $n \geq 3$, $e \in \Omega_n$, $\xi \in \Omega_n$ and $e \neq \pm\xi$. The set

$$\mathcal{L}_e(\xi) := \{\eta : \eta = \rho\, \xi,\ \rho \in \mathrm{SO}(n),\ \rho e = e\}$$

is called the *parallel of latitude* with pole e and passing ξ.

If it is not necessary to point out which point is in the parallel with pole e, then we write the parallel as $\mathcal{L}_e$ simply.