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Preface

Historically in the study of entire or meromorphic functions, one often considered the function and its derivative at the same time. Thus, as early as in the time of Laguerre, he compared the genus of an entire function $f(z)$ with that of its derivative $f'(z)$. In the proof of the Picard-Borel theorem on entire functions, Borel used the fact that an entire function $f(z)$ and its derivative $f'(z)$ have the same order. In Nevanlinna's theory of meromorphic functions, the inequality

$$m(r, \frac{f'}{f}) < 24 + 3\log^+ \left|\frac{1}{f(0)}\right| + 2\log^+ \frac{1}{r}$$

$$+4\log^+ R + 3\log^+ \frac{1}{R-r} + 4\log^+ T(R, f)$$

plays an important role. Here both $f(z)$ and $f'(z)$ appear in the form of the logarithmic derivative. In the Ahlfors-Shimizu characteristic function

$$T_0(r, f) = \int_0^r \frac{A(t, f)}{t} \mathrm{d}t,$$

$$A(r, f) = \frac{1}{\pi}\int_0^r \rho \mathrm{d}\rho \int_0^{2\pi} \frac{|f'(\rho e^{i\theta}|}{1 + |f(\rho e^{i\theta})|^2} \mathrm{d}\theta,$$

$f(z)$ and $f'(z)$ also both appear in the form of the spherical derivative.

Similar phenomenon occurs in the study of normal families. For a long time Motal believed that the following theorem is true:

Let $\mathcal{F}$ be a family of holomorphic functions in a domain D.

If each function $f(z)$ of the family $\mathcal{F}$ satisfies in D the condition

$$f(z) \neq 0, \ \ f^{(k)}(z) \neq 1,$$

then the family $\mathcal{F}$ is normal in D, where $k > 0$ is a fixed integer independent of $f(z)$.

This theorem was first proved by Miranda. The method used by Miranda was based upon Nevenlinna's theory of meromorphic functions and a theorem of Borel on nondecreasing functions. Next by using a different method, the method of Wiman-Valiron, Valiron generalized the above theorem by replacing the condition $f^{(k)}(z) \neq 1$ in D with the condition

$$\sum_{j=0}^{k} a_j f^{(k-j)}(z) \neq 1$$

in D, where the coefficients $a_j (j = 0, 1, \cdots, k)$ are constants with $a_0 \neq 0$. The expression on the left side of this inequality is a simple example of what we call a linear differential polynomial of $f(z)$.

In Nevenlinna's theory of meromorphic functions, the second fundamental inequality may be written in the following form:

$$\sum_{j=1}^{q} m(r, \frac{1}{f - a_j}) + m(r, f)$$
$$\leq 2T(r, f) - N_1(r) + S(r, f),$$

with

$$N_1(r) = 2N(r, f) - N(r, f') + N(r, \frac{1}{f'}),$$

where $a_j \, (j = 1, 2, \cdots, q, q \geq 2)$ are q distinct finite constants. Nevenlinna proved the problem of extending the second fundamental inequality by replacing the constants a_j ($j =$

$1, 2, \cdots, q$) with meromorphic functions $\phi_j(z)\,(j = 1, 2, \cdots, q)$, whose growth in certain sense is less rapid than that of $f(z)$. Such meromorphic functions $\phi_j(z)\,(j = 1, 2, \cdots, q)$ are now called small functions with respect to $f(z)$. Chuang nearly solved the problem by introducing a Wronskian of the form $W(f, \psi_1, \cdots, \psi_p)$, where $\psi_k(z)\,(k = 1, 2, \cdots, p)$ are p linearly independent small functions with respect to $f(z)$. The problem was finally solved by Steinmetz by using a Wronskian of the form $W(B_1, B_2, \cdots, B_m, b_1 f, b_2 f, \cdots, b_n f)$, where $B_h\,(h = 1, 2, \cdots, m)$ and $b_k\,(k = 1, 2, \cdots, n)$ are certain small functions with respect to $f(z)$. Here $W(f, \psi_1, \cdots, \psi_p)$ and $W(B_1, B_2, \cdots, B_m, b_1 f, b_2 f, \cdots, b_n f)$ are also examples of differential polynomials of $f(z)$, in which the first is linear and the second is nonlinear.

In general, a differential polynomial of a meromorphic function $f(z)$ is a polynomial of $f(z)$ and derivatives of $f^{(j)}(z)\,(j = 1, 2, \cdots, q)$ up to a certain order q, which possesses the following form:

$$P(z) = P(f, f', \cdots, f^{(q)})$$

$$= \sum_{k=1}^{n} a_k(z)(f)^{s_{k0}}(f')^{s_{k1}} \cdots (f^{(q)})^{s_{kq}},$$

where the coefficients $a_k(z)\,(k = 1, 2, \cdots, n)$ are small functions with respect to $f(z)$ and where $s_{kj}\,(k = 1, 2, \cdots, n;\ j = 0, 1, \cdots, q)$ are nonnegative integers. The simplest examples of differential polynomials of $f(z)$ are the successive derivatives $f^{(j)}(z)\,(j = 1, 2, \cdots)$.

Finally it should be pointed out that this book is not a systematic study of differential polynomials of meromorphic functions. It is rather a collection of research work, namely due to the author, involving differential polynomials of meromorphic functions.

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Contents

Chapter I
Inversion of a System of Differential Polynomials

1 Inversion Formulas

Consider n systems of meromorphic functions

$$(\phi_{j1}, \phi_{j2}, \cdots, \phi_{jn}) \, (j = 1, \cdots, n) \tag{1.1}$$

in a domain D with the determinant

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} \not\equiv 0. \tag{1.2}$$

Then there exists an unique system of linear differential equations

$$w_i' + a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n = 0 \, (i = 1, \cdots, n), \tag{1.3}$$

where the coefficients $a_{ik}(i, k = 1, 2, \cdots, n)$ are meromorphic functions in D, such that

$$w_1 = \phi_{j1}, \, w_2 = \phi_{j2}, \, w_n = \phi_{jn} \, (j = 1, 2, \cdots, n) \tag{1.4}$$

are n solutions of the system of differential equations (1.3). In fact, in order that this requirement holds, it is necessary and sufficient that for each $i(1 \leq i \leq n)$, we have identically

$$\phi'_{j1} + a_{i1}\phi_{j1} + a_{i2}\phi_{j2} + \cdots + a_{in}\phi_{jn} = 0 \,(j = 1, 2, \cdots, n). \quad (1.5)$$

By the condition (1.2), the system of equations (1.5) has an unique system of solutions $a_{i1}, a_{i2}, \cdots, a_{in}$, which are meromorphic functions in D.

Now let $f_1, f_2, \cdots, f_n$ be a system of meromorphic functions in D and $F_1, F_2, \cdots, F_n$ be the corresponding system of meromorphic functions in D defined by

$$F_i = f'_i + a_{i1}f_1 + a_{i2}f_2 + \cdots + a_{in}f_n \,(i = 1, \cdots, n). \quad (1.6)$$

Consider the determinant

$$\begin{vmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{vmatrix} \not\equiv 0. \quad (1.7)$$

For each integer $j(1 \leq j \leq n)$, let δ_j and Δ_j be the two determinants obtained by replacing the j-th column of Δ with $f_1, f_2, \cdots . f_n$ and with $F_1, F_2, \cdots, F_n$ respectively. We have the following theorem:

Theorem 1.1. *With the above notions, the identities*

$$f_i = \frac{\delta_1}{\Delta}\phi_{1i} + \frac{\delta_2}{\Delta}\phi_{2i} + \cdots + \frac{\delta_n}{\Delta}\phi_{ni} \,(i = 1, 2, \cdots, n) \quad (1.8)$$

and

$$\left(\frac{\delta_i}{\Delta}\right)' = \frac{\Delta_j}{\Delta} \,(j = 1, 2, \cdots, n) \quad (1.9)$$

hold in D, where the prime (′) denotes derivative.

Proof. The identities (1.8) are evident. To prove the identities (1.9), let $\Gamma : |z - z_0| < r$ be a circle belonging to D such that the functions $\phi_{ij}\,(i,j = 1,2,\cdots,n)$ and $f_i\,(i = 1,2,\cdots,n)$ are holomorphic in Γ and $\Delta \neq 0$ in Γ. Then the functions $\Delta_j/\Delta(j = 1,2,\cdots,n)$ are holomorphic in Γ, hence we can find holomorphic functions $c_j(j = 1,2,\cdots,n)$ in Γ satisfying the identities

$$c_j' = \frac{\Delta_j}{\Delta}\,(j = 1,2,\cdots,n) \tag{1.10}$$

in Γ. Evidently in Γ we have

$$c_1'\phi_{1i} + c_2'\phi_{2i} + \cdots + c_n'\phi_{ni} = F_i\,(i = 1,2,\cdots,n). \tag{1.11}$$

Now consider the system of holomorphic functions

$$w_i = c_1\phi_{1i} + c_2\phi_{2i} + \cdots + c_n\phi_{ni}\,(i = 1,2,\cdots,n) \tag{1.12}$$

in Γ. By (1.5) and (1.11), we have in Γ,

$$\begin{aligned}
&w_i' + a_{i1}w_1 + a_{i2}w_2 + \cdots + a_{in}w_n\\
&= c_1\phi_{1i}' + c_2\phi_{2i}' + \cdots + c_n\phi_{ni}' + c_1'\phi_{1i} + c_2'\phi_{2i}\\
&+ \cdots + c_n'\phi_{ni} = a_{i1}(c_1\phi_{11} + c_2\phi_{21} + \cdots + c_n\phi_{n1})\\
&+ a_{i2}(c_1\phi_{12} + c_2\phi_{22} + \cdots + c_n\phi_{n2}) + \cdots\\
&+ a_{in}(c_1\phi_{1n} + c_2\phi_{2n} + \cdots + c_n\phi_{nn})\\
&= c_1'\phi_{1i} + c_2'\phi_{2i} + \cdots + c_n'\phi_{ni} = F_i\,(i = 1,2,\cdots,n).
\end{aligned} \tag{1.13}$$

Consequently from (1.6) and (1.13), we have in Γ,

$$\begin{aligned}
&(f_i - w_i)' + a_{i1}(f_1 - w_1) + a_{i2}(f_2 - w_2)\\
&+ \cdots + a_{in}(f_n - w_n) = 0\,(i = 1,2,\cdots,n).
\end{aligned}$$

This shows that the system of holomorphic functions

$$W_1 = f_1 - w_1,\ W_2 = f_2 - w_2,\ \cdots,\ W_n = f_n - w_n \tag{1.14}$$

in Γ satisfies the system of equations (1.3). Hence by using a well known theorem on systems of linear differential equations, we can find constants $\gamma_1, \gamma_2, \cdots, \gamma_n$, such that

$$w_i = \gamma_1 \phi_{1i} + \gamma_2 \phi_{2i} + \cdots + \gamma_n \phi_{ni}\,(i = 1, 2, \cdots, n) \tag{1.15}$$

in Γ. From (1.12), (1.14), (1.15) and (1.10), we get the identities

$$f_i = C_1 \phi_{1i} + C_2 \phi_{2i} + \cdots + C_n \phi_{ni}\,(i = 1, 2, \cdots, n), \tag{1.16}$$

which hold in Γ, where

$$C_j = c_j + \gamma_j\,(j = 1, 2, \cdots, n) \tag{1.17}$$

are holomorphic functions in Γ satisfying the identities

$$C_j' = \frac{\Delta_j}{\Delta}\,(j = 1, 2, \cdots, n) \tag{1.18}$$

in Γ, by (1.10). By (1.16) we deduce the identities

$$C_j = \frac{\delta_j}{\Delta}\,(j = 1, 2, \cdots, n) \tag{1.19}$$

in Γ. (1.18) and (1.19) yield (1.9) in Γ. Since the left member and the right member of (1.9) are meromorphic functions in D, the identities (1.9) hold in D. This completes the proof of Theorem 1.1.

2 An Application of Theorem 1.1

First of all, let us introduce some notations as follows: Consider again the n systems of meromorphic functions (1.1) in a domain D satisfying the condition (1.2), and the system of linear differential equations (1.3). We define

$$\begin{aligned} L_{(1)i}(w_1, w_2, \cdots, w_n) &= w_i' + a_{i1}w_1 \\ &+ a_{i2}w_2 + \cdots + a_{in}w_n \ (i = 1, 2, \cdots, n), \end{aligned} \tag{1.20}$$

where $w_1, w_2, \cdots, w_n$ are considered as n meromorphic functions in D. Next we define

$$\begin{aligned} L_{(2)i}(w_1, w_2, \cdots, w_n) &= L_{(1)i}\{L_{(1)1}(w_1, w_2, \cdots, w_n), \\ &\cdots, L_{(1)n}(w_1, w_2, \cdots, w_n)\}(i = 1, 2, \cdots, n). \end{aligned} \tag{1.21}$$

In general, for any integer $p \geq 1$, we define

$$\begin{aligned} L_{(p+1)i}(w_1, w_2, \cdots, w_n) &= L_{(1)i}\{L_{(p)1}(w_1, w_2, \cdots, w_n), \\ &\cdots, L_{(p)n}(w_1, w_2, \cdots, w_n)\} \ (i = 1, 2, \cdots, n). \end{aligned} \tag{1.22}$$

On the other hand, let $f_1, f_2, \cdots, f_n$ be a system of meromorphic functions in the domain D. Then for any two integers $p(p \geq 1)$ and $j(1 \leq j \leq n)$, we define $\Delta_{(p)j}$ to be the determinant obtained by replacing the j-th column of the determinant Δ in (1.7), with

$$L_{(p)i}(f_1, f_2, \cdots, f_n) \ (i = 1, 2, \cdots, n). \tag{1.23}$$

With this notation, the identities (1.9) in Theorem 1.1 may be written in the form

$$\left(\frac{\delta_j}{\Delta}\right)' = \frac{\Delta_{(1)j}}{\Delta} \ (j = 1, 2, \cdots, n). \tag{1.24}$$

From (1.24) we deduce successively

$$\left(\frac{\delta_j}{\Delta}\right)'' = \left(\frac{\Delta_{(1)j}}{\Delta}\right)' = \frac{\Delta_{(2)j}}{\Delta} \ (j = 1, 2, \cdots, n), \tag{1.25}$$

in general, for any integer $p (p \geq 1)$ we have the identities

$$\left(\frac{\delta_j}{\Delta}\right)^{(p)} = \frac{\Delta_{(p)j}}{\Delta} \, (j = 1, 2, \cdots, n), \tag{1.26}$$

which hold in D.

Now let $a_k (k = 0, 1, \cdots, p; p \geq 1)$ be $p + 1$ meromorphic functions in the domain D with

$$a_p \not\equiv 0. \tag{1.27}$$

Consider the differential equation

$$\omega(u) = \alpha_p u^{(p)} + \alpha_{p-1} u^{(p-1)} \cdots + \alpha_1 u' + \alpha_0 u = 0 \tag{1.28}$$

and the system of differential equations

$$\begin{aligned} &\Omega_i(w_1, w_2, \cdots, w_n) \equiv \alpha_p L_{(p)i}(w_1, w_2, \cdots, w_n) \\ &+\alpha_{p-1} L_{(p-1)i}(w_1, w_2, \cdots, w_n) + \cdots \\ &+\alpha_1 L_{(1)i}(w_1, w_2, \cdots, w_n) + \alpha_0 w_i = 0 \, (i = 1, 2, \cdots, n). \end{aligned} \tag{1.29}$$

We are going to investigate the relationship between the solutions of the differential equation (1.28) and those of the system of differential equations (1.29). Let us first prove the following theorem:

Theorem 1.2. *Assume that we can find p linearly independent meromorphic functions in the domain D,*

$$u_k = u_k(z) \, (k = 1, 2, \cdots, p) \tag{1.30}$$

satisfying the differential equation (1.28). *Then in order that a system of meromorphic functions $f_1, f_2, \cdots, f_n$ in D satisfies the* *system of differential equations* (1.29), *it is necessary and sufficient that there exists a system of functions* $U_j (j = 1, 2, \cdots . n)$ *of the form*

$$U_j = \sum_{k=1}^{p} C_{jk} u_k \, (j = 1, 2, \cdots, n), \tag{1.31}$$

where $C_{jk} (j = 1, 2, \cdots, n; k = 1, 2, \cdots, p)$ *are constants, such that*

$$f_i = U_1 \phi_{1i} + U_2 \phi_{2i} + \cdots + U_n \phi_{ni} \, (i = 1, 2, \cdots, n) \tag{1.32}$$

in D, *in which* $\phi_{ji} (j, i = 1, 2, \cdots, n)$ *are the functions in* (1.1).

Proof. By the formula (1.26), we have in D,

$$\begin{aligned} \omega \left(\frac{\delta_j}{\Delta} \right) &= \alpha_p \frac{\Delta_{(p)j}}{\Delta} + \alpha_{p-1} \frac{\Delta_{(p-1)j}}{\Delta} + \cdots \\ &+ \alpha_1 \frac{\Delta_{(1)j}}{\Delta} + \alpha_0 \frac{\delta_j}{\Delta} = \frac{d_j}{\Delta} \, (j = 1, 2, \cdots, n), \end{aligned} \tag{1.33}$$

where d_j is the determinant obtained in replacing the j-th column of the determinant Δ in (1.7), by

$$\Omega_i (f_1, f_2, \cdots, f_n) \, (i = 1, 2, \cdots, n). \tag{1.34}$$

Now assume that the system of meromorphic functions $f_1, f_2, \cdots, f_n$ in D satisfies the system of differential equations (1.29). Thus the functions (1.34) are identically equal to zero in D, hence from (1.33), the functions

$$\frac{\delta_j}{\Delta} \, (j = 1, 2, \cdots, n) \tag{1.35}$$

are solutions of the differential equation (1.28). Consequently we have

$$\frac{\delta_j}{\Delta} = U_j \, (j = 1, 2, \cdots, n), \tag{1.36}$$

where $U_j (j = 1, 2, \cdots, n)$ are functions of the form (1.31). Finally by the formulas (1.8) in Theorem 1.1, we get (1.32).

Next assume that the system of meromorphic functions $f_1, f_2 \cdots, f_n$ in D can be expressed in the form (1.32), where $U_j (j = 1, 2, \cdots, n)$ are functions of the form (1.31). Then comparing (1.32) with (1.8), it can be see that we have (1.36). Hence the functions (1.35) are solutions of the differential equation (1.28). Consequently from (1.33), we have

$$\frac{d_j}{\Delta} = 0 \, (j = 1, 2, \cdots, n) \tag{1.37}$$

in D. The identities (1.37) can be written in the form

$$\psi_{i1}\Omega_1 + \psi_{i2}\Omega_2 + \cdots + \psi_{in}\Omega_n = 0 \, (j = 1, 2, \cdots, n), \tag{1.38}$$

where $\Omega_i \, (i = 1, 2, \cdots, n)$ are the functions (1.34). Consider the determinant

$$\delta = \begin{vmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1n} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \psi_{n1} & \psi_{n2} & \cdots & \psi_{nn} \end{vmatrix} \tag{1.39}$$

which, by a known rule, satisfies the identity

$$\Delta\delta = 1 \tag{1.40}$$

in D, which implies that δ is nonidentically equal to zero. Hence in D we have

$$\Omega_i = 0 \, (i = 1, 2, \cdots, n), \tag{1.41}$$

and the system of functions $f_1, f_2, \cdots, f_n$ satisfies the system of differential equations (1.29). Theorem 1.2 is now completely proved.