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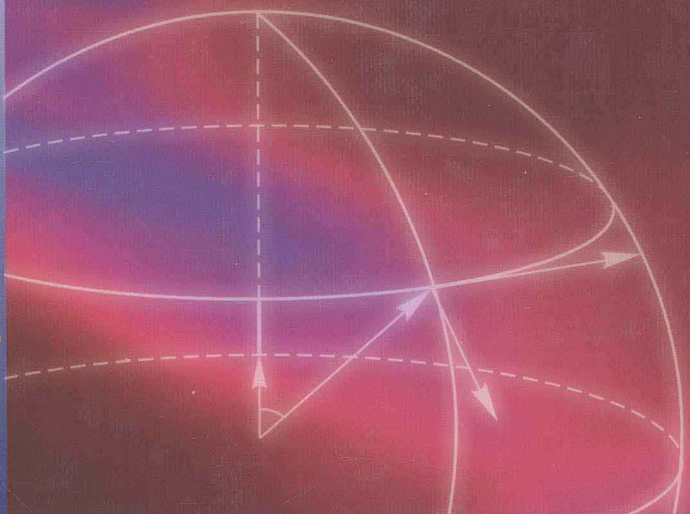

Volume I

Physics and Partial Differential Equations

物理学与偏微分方程（上册）

Tatsien Li
Tiehu Qin

Translated by Yachun Li

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WULIXUE YU PIANWEIFEN FANGCHENG

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Preface to the English Edition

The first volume of the Chinese edition of this book was published in July 1997, and the second volume was published in June 2000. In July 2000, upon the readers' request, we corrected several typographical errors and republished the first volume.

In this edition, minor typographical errors are corrected, and a small paragraph has been added to section 5.5.4 in Chapter 5, while the remaining text is unchanged.

We would like to take this opportunity to express our sincere thanks to our teachers, friends, and readers for their encouragement and support.

Tatsien Li
Tiehu Qin
Shanghai
September 2003

Preface to the Chinese Edition

The fundamental equations in many important physical and mechanical disciplines are partial differential equations. Although the names of these equations are well known, and although a considerable amount of research has been done on these equations, it is not an easy task to comprehensively and profoundly understand the related physical and mechanical background. The purpose of this book is to offer some help not only to teachers, graduate students, and senior undergraduate students engaged in studying, researching, and teaching applied partial differential equations, but also to scholars and researchers in other disciplines and application areas, so that they can become proficient in the use of important fundamental equations in modern physics, gain familiarity with the whys, wherefores, and derivation of these equations, understand some commonly used mathematical models more easily, and thus not only study and use partial differential equations more consciously, but also learn to grasp some significant problems in order to properly carry out their research. Therefore, our purpose in writing this book is to build a bridge between physics and partial differential equations.

In this book, starting with the most basic concepts of physics we focus on the whole process of establishing the fundamental equations for physical and mechanical disciplines such as electrodynamics, fluid dynamics, magnetohydrodynamics, reacting fluid dynamics, elastic mechanics, thermoelastic mechanics, viscoelastic mechanics, kinetic theory of gases, special relativity, and quantum mechanics. At the same time, we give a brief description of the mathematical structures and features of these equations, including their types and basic characteristics, the behavior of solutions, and some approaches commonly used to solve these equations. We also selectively introduce some worldwide research results from recent years, including those of the authors and their research group. We hope that readers who are unfamiliar with the related physical and mechanical disciplines can gain access to the core of these disciplines in an easy-to-digest way in a short time so as to complete as soon as possible their transition from physics to mathematics and from related physical and mechanical fields to their mathematical models described by partial differential equations. On the other hand, for readers who are more familiar with the related physical and mechanical disciplines, we hope that, through their in-depth understanding of the mathematical structures and features of the fundamental equations, they will ultimately see the advantage of effective mathematical tools and expressions in more clearly presenting the basic contents of physics, and consequently use modern mathematical concepts, methods, and tools more consciously to solve related physical and mechanical problems.

This book is divided into two volumes, each consisting of five chapters. The contents of each chapter are relatively independent; however, all of the chapters echo and relate with each other to a certain extent. Exercises and a bibliography are included in each chapter.

The vast majority of chapters are not meant to be difficult for those readers who have taken basic undergraduate courses in mathematics and physics. This book can be used as a textbook for graduate courses or elective senior undergraduate courses, as well as a reference book or extracurricular reading material.

Since the second half of 1987, the contents of this book have been continuously and successfully taught in Fudan University as both an elective senior undergraduate course and a required graduate degree course. The lecture notes have been constantly supplemented and revised, and it is on these that the final version of this book is based.

The authors would like to thank Higher Education Press for its enthusiastic support of the publication of this book, and to thank Professor Sixu Guo for careful and meticulous typesetting. Thanks also go to Dr. Zhijie Cai for his responsible and proficient typing of the entire manuscript, and to Dr. Yingqiu Gu for his assistance in conforming of all the physical units in this book to the international system of units (SI units). In particular, we are grateful to Minyou Qi, Professor of Mathematics in the Department of Mathematics at Wuhan University, and Guangjiong Ni, Professor of Physics in the Department of Physics at Fudan University. They have carefully reviewed the manuscript and supplied many helpful comments and suggestions. Their hard work enriched this book.

As mathematicians, the authors may have a superficial understanding of physics to a certain extent, so errors and omissions are inevitable. We hope that readers will not spare their comments and corrections.

Tatsien Li
Tiehu Qin
November 10, 1996

Contents

Chapter 1

Electrodynamics

1.1 Introduction

Electrodynamics is aimed at the study of fundamental properties of *electromagnetic fields*, that is, the laws of motion of electromagnetic fields and their interaction with charged substances.

Electric and magnetic fields were first introduced as description tools. Gradually it was realized that they also exist as substances which have their own laws of motion and can interact with charged substances. Moreover, they are also equipped with basic features of motion such as energy and momentum, which can be converted into each other in the interaction with charged substances.

In order to understand electromagnetic fields, we need to know how *electric field intensity* $\boldsymbol{E} = (E_1, E_2, E_3)$ and *magnetic induction intensity* $\boldsymbol{B} = (B_1, B_2, B_3)$ change with the spatial location (x_1, x_2, x_3) and time t; i.e., we need to grasp the functions $\boldsymbol{E}(t, x_1, x_2, x_3)$ and $\boldsymbol{B}(t, x_1, x_2, x_3)$, from which we can figure out the distribution and change laws of the electromagnetic fields, calculate the forces acting on charged substances by electromagnetic fields (given by the *Lorentz force formula*), and calculate energy and momentum of electromagnetic fields.

The laws of motion of electromagnetic fields are described by a system of partial differential equations in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$; these are called *Maxwell's equations*. Under prescribed conditions for specific problems, we can solve $\boldsymbol{E}(t, x_1, x_2, x_3)$ and $\boldsymbol{B}(t, x_1, x_2, x_3)$ from Maxwell's equations so as to determine the electromagnetic field under consideration and all of its features. Just like $F = ma$ in Newtonian mechanics, Maxwell's equations, the fundamental equations in electrodynamics, are the basis and starting point for any discussion related to electromagnetic fields.

Therefore, the study of electrodynamics can be roughly summarized as follows:

(a) To establish Maxwell's equations according to the basic features of electromagnetic fields, which are the basic equations and general framework of electrodynamics.

(b) To simplify Maxwell's equations according to various specific situations, so as to obtain their solutions and carry on the corresponding physical interpretation.

We will focus on (a) while giving some explanations for selective points of (b).

1.2 Preliminaries

First we recall some necessary preliminaries to be used in the establishment of Maxwell's equations, including electric field, magnetic field, and electromagnetic induction, etc., and we give their corresponding mathematical forms. In the next section, we will combine them organically into the desired Maxwell's equations.

1.2.1 Coulomb's Law, Divergence, and Curl of Electrostatic Field

1.2.1.1 Coulomb's Law; Electric Field Intensity

Coulomb's law resulted from experiments and is the foundation of the theory of electrostatics. It is stated as follows: Let q and q_1 be two point electric charges at rest in a vacuum, and let $\boldsymbol{r}_1$ be the position vector in the direction from charge q_1 to charge q, with distance $r_1 = |\boldsymbol{r}_1|$. Then charge q_1 acts upon charge q with a force

$$\boldsymbol{F} = k\frac{qq_1\boldsymbol{r}_1}{r_1^3}, \tag{1.1}$$

where $k > 0$ is a constant depending on the system of units. In the international standard system of units (SI units), which we employ here, force, distance, and charge are measured by Newton (N), meter (m), and Coulomb (C), respectively, and

$$k = \frac{1}{4\pi\varepsilon_0}, \tag{1.2}$$

where $\varepsilon_0 = 8.854\,19 \times 10^{-2}\,\mathrm{C}^2/(\mathrm{N}\cdot\mathrm{m}^2)$ is called the *dielectric constant* in a vacuum.

One should note that Coulomb's law is valid only for a point charge at rest in a vacuum.

For the case of several point charges q, q_i $(i = 1,\ldots,n)$, according to the principle of superposition of vectors, the force upon q due to other charges q_i $(i = 1,\ldots,n)$ is given by

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{qq_i\boldsymbol{r}_i}{r_i^3},$$

where $\boldsymbol{r}_i$ is the position vector pointing from q_i to q, $r_i = |\boldsymbol{r}_i|$.

Similarly, the force acting on a point charge q due to continuous distribution of charges in the spatial domain Ω with charge density $\rho(x_1,x_2,x_3)$, from the principle of superposition, can be given by

$$\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\int_\Omega \frac{q\rho\boldsymbol{r}}{r^3}\mathrm{d}V,$$

where $\boldsymbol{r}$ is the position vector pointing from volume element $\mathrm{d}V$ to point charge q, $r = |\boldsymbol{r}|$.

According to Coulomb's law, the space around the charge has a special physical property: any charge in this space has force upon it. Any space with this property is known as an *electric field*, which can be described mathematically by the vector field indicating the forces upon charges. Here the electric field is due to the existence of charges. However, we will see later that an electric field is a form of existence of substances, which can exist independently without charges (for example, changing a magnetic field can generate an electric field). The electric field generated by the static charges is called the *electrostatic field*.

The force on a charge is different from point to point in an electric field. In order to describe the force in the electric field, we take the force on a static unit positive charge (test charge) at a particular point to measure the intensity of the electric field at this point, which is called the *electric field intensity*, denoted by $\boldsymbol{E} = (E_1, E_2, E_3)$. In the case of the electrostatic field, it is a function with respect to only (x_1, x_2, x_3). For an electric field evolving with time t, electric field intensity can be defined in a similar way, but the test charge must be static, and in this case, it is also a function with respect to t.

From Coulomb's law, a static charge q in the electric field with intensity $\boldsymbol{E}$ experiences a force

$$\boldsymbol{F} = q\boldsymbol{E}. \tag{1.3}$$

As to the actual measurement of electric field intensity, we should be careful that the original electric field is not changed much by introducing the test charge. Therefore, it is not necessary to take a unit positive charge as the test charge. Instead, we can take a smaller point charge q and determine the electric field intensity $\boldsymbol{E}$ from $\boldsymbol{E} = \boldsymbol{F}/q$.

From (1.1), (1.2), the intensity of an electric field caused by a single point charge q_1 is

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_1\boldsymbol{r}_1}{r_1^3}, \tag{1.4}$$

where $\boldsymbol{r}_1$ is the vector centered at q_1; similarly, the intensity of an electric field caused by several point charges q_i $(i = 1, \ldots, n)$ can be given by

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i\boldsymbol{r}_i}{r_i^3}; \tag{1.5}$$

while for an electric field generated by continuous distribution of charges in Ω with density ρ, the intensity can be obtained as

$$\boldsymbol{E}(P) = \frac{1}{4\pi\varepsilon_0}\int_{\Omega}\frac{\rho(P')\boldsymbol{r}_{P'P}}{r_{P'P}^3}\mathrm{d}V_{P'}, \tag{1.6}$$

where $P : (x_1, x_2, x_3)$, $P' : (x_1', x_2', x_3')$, $\boldsymbol{r}_{P'P} = \overrightarrow{P'P} = (x_1 - x_1', x_2 - x_2', x_3 - x_3')$, $\mathrm{d}V_{P'} = \mathrm{d}x_1'\mathrm{d}x_2'\mathrm{d}x_3'$.

The above formulas are valid only for the electric field generated by static charges.

1.2.1.2 Gauss's Law

First, we introduce the concept of *electric flux*.

We can define electric field lines after determining electric field intensity $\boldsymbol{E}$. The *electric field line* is the integral curve of vector field $\boldsymbol{E}$, i.e., the curve tangential to $\boldsymbol{E}$ everywhere, which satisfies

$$\frac{\mathrm{d}x_1}{E_1} = \frac{\mathrm{d}x_2}{E_2} = \frac{\mathrm{d}x_3}{E_3}, \tag{1.7}$$

whose direction is given by that of $\boldsymbol{E}$. Electric field lines should fill the whole space. However, in order to reflect the strength of distribution of the electric field, we normally follow this rule: the electric field lines are dense where the electric field intensity is large, and the electric field lines are sparse where the electric field intensity is small. We formulate, at a point in the electric field, that the number of electric field lines passing through a unit surface area is $\pm|\boldsymbol{E}| = \boldsymbol{E}\cdot\boldsymbol{n}$, where $\boldsymbol{n}$ is the outward unit normal to the surface, and $\boldsymbol{E}\cdot\boldsymbol{n}$ is the inner product (scalar product) of $\boldsymbol{E}$ and $\boldsymbol{n}$. If the direction of $\boldsymbol{n}$ is in accordance with $\boldsymbol{E}$, it takes a plus sign; otherwise it takes a minus sign. Then for a general element of surface area $\mathrm{d}S$ with outward unit normal $\boldsymbol{n}$, the number of electric field lines passing through $\mathrm{d}S$ along $\boldsymbol{n}$ is $\boldsymbol{E}\cdot\boldsymbol{n}\mathrm{d}S$, which is called the *electric flux* passing through $\mathrm{d}S$ along $\boldsymbol{n}$. Thus the electric flux passing through an arbitrary surface S along $\boldsymbol{n}$ is $\int_S \boldsymbol{E}\cdot\boldsymbol{n}\mathrm{d}S$, where $\mathrm{d}S$ is the surface element of S. From the above definition we can see that the electric flux is actually the flux of vector field $\boldsymbol{E}$ passing through the corresponding surface.

Gauss's Law. *In an electrostatic field, the outward electric flux passing through an arbitrary closed surface Γ is equal to the algebraic sum of charges enclosed in the surface Γ divided by ε_0.*

Based on Gauss's law, if Γ contains only point charges with algebraic sum Q, then

$$\int_\Gamma \boldsymbol{E}\cdot\boldsymbol{n}\mathrm{d}S = \frac{1}{\varepsilon_0}Q; \tag{1.8}$$

for continuously distributed charges with density ρ, it holds that

$$\int_\Gamma \boldsymbol{E}\cdot\boldsymbol{n}\mathrm{d}S = \frac{1}{\varepsilon_0}\int_\Omega \rho\mathrm{d}V, \tag{1.9}$$

where Ω is the domain enclosed by Γ, and $\boldsymbol{n}$ is the outward unit normal to Γ.

Proof of Gauss's Law. From the principle of superposition, it is sufficient to prove the case when the inside point charge is a unit positive charge, and, without loss of generality, we assume that the charge is at the origin. Then from (1.5), the electric field intensity is

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{r}}{r^3}, \tag{1.10}$$

where $\boldsymbol{r} = (x_1, x_2, x_3)$. Therefore

$$\boldsymbol{E}\cdot\boldsymbol{n}\mathrm{d}S = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\cos\theta\mathrm{d}S,$$

where θ is the angle between the outward unit normal $\boldsymbol{n}$ to $\mathrm{d}S$ and $\boldsymbol{r}$. It is easy to see that

$$\mathrm{d}\omega = \frac{\cos\theta}{r^2}\mathrm{d}S$$

is the solid angle formed by $\mathrm{d}S$ with respect to the origin, whose sign depends on whether θ is an acute or an obtuse angle. Since any closed surface forms a solid angle of 4π with respect to any interior point, we have

$$\int_\Gamma \boldsymbol{E}\cdot\boldsymbol{n}\mathrm{d}S = \frac{1}{4\pi\varepsilon_0}\int_\Gamma \mathrm{d}\omega = \frac{1}{\varepsilon_0}.$$

The proof is completed. □

As a result, the electric flux passing through a closed surface depends only on the total charges inside the surface and has nothing to do with either the distribution of charges or the outside charges!

Formula (1.9) is the integral form of Gauss's law. Now we come to its differential form. By Green's formula

$$\int_{\Gamma} \boldsymbol{E}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_{\Omega} \operatorname{div}\boldsymbol{E}\,\mathrm{d}V, \tag{1.11}$$

we obtain from (1.9) that

$$\int_{\Omega} \operatorname{div}\boldsymbol{E}\,\mathrm{d}V = \frac{1}{\varepsilon_0}\int_{\Omega} \rho\,\mathrm{d}V,$$

which holds for arbitrary domain Ω in the electric field, and thus we obtain the differential form of Gauss's law

$$\operatorname{div}\boldsymbol{E} = \frac{\rho}{\varepsilon_0}, \tag{1.12}$$

which is Gauss's law for the case of continuously distributed charges.

For point charges, we can also rewrite (1.8) as (1.12) by using the δ-function. For simplicity, we suppose that there is only one charge Q located at the origin, with density $Q\delta(x_1,x_2,x_3)$. According to (1.8) and (1.11), we still have

$$\operatorname{div}\boldsymbol{E} = \frac{1}{\varepsilon_0}Q\delta(x_1,x_2,x_3), \tag{1.13}$$

which is the differential form of Gauss's law for the case of point charges.

From (1.12), (1.13), we know that **the electrostatic field is a field with source**, whose source is the charge. Every unit positive charge issues an electric flux of $\frac{1}{\varepsilon_0}$, while every unit negative charge absorbs an electric flux of $\frac{1}{\varepsilon_0}$. Later on, we will see that this is an important difference between electrostatic field and magnetostatic field, and between general electric field and magnetic field as well.

Next we will show that electrostatic field is irrotational. First we prove that, for an arbitrary closed curve l, it holds that

$$\int_{l} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = 0. \tag{1.14}$$

This implies that the circulation of electric field intensity $\boldsymbol{E}$ along an arbitrary closed curve l is zero, i.e. the work done by the electrostatic field along any closed curve is zero. As before, we prove (1.14) only for the case when the electrostatic field is generated by a single positive unit charge at the origin. The electric field intensity $\boldsymbol{E}$ is then given by (1.10), and hence

$$\oint_{l} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = \frac{1}{4\pi\varepsilon_0}\oint_{l}\frac{\boldsymbol{r}}{r^3}\cdot\mathrm{d}\boldsymbol{r}.$$

where $\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} = \frac{1}{2}\mathrm{d}(\boldsymbol{r}\cdot\boldsymbol{r}) = \frac{1}{2}\mathrm{d}(r^2) = r\mathrm{d}r$, and therefore

$$\oint_{l} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = \frac{1}{4\pi\varepsilon_0}\oint_{l}\frac{1}{r^2}\mathrm{d}r = -\frac{1}{4\pi\varepsilon_0}\oint_{l}\mathrm{d}\left(\frac{1}{r}\right) = 0,$$

so (1.14) is true.

We now turn to the differential form of (1.14). From the Stokes formula, (1.14) can be rewritten as

$$\int_S \operatorname{rot} \boldsymbol{E} \cdot \boldsymbol{n} \mathrm{d}S = 0,$$

where S is an arbitrary surface enclosed by l. Due to the arbitrariness of l and S, we immediately arrive at

$$\operatorname{rot} \boldsymbol{E} = \boldsymbol{0}, \tag{1.15}$$

which implies that **the electrostatic field is an irrotational field.**

Now we introduce the concept of potential of an electrostatic field by using its irrotationality. Since the vector field $\boldsymbol{E}$ is irrotational, there exists a scalar function $\phi(x_1, x_2, x_3)$ such that

$$\boldsymbol{E} = -\operatorname{grad} \phi; \tag{1.16}$$

here ϕ (which can differ up to an arbitrary additive constant) is called the *electrostatic field potential* (see Lemma 1.2 in section 1.6 of this chapter for reference). The minus sign on the right-hand side of (1.16) is used to indicate that the electric field intensity points toward the direction in which the electric potential decreases.

In the SI units, the unit of electric potential is Volt (V), and the unit of electric field intensity is Volt/meter (V/m) accordingly.

Actually, the potential function ϕ of the electrostatic field can be given by

$$\phi(x_1, x_2, x_3) = -\int_{(x_1^0, x_2^0, x_3^0)}^{(x_1, x_2, x_3)} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} + \phi_0, \tag{1.17}$$

where (x_1^0, x_2^0, x_3^0) is any given point inside the electric field, and ϕ_0 is an arbitrary constant. From (1.15) we know that the integral on the right-hand side of the above formula is independent of the integral path.

It is easy to verify directly that for the electrostatic field generated by the point charge Q at the origin, if it is assumed that the potential ϕ vanishes at infinity, then

$$\phi(x_1, x_2, x_3) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r},$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Then, for an electrostatic field generated by continuously distributed charges with density ρ in Ω, the potential ϕ is

$$\phi(P) = \frac{1}{4\pi\varepsilon_0} \int_\Omega \frac{\rho(P')}{r_{P'P}} \mathrm{d}V_{P'}.$$

We can summarize in the following: **the electrostatic field is an irrotational field with source, whose divergence is $\frac{\rho}{\varepsilon_0}$ and curl is zero**. This is a corollary of Coulomb's law. We will see later on that, although Coulomb's law is valid only for the electrostatic field and cannot apply to general fields, the conclusion that the divergence of the electric field intensity is $\frac{\rho}{\varepsilon_0}$ is still true for general fields, while the irrotationality is true only for the electrostatic field.

1.2.2 Ampère–Biot–Savart Law; Divergence and Curl of Magnetostatic Field

1.2.2.1 Electric Current Density,; Conversation Law of Charges

If the charge is not at rest, then its directional flow forms an electric current. The electric current density vector $\boldsymbol{j}$ is introduced to describe the state of the current, which is a function of t, x_1, x_2, x_3, measuring the current flow of a point in the conductor at a particular time. Its definition is the following: The *current density* $\boldsymbol{j}$ is a vector in the direction of the current flow with magnitude equal to the amount of the charge across a unit area perpendicular to the direction of current flow in unit time. Thus, for the surface element $\mathrm{d}S$ with unit normal vector $\boldsymbol{n}$, the charge across $\mathrm{d}S$ in the direction of $\boldsymbol{n}$ in unit time (i.e., current) is

$$\mathrm{d}I = \boldsymbol{j} \cdot \boldsymbol{n} \mathrm{d}S.$$

In SI units, the unit of current is Ampère (A), and 1 Ampère= 1 Coulomb/second (C/s).

The *conservation law of electric charges* tells us the following: The amount of charges is conserved. Therefore, the increased amount of charges in any closed surface Γ in unit time is equal to the amount of charges flowing into the surface Γ during the same time period, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} \rho \mathrm{d}V = -\int_{\Gamma} \boldsymbol{j} \cdot \boldsymbol{n} \mathrm{d}S, \tag{1.18}$$

where Ω is the domain enclosed by Γ, and $\boldsymbol{n}$ is the outward unit normal to Γ. Then, by Green's formula (1.11), it follows that

$$\int_{\Omega} \frac{\partial}{\partial t} \rho \mathrm{d}V = -\int_{\Omega} \operatorname{div} \boldsymbol{j} \mathrm{d}V. \tag{1.19}$$

Due to the arbitrariness of domain Ω, (1.19) implies that

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \boldsymbol{j} = 0. \tag{1.20}$$

This is the differential form of the conservation law of electric charges, which is known as the *continuity equation of electric current*. If we denote the macroscopic velocity of current by $\boldsymbol{v}$, it is easy to see that $\boldsymbol{j} = \rho \boldsymbol{v}$. Inserting it into (1.20), we can see that it has the same form as the continuity equation in continuum mechanics (see Chapters 2 and 5).

In the case of stationary current, although there are moving charges, the distribution of charge and current is time independent everywhere, and then we have

$$\operatorname{div} \boldsymbol{j} = 0, \tag{1.21}$$

which implies that the stationary current has no source. The integral form of (1.21) is

$$\int_{\Gamma} \boldsymbol{j} \cdot \boldsymbol{n} \mathrm{d}S = 0, \tag{1.22}$$

where Γ denotes any closed surface in the electric field.

In the case of stationary current, the charges in the conductor form a time-independent steady distribution. Experiments show that the electric field generated by a steady current still obeys Coulomb's law, and then can be treated as an electrostatic field, whose electric field intensity is

$$\boldsymbol{E}(P) = \frac{1}{4\pi\varepsilon_0}\int_\Omega \frac{\rho(P')\boldsymbol{r}_{P'P}}{r^3_{P'P}}\mathrm{d}V_{P'}. \tag{1.23}$$

At this time, $\boldsymbol{E}$ is still an irrotational field with source, satisfying

$$\operatorname{div}\boldsymbol{E} = \frac{\rho}{\varepsilon_0},$$
$$\operatorname{rot}\boldsymbol{E} = \boldsymbol{0}.$$

In other words, the electric field generated by a steady current still obeys the law of electrostatic field.

1.2.2.2 Ampère–Biot–Savart Law; Magnetic Induction Intensity

It has been found that there is force upon an electrified conducting wire when it is near another conducting wire with electric current flow. If the two currents are moving in the same direction, the two wires attract each other; otherwise, they repel each other. Generally speaking, there is force upon a current element $\boldsymbol{j}dV$ when it is close to another current. Any physical object with this property is called a *magnetic field*, which can be mathematically described by the vector field reflecting the force upon current elements. Later on, we will see that the introduction of magnetic field is not only as a tool for dealing with problems, but also as a reflection of objective physical phenomena. A magnetic field is not necessarily generated by electric current; changing an electric field can also generate a magnetic field.

We consider a steady electric current distribution $\boldsymbol{j}(x_1,x_2,x_3)$. Experimental results show that the force upon the current element $\boldsymbol{j}(P)\mathrm{d}V_P$ at point P in this distribution due to current element $\boldsymbol{j}(P')\mathrm{d}V_{P'}$ at point P' is given by

$$\frac{\mu_0}{4\pi}\boldsymbol{j}(P)\mathrm{d}V_P \times \left(\frac{\boldsymbol{j}(P')\mathrm{d}V_{P'} \times \boldsymbol{r}_{P'P}}{r^3_{P'P}}\right), \tag{1.24}$$

where $\mu_0 = 4\pi \times 10^{-7}$ Volt · second/(Amp · meter) (V·s/(A·m)) is the *magnetic conductivity* in a vacuum, and the symbol "×" between vectors represents the outer product (vector product). Thus the total force upon current element $\boldsymbol{j}(P)\mathrm{d}V_P$ at point P is

$$\mathrm{d}\boldsymbol{F}(P) = \frac{\mu_0}{4\pi}\boldsymbol{j}(P)\mathrm{d}V_P \times \int_\Omega \frac{\boldsymbol{j}(P')\mathrm{d}V_{P'} \times \boldsymbol{r}_{P'P}}{r^3_{P'P}}, \tag{1.25}$$

where Ω is the domain of the current distribution. This is the *Ampère–Biot–Savart law*. Let

$$\boldsymbol{B}(P) = \frac{\mu_0}{4\pi}\int_\Omega \frac{\boldsymbol{j}(P')\mathrm{d}V_{P'} \times \boldsymbol{r}_{P'P}}{r^3_{P'P}}. \tag{1.26}$$

Equation (1.25) can be rewritten as

$$\mathrm{d}\boldsymbol{F}(P) = \boldsymbol{j}(P)\mathrm{d}V_P \times \boldsymbol{B}(P), \tag{1.27}$$

where $\boldsymbol{B}(P)$ is called the *magnetic induction intensity* at point P, with unit Tesla (T), 1 Tesla = 1 N/(A·m). This relation shows that the force upon the current element depends not only on the magnitude of $\boldsymbol{B}$ but also on its direction. Moreover, the magnitude of $\boldsymbol{B}$ at one point equals the maximum force upon the unit current element at this point, and its direction is either in the same or the opposite direction of the current element when the force is zero.

Formally, formula (1.27) is similar to the the following formula of the force upon the charge element $\rho(P)\mathrm{d}V_P$ in an electrostatic field:

$$\mathrm{d}\boldsymbol{F}(P)=\rho(P)\mathrm{d}V_P\boldsymbol{E}(P), \tag{1.28}$$

while the definition (1.26) of magnetic induction intensity $\boldsymbol{B}$ is similar to the following definition of the electric field intensity:

$$\boldsymbol{E}(P)=\frac{1}{4\pi\varepsilon_0}\int_\Omega \frac{\rho(P')\boldsymbol{r}_{P'P}}{r_{P'P}^3}\mathrm{d}V_{P'}. \tag{1.29}$$

The magnetic field $\boldsymbol{B}$ generated by steady electric current is a function with respect to only (x_1,x_2,x_3), which is called the *magnetostatic field*.

1.2.2.3 Ampère Theorem

For the magnetostatic field, its magnetic induction intensity $\boldsymbol{B}$ satisfies the following.

Ampère Theorem. *For any closed curve l, it holds that*

$$\oint_l \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l}=\mu_0\int_S \boldsymbol{j}\cdot\boldsymbol{n}\mathrm{d}S, \tag{1.30}$$

where S is any surface surrounded by l, whose direction of the outward unit normal $\boldsymbol{n}$ forms a right-hand coordinate system with respect to the rotary direction of l.

Actually, the surface integral on the right-hand side of (1.30) is independent of the choice of surface S, which can be deduced from the continuity equation (1.22) of steady currents.

Formula (1.30) is the integral form of the Ampère theorem. From the Stokes formula

$$\oint_l \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{l}=\int_S \mathrm{rot}\,\boldsymbol{B}\cdot\boldsymbol{n}\mathrm{d}S, \tag{1.31}$$

using formula (1.30), and noticing the arbitrariness of S, we have

$$\mathrm{rot}\,\boldsymbol{B}=\mu_0\boldsymbol{j}. \tag{1.32}$$

which is the differential form of the Ampère theorem. It is equivalent to the integral form (1.30) if all the quantities under consideration are smooth. This relation shows that **the magnetostatic field is a rotational field, whose curl is $\mu_0\boldsymbol{j}$.**

Proof of the Ampère Theorem. We prove the differential form (1.32). We denote the gradient, curl, and divergence at (x_1,x_2,x_3) and (x_1',x_2',x_3') by grad, rot, div and grad$'$, rot$'$, div$'$, respectively.

Noticing that

$$\frac{\boldsymbol{r}_{P'P}}{r_{P'P}^3}=-\mathrm{grad}\frac{1}{r_{P'P}},$$