许宝騄文集

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内 容 简 介

许宝騄先生是我国已故的著名数学家．他对我国概率论和数理统计学的发展作出了重要的贡献，有很高的学术成就，共有论文三十九篇，其中包括解放后在国内发表的十三篇．本文集从中选了十九篇论文，基本上反映了各个时期许先生在各个领域的工作．

读者对象是高等学校数学系高年级学生、研究生和数学工作者．

许宝騄文集

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许　宝　騄
(1910—1970)

前　言

许宝騄教授是我国著名数学家、北京大学一级教授、中国科学院学部委员，曾任第四届全国政协委员．他祖籍浙江杭州，一九一〇年九月一日生于北京，在“文化革命”期间曾受到林彪、“四人帮”极左路线的残酷迫害．于一九七〇年十二月十八日在北京不幸病逝，终生未婚．今年是许先生诞生七十周年，我们编辑出版这部《许宝騄论文集》，表示对他的深切怀念．

许先生最初曾研读化学（1928—1930，燕京大学化学系）．一九三〇年入清华大学数学系改学数学，以后获清华大学理学士学位．于清华毕业后，在北京大学数学系担任了两年助教．一九三六年经考试公费赴英留学，为伦敦大学研究生，同时也在剑桥大学学习．赴英后的第三年开始在伦敦大学兼任讲师．一九三八年取得哲学博士学位．一九四〇年又获科学博士学位．同年回国，受聘为北京大学教授，在昆明西南联大任教．一九四五年他应邀先后在美国加州（贝克利）大学、哥伦比亚大学和北卡罗林纳大学讲学，任访问教授．一九四七年十月，他回到北京，迎接解放．从此以后，他一直在北京大学任教．

许先生是一位爱国科学家．他热爱中国共产党，热爱社会主义祖国．解放以前，他就和党的地下工作者有接触．一九四五年在重庆参加了地下革命组织“民主革命同盟”，反对蒋介石的反动统治．北京解放以后，他致电美国数学界的同事，表示他对于北京的解放十分高兴．同时，以巨大的热情投入新中国文教科学事业的建设工作．一九五〇年，他久病初愈，就开始讲课，带研究生，主持讨论班．除此之外，还利用业余时间帮助同志们学俄文．一九五一年他再次病倒住院，组织上曾多次建议他出国疗养，但都被婉言谢绝了．他决心留在国内，带病坚持工作．这一时期，他校阅了一些俄文翻译的基础课教材．一九五六年以后，他的行动已很不方便，但是，直到“文化革命”以前，他的教学、科研活动一刻也没有停止．在他卧室的外间，挂起了一块黑板，他在那里继续给高年级学生、研究生和青年教师讲课．同时，在病床上坚持科学研究．在林彪、“四人帮”横行全国的那些日子里，许先生遭到残酷迫害；他在身患重病又备受折磨的极为困难的条件下，仍未放弃工作．约在去世前一个多月，他完成了关于实验设计和代数编码之间的联系的那篇论文．他逝世以后，人们在他的床前看到了一叠叠的算草．为了发展祖国的科学，为了培养年轻一代的数学工作者，他忘我地劳动，直至生命的最后一息．这种献身祖国、献身科学的精神，是值得我们永远学习的．

许先生对我国概率论、数理统计两学科的发展作出了重要贡献．一九五六年周恩来总理主持制订的我国第一个科学规划中，把概率统计列为数学学科的重点方向之一．这一年，根据教育部的统一安排，一些兄弟院校的教师、学生共四十多人集中到北大，在许宝騄教授的主持下从事概率统计的学习和研究．同时，北京大学成立了由许先生任主任的全国第一个概率统计教研室．到“文化革命”以前，北大的概率统计专门化共培养了八届学生．许先生主持制订了专门化学生的培养计划和教学大纲，指导过五届毕业论文．一些专门化课程的教材也是以他的讲稿为蓝本整理而成的．许先生领导的讨论班，不仅有北大的

教师和学生，还有兄弟院校的进修教师、学生和中国科学院数学研究所的同志参加．讨论班的内容曾涉及极限定理、马氏过程、多元分析、实验设计、次序统计量、过程统计、判决函数和组合数学等各个专题．为了更好地发展我国概率统计学科，许先生曾提出过一些很好的意见．他曾经建议在北大数学系设立统计实验室．一九六三年，他又提出筹办一个概率统计的学术刊物，并表示如果经费有困难，他本人可以贡献一部分．可惜，由于种种原因，他的这些愿望没有实现．令人愤慨的是，在林彪、“四人帮”的煽动下，这些宝贵意见还被当作许先生“走资本主义道路”的材料来横加批判．这使我们今天想起来还感到十分痛心．

许先生对待教学工作极为认真．他作风严谨，讲课条理清晰．他的《抽象积分论》、《矩阵论》和《点集拓扑》的讲稿都有鲜明的特点，对我们的教学工作至今仍有很大的参考价值．一九五六年以后，他的主要精力用在培养青年教师和高年级学生方面．他循循善诱，能使不同程度的人在自己的原有水平上都得到提高．在讨论班上，一个专题报告之后，他常把报告的内容重新整理，提出新的方法和进一步研究的方向，把青年人直接引向科研前线．他十分注意鼓励青年人在讨论班上大胆发表自己的意见，也十分注意鼓励青年人进行科研实践，告诫大家不要做那种眼高手低、最后一事无成的人．青年人取得了成果，他总是十分高兴．他不仅认真审查论文的原稿，而且在经过认真的思考以后再和作者在一起进行深入的讨论，提出中肯的意见，力求把结果向前推进．特别应该指出的是，在教学、科研的正常秩序受到冲击的情况下，他也从不随波逐流，总是在力所能及的范围内勤勤恳恳地工作，帮助青年人树立良好的学风．正因为如此，他受到他的学生们的一致的尊敬和爱戴．

许先生有很高的学术成就．他对中国古典文学很有修养，他精通英、法、德、俄等多种语言，对概率统计学科有杰出的贡献．为了纪念他，一九七九年美国《数理统计年鉴》邀请了一些著名学者、许先生过去的同事和学生撰文介绍他的生平，高度评价了他在概率论和数理统计两方面的工作．

许先生共有学术论文三十九篇，其中解放以后在国内发表的十三篇．由于篇幅的限制，我们只从他的全部论文中选出了十九篇，编成这部文集出版．选编的原则一是力求反映许先生在各个时期的工作，一是尽可能反映许先生在数学各个领域的工作．许先生论文集的出版，将使他的重要研究成果得以更好流传，并将激励我们在党中央的领导下树立信心，埋头苦干，为实现四个现代化，迅速提高中华民族的科学文化水平而努力奋斗．

江泽涵　段学复
北京大学数学系
一九八〇年二月

许宝騄先生论文目录(全部)

1. On the Limit of a Sequence of Point Sets. Bull. Amer. Math. Soc., **41** (1935), 502—504.
2. Contribution to the Theory of "Student's" t-Test as Applied to the Problem of Two Samples. *Stat. Res. Mem.*, **2** (1938), 1—24.

3. On the Best Unbiassed Quadratic Estimate of the Variance. *Stat. Res. Mem.,* **2** (1938), 91—104.
4. Notes on Hotelling's Generalized T. *Ann. Math. Stat.,* **9** (1938), 231—243.
5. A New Proof of the Joint Product Moment Distribution. *Proc. Cambridge Philoso. Soc.,* **35** (1939), 336—338.
6. On the Distribution of Roots of Certain Determinantal Equations. *Ann. Eugenics,* **9** (1939), 250—258.
7. On n-Fold Iterated Limits. *Chinese Math. Soc.,* **2** (1940), 40—53.
8. An Algebraic Derivation of the Distribution of Rectangular Coordinates. *Proc. Edinburgh Math. Soc.,* **6** (1940), 185—198.
9. On Generalized Analysis of Variance. *Biometrika,* **31** (1940), 221—237.
10. On the Limiting Distribution of Roots of a Determinantal Equation. *J. London Math. Soc.* **16** (1941), 183—194.
11. On the Limiting Distribution of the Canonical Correlations. *Biometrika,* **32** (1941), 38—45.
12. Analysis of Variance from the Power Function Standpoint. *Biometrika,* **32** (1941), 62—69.
13. Canonical Reduction of the General Regression Problem. *Ann. Eugenics,* **11** (1941), 42—46.
14. On the Problem of Rank and the Limiting Distribution of Fisher's Test Function. *Ann. Eugenics,* Vol. **11** (1941), 39—41.
15. The Limiting Distribution of a General Class of Statistics. *Acad. Sinica Science Record,* **1** (1942), 37—41.
16. Some Simple Facts about the Separation of Degrees of Freedom in Factorial Experiments. *Sankhya,* **6** (1943), 253—254.
17. The Approximate Distributions of the Mean and Variance of a Sample of Independent Variables. *Ann. Math. Stat.,* **16** (1945), 1—29.
18. On the Approximate Distribution of Ratios. *Ann. Math. Stat.,* **16** (1945), 204—210.
19. On the Power Functions of the E^2-Test and the T^2-Test. *Ann. Math. Stat.,* **16** (1945), 278—286.
20. On a Factorization of Pseudo-Orthogonal Matrices. *Quart. J. Math. Oxford Ser.,* **17** (1946), 162—165.
21. Sur un Théorème de Probabilités Dénombrables. (with K. L. Chung). *C. R. Acad. Sci. Paris,* **223** (1946), 467—469.
22. On the Asymptotic Distributions of Certain Statistics Used in Testing the Independence between Successive Observations from a Normal Population. *Ann. Math. Stat.,* **17** (1946), 350—354.
23. Complete Convergence and the Law of Large Numbers. (with H. Robbins). *Proc. Nat. Acad. Sci. (U.S.A.),* **33** (1947), 25—31.
24. The Limiting Distribution of Functions of Sample Means and Application to Testing Hypotheses. *Proc. Berkeley Symp. Math. Stat. Prob.* (1949), 359—402.
25. 绝对动差与特征函数. 数学学报,第一卷,第三期(1951), 257—280.
26. 独立机变数的化简系数. 科学记录,第四卷,第三期(1952), 197—200.
27. On Symmetric, Orthgonal and Skem-Symmetric Matrices. *Proc. Edinburgh Math. Soc. Ser. 2,* **10**, Part 1 (1953), 37—44.

28. 在零点的邻区内彼此相等的特征函数. 数学学报,第四卷,第一期(1954), 21—32.
29. 论矩阵的一种变换. 数学学报,第五卷,第三期 (1955), 333—346.
30. 论矩阵偶的一种变换. 北京大学学报(自然科学),第一卷, 第一期 (1955), 1—16.
31. 一个厄密方阵及一个对称或斜称方阵的联合变换. 北京大学学报(自然科学), 第三卷, 第二期 (1957), 167—209.
32. L 族内的分布函数的绝对连续性.北京大学学报(自然科学),第四卷,第二期(1958), 145— 150.
33. 欧氏空间上纯间断的时齐马可夫过程的概率转移函数的可微性. 北京大学学报(自然科学),第四卷,第三期 (1958), 257—270.
34. 一个不是 L_3 方案的 M_3 结合方案,其 $s=6$. 数学学报,第十四卷,第一期 (1964), 177—178.
35. A General Weak Limit Theorem for Independent Distributions. Appendix III in Limit Distributions of Sums of Independent Random Variables, revised edition, (Eds. B. V. Gnedenko and A. N. Kolmogorov), (Translated by K. L. Chung), (1968).
36. 变叙的项的极限分布(许先生所主持讨论班的成果, 以"班成"名义发表). 数学学报,第十四卷. 第五期 (1964), 694—714.
37. 部分平衡不完全区组设计(许先生所主持讨论班的成果,以"班成"名义发表). 数学进展,第七卷.第三期 (1964), 240—281.
38. 随机矩阵的重合性质(合作者: 陈家鼎、郑忠国). 北京大学学报 (自然科学), 第一期 (1979), 21—47.
39. BIB 矩阵与单纯码及正交码(遗稿). 许宝騄论文集(1981),211—225.

目　　录

CONTRIBUTION TO THE THEORY OF "STUDENT'S" t-TEST AS APPLIED TO THE PROBLEM OF TWO SAMPLES[1)]

I. Introduction

Consider two unconnected normal populations, Π_1 and Π_2, with means ξ_1, ξ_2 and variances σ_1^2, σ_2^2 respectively, and suppose that a sample is drawn from each of the populations. The general question whether the populations are alike or not presents itself in three different phases which we may describe as follows in the terminology commonly employed in testing statistical hypotheses:

(A) to test the hypothesis H_1, that $\xi_1 = \xi_2$ and $\sigma_1^2 = \sigma_2^2$, against the set of all the alternative hypotheses which specify nothing except that $\xi_1 \neq \xi_2$ or $\sigma_1^2 \neq \sigma_2^2$ or both;

(B) to test the hypothesis H_2, that $\xi_1 = \xi_2$ while the alternatives specify nothing except that $\xi_1 \neq \xi_2$;

(C) in connexion with Π_1 and Π_2 it is assumed as given that σ_1^2 and σ_2^2 have the same (though unknown) value, to test the hypothesis H_3, that $\xi_1 = \xi_2$, against the set of alternatives that $\xi_1 \neq \xi_2$.

It is to be noticed that if either H_1 or H_3 is true, then the populations Π_1 and Π_2 will be identical, while if H_2 is the hypothesis under test, we are only interested in whether or not the means are the same, neglecting altogether any difference that may exist between the population variances.*

R. A. Fisher (1925) was the first to prove that whenever the two normal populations are identical a certain statistic calculated from the two samples will follow exactly the t-distribution of "Student". The square of this t we shall later denote by u_1 and consider it in detail. Thus, without discriminating the hypotheses H_1 and H_3, Fisher introduced the t-test as the criterion for the identity of Π_1 and Π_2. As is well known, the t-test used for this purpose is an exact one in the sense that the distribution of t is entirely known under the assumption that H_1 is true. Later on R. Sato (1937), using the method of treating composite hypotheses due to Neyman (1935), obtained some general results which involve the fact that, if the hypothesis to be tested is H_3, then the t-test is the uniformly most powerful of all the unbiassed exact tests that can possibly be constructed. It follows that the t-test completely answers the question (C).

Fisher (1934, p. 122) expressed the opinion that for the hypothesis H_1 the same t-test should be used, while Neyman and Pearson (1930) suggested that the λ-test might be used. The properties of λ have been studied by P. V. Sukhatme (1935), who has also prepared a table of 5% significance level of λ. These are the attempts to

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1) 原载：*Stat. Res. Mem.*, **2**(1938), 1—24.

* Such a question might arise, for example, if we wished to know whether one variety of sugar beet had on the average a higher sugar content than a second, differences in the variability among individual plants being immaterial.

answer the question (A).

As to the testing of the hypothesis H_2, criteria have been suggested by Fisher (1936) and M. S. Bartlett (cf. Welch, 1938), the former based his criterion on fiducial arguments and the latter devised some tests that have the advantage of being exact.

The purpose of this paper is to give a thorough survey of the possibilities that certain test criteria, namely, $u_1 (= t^2)$ and u_2 (which is closely related to u_1), may be advantageously used for H_1 or H_2. The whole discussion is, of course, dependent on a knowledge of the distributions of u_1 and u_2, and thence of their power functions.* in terms of the parameters ξ_i and σ_i. Unfortunately most of the results obtained below are of a rather negative character. The only positive conclusion arrived at is that, in the case where the sample sizes are the same, the test $u_1 (= u_2$ here) may be safely used in testing the hypothesis H_2. The same problem has been taken up by B. L. Welch (1938); the results obtained by him by an approximate method are in full accordance with the general results described below.

II. THE DISTRIBUTION OF A STATISTIC u

(1) We have assumed that two independent normal populations have means ζ_1, ζ_2 and variances σ_1^2, σ_2^2 respectively. Let samples of $n_j (j = 1, 2)$ individuals be drawn, giving rise to $\bar{x}_j$, the sample means, and $\Sigma_j = \Sigma(x - \bar{x}_j)^2$, the total variations about the means within the samples, for $j = 1, 2$. Denote by

$$\delta = \xi_1 - \xi_2, \qquad \theta = \sigma_1^2/\sigma_2^2, \tag{1}$$

$$\sigma^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}, \qquad \lambda = \frac{\delta^2}{2\sigma^2}. \tag{2}$$

Thus H_1 is the hypothesis that $\lambda = 0$ and $\theta = 1$, H_2 is the hypothesis that $\lambda = 0$, while the hypothesis H_3 states that $\lambda = 0$ under the *a priori* assumption that $\theta = 1$.

Instead of considering t we shall always deal with its square. Accordingly we put,

$$u = (\bar{x}_1 - \bar{x}_2)^2/(A_1\Sigma_1 + A_2\Sigma_2), \tag{3}$$

where A_1 and A_2 are some known positive constants. Define also B_1 and B_2 by

$$B_j\sigma^2 = A_j\sigma_j^2 \quad (j = 1,\ 2). \tag{4}$$

Later on we shall identify u with some particular tests by giving special values to A_1 and A_2.

A method of finding the distribution of the ratio of two independent random variates is due to H. Cramér (1937, p. 46). If $z = X_1/X_2$ and if X_1 and X_2 are independent, then under certain restrictions the elementary probability law of z is given by

$$p(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \phi_1(t)\phi_2'(-tz)dt, \tag{5}$$

where $\phi_j(t)$ is the characteristic function of $X_j (j = 1, 2)$ and $\phi_2'(-tz)$ denotes the function

$$\frac{d}{dy}\phi_2(y)\Big|_{y=-tz}.$$

* For a definition of the power function of a test and a discussion of some of its properties see Neyman and Pearson (1936, 1938).

We shall use the formula (5) to get the distribution of u, without, however, stopping to prove the legitimacy of doing so; the reader can easily satisfy himself in this respect.

The numerator and denominator of u have respectively the characteristic functions

$$\phi_1(t) = e^{-\lambda} e^{\lambda/(1-2\sigma^2 it)}(1-2\sigma^2 it)^{-\frac{1}{2}}, \tag{6}$$

$$\begin{aligned}\phi_2(t) &= (1-2\sigma_1^2 A_1 it)^{-\frac{1}{2}(n_1-1)}(1-2\sigma_2^2 A_2 it)^{-\frac{1}{2}(n_2-1)} \\ &= (1-2B_1\sigma^2 it)^{-\frac{1}{2}(n_1-1)}(1-2B_2\sigma^2 it)^{-\frac{1}{2}(n_2-1)}.\end{aligned} \tag{7}$$

Hence

$$\begin{aligned}\phi_2'(-tu) = \sigma^2 i[&B_1(n_1-1)(1+2B_1\sigma^2 itu)^{-\frac{1}{2}(n_1+1)}(1+2B_2\sigma^2 itu)^{-\frac{1}{2}(n_2-1)} \\ &+ B_2(n_2-1)(1+2B_1\sigma^2 itu)^{-\frac{1}{2}(n_1-1)}(1+2B_2\sigma^2 itu)^{-\frac{1}{2}(n_2+1)}].\end{aligned} \tag{8}$$

Substituting (6) and (8) into (5) and making the transformation $2\sigma^2 t = \tau$ in the resulting integral, we get the elementary probability law of u:

$$p(u) = B_1(n_1-1)J_1 + B_2(n_2-1)J_2, \tag{9}$$

where

$$J_1 = \frac{e^{-\lambda}}{4\pi}\int_{-\infty}^{\infty} e^{\lambda/(1-i\tau)}(1-i\tau)^{-\frac{1}{2}}(1+B_1 i\tau u)^{-\frac{1}{2}(n_1+1)}(1+B_2 i\tau u)^{-\frac{1}{2}(n_2-1)}d\tau, \tag{10}$$

$$J_2 = \frac{e^{-\lambda}}{4\pi}\int_{-\infty}^{\infty} e^{\lambda/(1-i\tau)}(1-i\tau)^{-\frac{1}{2}}(1+B_1 i\tau u)^{-\frac{1}{2}(n_1-1)}(1+B_2 i\tau u)^{-\frac{1}{2}(n_2+1)}d\tau. \tag{11}$$

(2) *The expansion of $p(u)$ into an infinite series.* If X_1 and X_2 are distributed independently as χ^2's with f_1 and f_2 degrees of freedom respectively, then it is well known that their ratio $z = X_1/X_2$ will have the elementary probability law

$$p(z) = \frac{1}{B\left(\frac{1}{2}f_1, \frac{1}{2}f_2\right)} z^{\frac{1}{2}f_1-1}(1+z)^{-\frac{1}{2}(f_1+f_2)} \quad \text{for } z > 0, \tag{12}$$

while $p(z) = 0$ for $z < 0$. On the other hand, we may apply (5) to find $p(z)$. Thus, if $\phi_j(t)$ is the characteristic function of X_j for $j = 1, 2$, then

$$\phi_1(t) = (1-2it)^{-\frac{1}{2}f_1}, \tag{13}$$

$$\phi_2'(-tz) = f_2 i(1+2itz)^{-\frac{1}{2}f_2}. \tag{14}$$

Substituting (13) and (14) into (5) and putting $2t = \tau$ in the resulting integral, we get

$$p(z) = \frac{f_2}{4\pi}\int_{-\infty}^{\infty}(1-i\tau)^{-\frac{1}{2}f_1}(1+i\tau z)^{-\frac{1}{2}f_2-1}d\tau. \tag{15}$$

The right-hand sides of (12) and (15) must be identical; we have therefore

$$\begin{aligned}\frac{f_2}{4\pi}\int_{-\infty}^{\infty}(1-i\tau)^{-\frac{1}{2}f_1}(1+i\tau z)^{-\frac{1}{2}f_2-1}d\tau &= \frac{1}{B\left(\frac{1}{2}f_1, \frac{1}{2}f_2\right)} z^{\frac{1}{2}f_1-1}(1+z)^{-\frac{1}{2}(f_1+f_2)} \text{ for } z \geqslant 0 \\ &= 0 \text{ for } z < 0,\end{aligned} \tag{16}$$

identically in z, f_1 and f_2, at least when f_1 and f_2 are positive integers.

With the help of (16) we can evaluate the integral, say,

$$\Psi(z, f_2) = \frac{f_2 e^{-\lambda}}{4\pi}\int_{-\infty}^{\infty} e^{\lambda/(1-i\tau)}(1-i\tau)^{-\frac{1}{2}}(1+i\tau z)^{-\frac{1}{2}f_2-1}d\tau, \tag{17}$$

where f_2 is some positive integer. In fact, the integrand of (17) is $\sum_{k=0}^{\infty}\phi_k(\tau)$, where

$$\phi_k(\tau) = \frac{\lambda^k}{k!}(1-i\tau)^{-k-\frac{1}{2}}(1+i\tau z)^{-\frac{1}{2}f_2-1}. \tag{18}$$

We have

$$\left|\sum_{k=0}^{n}\phi_k(\tau)\right| \leqslant \sum_{k=0}^{\infty}|\phi_k(\tau)| = \sum_{k=0}^{\infty}\frac{\lambda^k}{k!}(1+\tau^2)^{-\frac{1}{2}k-\frac{1}{4}}(1+\tau^2z^2)^{-\frac{1}{4}f_2-\frac{1}{2}}$$

$$= e^{\lambda/(1+\tau^2)}(1+\tau^2)^{-\frac{1}{4}}(1+\tau^2z^2)^{-\frac{1}{4}f_2-\frac{1}{2}} \leqslant e^{\lambda}(1+\tau^2)^{-\frac{1}{4}}(1+\tau^2z^2)^{-\frac{1}{4}f_2-\frac{1}{2}}. \tag{19}$$

The last written function being summable over $(-\infty, \infty)$ whenever $z \neq 0$, the series $\sum_{k=0}^{\infty}\phi_k(\tau)$ may be integrated term-by-term;* thus we have

$$\Psi(z, f_2) = \frac{f_2e^{-\lambda}}{4\pi}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\int_{-\infty}^{\infty}(1-i\tau)^{-k-\frac{1}{2}}(1+i\tau z)^{-\frac{1}{2}f_2-1}d\tau. \tag{20}$$

Applying (16) to (20) with $f_1 = 2k+1$, we get

$$\Psi(z, f_2) = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}\frac{1}{B\left(k+\frac{1}{2}, \frac{1}{2}f_2\right)}z^{k-\frac{1}{2}}(1+z)^{-\frac{1}{2}(f_2+1)-k} \quad \text{for } z>0$$

$$= 0 \quad \text{for } z<0, \tag{21}$$

identically in z and f_2, at least when f_2 is a positive integer.

We are now in the position to evaluate J_1 and J_2. Suppose $0 < B_1 \leqslant B_2$, then from the identity

$$1 + B_2 i\tau u = \frac{B_2}{B_1}(1+B_1 i\tau u)\left(1 - \frac{1-B_1/B_2}{1+B_1 i\tau u}\right) \tag{22}$$

we have

$$(1+B_2 i\tau u)^{-\frac{1}{2}(n_2-1)} = \left(\frac{B_1}{B_2}\right)^{\frac{1}{2}(n_2-1)}(1+B_1 i\tau u)^{-\frac{1}{2}(n_2-1)}$$

$$\times\sum_{h=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}(n_2-1)+h\right)}{\Gamma\left(\frac{1}{2}(n_2-1)\right)h!}\left(1-\frac{B_1}{B_2}\right)^h(1+B_1 i\tau u)^{-h}. \tag{23}$$

Substituting (23) into (10) we get

$$J_1 = \frac{e^{-\lambda}}{4\pi}\left(\frac{B_1}{B_2}\right)^{\frac{1}{2}(n_2-1)}\int_{-\infty}^{\infty}\sum_{h=0}^{\infty}\phi_h(\tau)d\tau, \tag{24}$$

where

$$\phi_h(\tau) = \frac{\Gamma\left(\frac{1}{2}(n_2-1)+h\right)}{\Gamma\left(\frac{1}{2}(n_2-1)\right)h!}\left(1-\frac{B_1}{B_2}\right)^h e^{\lambda/(1-i\tau)}(1-i\tau)^{-\frac{1}{2}}(1+B_1 i\tau u)^{-\frac{1}{2}N-h}, \tag{25}$$

$$N = n_1 + n_2. \tag{26}$$

In order to show that we can reverse the order of integration and summation in (24), we have

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* By the Lebesgue theorem of term-by-term integration; cf. e.g. Kestelman (1937, p. 137, Theorem 214).

$$\left|\sum_{h=0}^{n} \phi_h(\tau)\right| \leqslant \sum_{h=0}^{\infty} |\phi_h(\tau)| = e^{\lambda/(1+\tau^2)}(1+\tau^2)^{-\frac{1}{4}}(1+B_1^2\tau^2u^2)^{-\frac{1}{2}N}$$

$$\times \sum_{h=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_2-1)+h\right)}{\Gamma\left(\frac{1}{2}(n_2-1)\right)h!}\left(1-\frac{B_1}{B_2}\right)^h (1+B_1^2\tau^2u^2)^{-\frac{1}{2}h}$$

$$\leqslant e^{\lambda}(1+\tau^2)^{-\frac{1}{4}}(1+B_1^2\tau^2u^2)^{-\frac{1}{2}N}\left[1-\frac{1-B_1/B_2}{1+B_1^2\tau^2u^2}\right]^{-\frac{1}{2}(n_2-1)}$$

$$\leqslant e^{\lambda}\left(\frac{B_1}{B_2}\right)^{-\frac{1}{2}(n_2-1)}(1+\tau^2)^{-\frac{1}{4}}(1+B_1^2\tau^2u^2)^{-\frac{1}{2}N}.$$

Provided $u \neq 0$, the last written function is summable over $(-\infty, \infty)$; therefore*

$$J_1 = \frac{e^{-\lambda}}{4\pi}\left(\frac{B_1}{B_2}\right)^{\frac{1}{2}(n_2-1)} \sum_{h=0}^{\infty}\int_{-\infty}^{\infty} \phi_h(\tau)d\tau$$

$$= \frac{e^{-\lambda}}{4\pi}\left(\frac{B_1}{B_2}\right)^{\frac{1}{2}(n_2-1)} \sum_{h=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_2-1)+h\right)}{\Gamma\left(\frac{1}{2}(n_2-1)\right)h!}\left(1-\frac{B_1}{B_2}\right)^h$$

$$\times \int_{-\infty}^{\infty} e^{\lambda/(1-i\tau)}(1-i\tau)^{-\frac{1}{2}}(1+B_1i\tau u)^{-\frac{1}{2}N-h}d\tau. \tag{27}$$

Comparing (20) with (27) we get immediately

$$J_1 = \left(\frac{B_1}{B_2}\right)^{\frac{1}{2}(n_2-1)} \sum_{h=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_2-1)+h\right)}{\Gamma\left(\frac{1}{2}(n_2-1)\right)h!}\left(1-\frac{B_1}{B_2}\right)^h \frac{1}{N+2h-2}\,\Psi(B_1u, N+2h-2). \tag{28}$$

Similarly we have

$$J_2 = \left(\frac{B_1}{B_2}\right)^{\frac{1}{2}(n_2+1)} \sum_{h=0}^{\infty} \frac{\Gamma\left(\frac{1}{2}(n_2+1)+h\right)}{\Gamma\left(\frac{1}{2}(n_2+1)\right)h!}\left(1-\frac{B_1}{B_2}\right)^h \frac{1}{N+2h-2}\,\Psi(B_1u, N+2h-2). \tag{29}$$

From (9), (28) and (29) it follows after some reduction that

$$p(u) = B_1^{\frac{1}{2}(n_2+1)} B_2^{-\frac{1}{2}(n_2-1)} \sum_{h=0}^{\infty} C_h\left(1-\frac{B_1}{B_2}\right)^h \Psi(B_1u, N+2h-2). \tag{30}$$

wherein we have put

$$C_h = \frac{\Gamma\left(\frac{1}{2}(n_2-1)+h\right)}{\Gamma\left(\frac{1}{2}(n_2-1)\right)h!}. \tag{31}$$

Formula (30) is true only when $B_1 \leqslant B_2$. In the case $B_2 \leqslant B_1$ we have only to interchange the indices 1 and 2. Thus

$$p(u) = B_1^{-\frac{1}{2}(n_1-1)} B_2^{\frac{1}{2}(n_1+1)} \sum_{h=0}^{\infty} C'_h\left(1-\frac{B_2}{B_1}\right)^h \Psi(B_2u, N+2h-2) \tag{32}$$

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* See footnote on p. 4.

for $B_2 \leqslant B_1$, where

$$C'_h = \frac{\Gamma\left(\frac{1}{2}(n_1-1)+h\right)}{\Gamma\left(\frac{1}{2}(n_1-1)\right)h!}. \tag{33}$$

With the functions Ψ given by (21) the right-hand side of (30) is a repeated series of positive terms. It is therefore identical with the double series

$$p(u) = \left(\frac{B_1}{B_2}\right)^{\frac{1}{2}(n_2-1)} e^{-\lambda} \sum_{h,k=0}^{\infty} C_h \left(1-\frac{B_1}{B_2}\right)^h \frac{\lambda^k}{k!} p_{hk}(u) \quad \text{for } B_1 \leqslant B_2, \tag{34}$$

where

$$p_{hk}(u) = \frac{B_1^{k+\frac{1}{2}} u^{k-\frac{1}{2}}}{B\left(k+\frac{1}{2}, \frac{1}{2}(N-2)+h\right)} (1+B_1 u)^{-\frac{1}{2}(N-1)-h-k}. \tag{35}$$

Similarly we have

$$p(u) = \left(\frac{B_2}{B_1}\right)^{\frac{1}{2}(n_1-1)} e^{-\lambda} \sum_{h,k=0}^{\infty} C'_h \left(1-\frac{B_2}{B_1}\right)^h \frac{\lambda^k}{k!} p'_{hk}(u) \quad \text{for } B_2 \leqslant B_1, \tag{36}$$

where

$$p'_{hk}(u) = \frac{B_2^{k+\frac{1}{2}} u^{k-\frac{1}{2}}}{B\left(k+\frac{1}{2}, \frac{1}{2}(N-2)+h\right)} (1+B_2 u)^{-\frac{1}{2}(N-1)-h-k}. \tag{37}$$

If, as is permissible, we sum the double series (34) first with respect to h and then with respect to k, we shall also obtain

$$p(u) = \left(\frac{B_1}{B_2}\right)^{\frac{1}{2}(n_2-1)} e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{1}{B\left(k+\frac{1}{2}, \frac{1}{2}(N-2)\right)} B_1^{k+\frac{1}{2}} u^{k-\frac{1}{2}} (1+B_1 u)^{-\frac{1}{2}(N-1)-k}$$
$$\times F\left(\frac{1}{2}(n_2-1), \frac{1}{2}(N-1)+k, \frac{1}{2}(N-2), \frac{1-B_1/B_2}{1+B_1 u}\right), \tag{38}$$

and a similar formula from (36).

Let us indicate rapidly a few particular cases of $p(u)$.

Case (i) $\lambda = 0$: Denoting the corresponding distribution by $p_0(u)$ we have

$$p_0(u) = \left(\frac{B_1}{B_2}\right)^{\frac{1}{2}(n_2-1)} \sum_{h=0}^{\infty} C_h \left(1-\frac{B_1}{B_2}\right)^h p_h(u), \tag{39}$$

where

$$p_h(u) = \frac{B_1^{\frac{1}{2}} u^{-\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{1}{2}(N-2)+h\right)} (1+B_1 u)^{-\frac{1}{2}(N-1)-h}. \tag{40}$$

Case (ii) $B_1 = B_2 = B$: Using the notation $p(u|B)$ in this connexion we have

$$p(u|B) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \bar{p}_k(u), \tag{41}$$

where

$$\bar{p}_k(u) = \frac{B^{k+\frac{1}{2}} u^{k-\frac{1}{2}}}{B\left(k+\frac{1}{2}, \frac{1}{2}(N-2)\right)} (1+Bu)^{-\frac{1}{2}(N-1)-k}. \tag{42}$$

Case (iii) $\lambda = 0$ and $B_1 = B_2 = B$:

$$p_0(u|B) = \frac{B^{\frac{1}{2}} u^{-\frac{1}{2}}}{B\left(\frac{1}{2}, \frac{1}{2}(N-2)\right)} (1 + Bu)^{-\frac{1}{2}(N-1)}. \tag{43}$$

Case (iv) $B_1 = 0$: Putting $B_1 = 0$ in (9) and (11) and remembering (21) we get

$$p(u|B_1 = 0) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{1}{B\left(k + \frac{1}{2}, \frac{1}{2}(n_2 - 1)\right)} B_2^{k+\frac{1}{2}} u^{k-\frac{1}{2}} (1 + B_2 u)^{-\frac{1}{2}n_2 - k}. \tag{44}$$

(44) is also the limiting form of (38) as B_1 tends to the limit zero.

Case (v) $\lambda = 0$ and $B_1 = 0$:

$$p_0(u|B_1 = 0) = \frac{1}{B\left(\frac{1}{2}, \frac{1}{2}(n_2 - 1)\right)} B_2^{\frac{1}{2}} u^{-\frac{1}{2}} (1 + B_2 u)^{-\frac{1}{2}n_2}. \tag{45}$$

(3) ***Finite expressions of*** $p_0(u)$. If both n_1 and n_2 are odd, then the integrands of (10) and (11) are uniform functions of τ in the upper half plane having only poles at the points $\tau = i/B_1 u$ and $i/B_2 u$. Accordingly the integrals in (10) and (11) can be evaluated by the calculus of residues, resulting in finite analytic formulae for J_1 and J_2. In the case $\lambda = 0$ we shall show how to get the finite expression for $p_0(u)$ without, however, resorting to the method of residues.

Putting $\lambda = 0$ and $n_j = 2m_j + 1$ $(j = 1, 2)$ in (10) and (11) we get

$$J_1 = \frac{1}{4\pi} \int_{-\infty}^{\infty} (1 - i\tau)^{-\frac{1}{2}} (1 + B_1 i\tau u)^{-(m_1+1)} (1 + B_2 i\tau u)^{-m_2} d\tau, \tag{46}$$

$$J_2 = \frac{1}{4\pi} \int_{-\infty}^{\infty} (1 - i\tau)^{-\frac{1}{2}} (1 + B_1 i\tau u)^{-m_1} (1 + B_2 i\tau u)^{-(m_2+1)} d\tau. \tag{47}$$

If we denote by

$$I_{r_1 r_2} = \frac{1}{4\pi} \int_{-\infty}^{\infty} (1 - i\tau)^{-\frac{1}{2}} (1 + B_1 i\tau u)^{-r_1} (1 + B_2 i\tau u)^{-r_2} d\tau, \tag{48}$$

then from (9), (46) and (47) we have

$$p_0(u) = 2m_1 B_1 I_{m_1+1, m_2} + 2m_2 B_2 I_{m_1, m_2+1}. \tag{49}$$

From the identity (22) follows easily the recurrence formula

$$I_{r_1 r_2} = \frac{1}{\Delta} (B_1 I_{r_1, r_2 - 1} - B_2 I_{r_1 - 1, r_2}), \tag{50}$$

where

$$\Delta = B_1 - B_2 \tag{51}$$

provided $\Delta \neq 0$. On the other hand, from (16) we have, on putting $f_1 = 1$ and $f_2 = 2(r-1)$,

$$\left.\begin{aligned} I_{r0} &= \frac{1}{2(r-1)B\left(\frac{1}{2}, r-1\right)} (B_1 u)^{-\frac{1}{2}} (1 + B_1 u)^{\frac{1}{2} - r}, \\ I_{0r} &= \frac{1}{2(r-1)B\left(\frac{1}{2}, r-1\right)} (B_2 u)^{-\frac{1}{2}} (1 + B_2 u)^{\frac{1}{2} - r}. \end{aligned}\right\} \tag{52}$$

With the help of (49), (50) and (52) we can compute $p_0(u)$ by repeated steps.

Below we give, for $n_1 = 9$ and $n_2 = 5$, the scheme of computation for $p_0(u)$:

$$p_0(u) = 8B_1I_{52} + 4B_2I_{43} \qquad \text{(i)}$$

$$= \frac{1}{\Delta}(8B_1^2I_{51} - 8B_1B_2I_{42} + 4B_1B_2I_{42} - 4B_2^2I_{33}) \qquad \text{(ii)}$$

$$= \frac{1}{\Delta}(8B_1^2I_{51} - 4B_1B_2I_{42} - 4B_2^2I_{33}) \qquad \text{(iii)}$$

$$= \frac{1}{\Delta^2}(\underline{8B_1^3I_{50}} - 8B_1^2B_2I_{41} - 4B_1^2B_2I_{41} + 4B_1B_2^2I_{32} - 4B_1B_2^2I_{32} + 4B_2^3I_{23}) \qquad \text{(iv)}$$

$$= \frac{1}{\Delta^2}(-12B_1^2B_2I_{41} + 4B_2^3I_{23}) \qquad \text{(v)}$$

$$= \frac{1}{\Delta^3}(-\underline{12B_1^3B_2I_{40}} + 12B_1^2B_2^2I_{31} + 4B_1B_2^3I_{22} - 4B_2^4I_{13}) \qquad \text{(vi)}$$

$$= \frac{1}{\Delta^4}(\underline{12B_1^3B_2^2I_{30}} - 12B_1^2B_2^3I_{21} + 4B_1^2B_2^3I_{21} - 4B_1B_2^4I_{12} - 4B_1B_2^4I_{12} + \underline{4B_2^5I_{03}}) \qquad \text{(vii)}$$

$$= \frac{1}{\Delta^4}(-8B_1^2B_2^3I_{21} - 8B_1B_2^4I_{12}) \qquad \text{(viii)}$$

$$= \frac{1}{\Delta^5}(-\underline{8B_1^3B_2^3I_{20}} + 8B_1^2B_2^4I_{11} - 8B_1^2B_2^4I_{11} + \underline{8B_1B_2^5I_{02}}). \qquad \text{(ix)}$$

Explanation. To obtain the lines (ii), (iv), (vi), (vii) and (ix) apply formula (50) to the lines immediately preceding; the lines (iii), (v) and (viii) are obtained by collecting similar terms in the lines immediately preceding. An underlined term cannot be reduced any further and has therefore not been copied in the subsequent lines, but is, of course, a term of the final expression of $p_0(u)$.

Thus, collecting all the underlined terms, we have, for $n_1 = 9$ and $n_2 = 5$,

$$p_0(u) = 8\frac{B_1^3}{\Delta^2}I_{50} - 12\frac{B_1^3B_2}{\Delta^3}I_{40} + 12\frac{B_1^3B_2^2}{\Delta^4}I_{30} - 8\frac{B_1^3B_2^3}{\Delta^5}I_{20} + 8\frac{B_1B_2^5}{\Delta^5}I_{02} + 4\frac{B_2^5}{\Delta^4}I_{03}. \qquad (53)$$

The formula (53) is true only when $B_1 \neq B_2$. If B_2 is held fixed and B_1 allowed to approach B_2 as its limit, the right-hand side of (53) will, of course, tend to the limit given in (43), namely

$$\frac{B_2^{\frac{1}{2}}u^{-\frac{1}{2}}}{B\left(\frac{1}{2}, 6\right)}(1 + B_2u)^{-\frac{13}{2}},$$

but in a rather complicated manner. Even the way in which the integral of (53), taken between the limits 0 and ∞, becomes unity is intricate enough. Thus

$$\int_0^\infty p_0(u)du = \frac{B_1^2}{\Delta^2} - \frac{2B_1^2B_2}{\Delta^3} + \frac{3B_1^2B_2^2}{\Delta^4} - \frac{4B_1^2B_2^3}{\Delta^5} + \frac{4B_1B_2^4}{\Delta^5} + \frac{B_2^4}{\Delta^4}$$

and

$$-\frac{4B_1^2B_2^3}{\Delta^5} + \frac{4B_1B_2^4}{\Delta^5} = -\frac{4B_1B_2^3}{\Delta^4},$$

$$-\frac{4B_1B_2^3}{\Delta^4} + \frac{3B_1^2B_2^2}{\Delta^4} + \frac{B_2^4}{\Delta^4} = \frac{3B_1B_2^2}{\Delta^3} - \frac{B_2^3}{\Delta^3},$$

$$\frac{3B_1B_2^2}{\Delta^3}-\frac{B_2^3}{\Delta^3}-\frac{2B_1^2B_2}{\Delta^3}=-\frac{2B_1B_2}{\Delta^2}+\frac{B_2^2}{\Delta^3},$$

$$-\frac{2B_1B_2}{\Delta^2}+\frac{B_2^2}{\Delta^2}+\frac{B_1^2}{\Delta^2}=\frac{(B_1-B_2)^2}{\Delta^2}=1.$$

In the manner described above we have computed $p_0(u)$ for several particular values of n_1 and n_2. The results are given below, the expressions I_{r0} and I_{0r} being defined in (52).

$n_1=5,\ \ n_2=15$:

$$p_0(u)=\frac{B_1^8}{\Delta^7}\left(4I_{30}-14\frac{B_2}{\Delta}I_{20}\right)+\frac{B_2^3}{\Delta^2}\sum_{r=2}^{8}(9-r)(2r-2)\left(\frac{B_1}{\Delta}\right)^{8-r}I_{0r}; \qquad (54)$$

$n_1=3,\ \ n_2=5$:

$$p_0(u)=2\frac{B_1^3}{\Delta^2}I_{20}-\frac{B_2^2}{\Delta}\left(2\frac{B_1}{\Delta}I_{02}+4I_{03}\right); \qquad (55)$$

$n_1=n_2=3$:

$$p_0(u)=\frac{2}{\Delta}(B_1^2I_{20}-B_2^2I_{02}); \qquad (56)$$

$n_1=n_2=5$:

$$p_0(u)=\frac{4}{\Delta^2}(B_1^3I_{30}+B_2^3I_{03})-\frac{4B_1B_2}{\Delta^3}(B_1^2I_{20}-B_2^2I_{02}); \qquad (57)$$

$n_1=n_2=7$:

$$p_0(u)=\frac{6}{\Delta^3}(B_1^4I_{40}-B_2^4I_{04})-\frac{12B_1B_2}{\Delta^4}(B_1^3I_{30}+B_2^3I_{03})+\frac{12B_1^2B_2^2}{\Delta^5}(B_1^2I_{20}-B_2^2I_{02}); \qquad (58)$$

$n_1=n_2=9$:

$$p_0(u)=\frac{8}{\Delta^4}(B_1^5I_{50}+B_2^5I_{05})-\frac{24B_1B_2}{\Delta^5}(B_1^4I_{40}+B_2^4I_{04})$$
$$+\frac{40B_1^2B_2^2}{\Delta^6}(B_1^3I_{30}-B_2^3I_{03})-\frac{40B_1^3B_2^3}{\Delta^7}(B_1^2I_{20}-B_2^2I_{02}); \qquad (59)$$

$n_1=n_2=11$:

$$p_0(u)=\frac{10}{\Delta^5}(B_1^6I_{60}-B_2^6I_{06})-\frac{40B_1B_2}{\Delta^6}(B_1^5I_{50}+B_2^5I_{05})+\frac{90B_1^2B_2^2}{\Delta^7}(B_1^4I_{40}-B_2^4I_{04})$$
$$-\frac{140B_1^3B_2^3}{\Delta^8}(B_1^3I_{30}+B_2^3I_{03})+\frac{140B_1^4B_2^4}{\Delta^9}(B_1^2I_{20}-B_2^2I_{02}). \qquad (60)$$

III. The power function of u

(1) From the definitions (1), (2) and (4) of the quantities θ, σ^2 and the Bi we have

$$B_1=\frac{n_1n_2A_1\theta}{n_1+n_2\theta},\qquad B_2=\frac{n_1n_2A_2}{n_1+n_2\theta}, \qquad (61)$$

so that

$$B_1/B_2=\rho\theta,\quad \text{where}\quad \rho=A_1/A_2. \qquad (62)$$