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人工边界方法

Artificial Boundary Method

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内 容 简 介

人工边界方法是求解无界区域上偏微分方程（组）数值解的一个重要和有效的方法。人工边界方法的核心问题是在人工边界上如何对已知的问题找出问题的解满足的准确（或者高精度近似）的边界条件。借助于人工边界方法，我们可将无界区域上的问题简化为有界区域上的问题进行数值计算。本书系统地介绍了人工边界方法的计算格式及其理论基础。本书可以作为科学与工程计算专业研究生课程的教材，亦可以作为科学与工程计算专业科学技术人员的参考书。

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Preface

The artificial boundary method is an effective numerical method for solving partial differential equations on unbounded domains by applying artificial boundary conditions (ABCs) on the boundaries of the reduced bounded domains. With more than 30 years development, the artificial boundary method has reached maturity in recent years. It has been applied to various problems in scientific and engineering computations, and theoretical issues such as the convergence and error estimates of the artificial boundary method have been solved gradually. Based on the research works by the authors over many years and the works by other researchers, we have collected the methods and theories of the artificial boundary method and have presented them in this book.

The partial contents of this book were taught in the fall, 2005 and the spring, 2007 in the Department of Mathematical Sciences of Tsinghua University and the Department of Mathematics of University of Science and Technology of China, respectively.

This book has nine chapters, as listed below.

Chapter 1: Global ABCs for the Exterior Problem of Second Order Elliptic Equations

Chapter 2: Global ABCs for the Navier System and Stokes System

Chapter 3: Global ABCs for Heat and Schrödinger Equations

Chapter 4: Absorbing Boundary Conditions for Wave Equation, Klein-Gordon Equation, and Linear KdV Equation

Chapter 5: Local ABCs

Chapter 6: Discrete ABCs

Chapter 7: Implicit ABCs

Chapter 8: Nonlinear ABCs

Chapter 9: Applications to Problems with Singularity

We have striven for accuracy and elegance in writing the book. However, errors are inevitable. We would be most grateful to learn of any errors in the book for the revision of future printing.

This book has benefited from works of other researchers, including our co-authors: Long-An Ying, Weizhu Bao, Zhongyi Huang, Chunxiong Zheng, Zhizhong Sun, Jicheng Jin, Dongsheng Yin, and Zhenli Xu. Professor Hermann Brunner has read through all the chapters of this book, and made numerous suggestions for improving the manuscript. We wish to express our appreciation for his kind help.

Houde Han, Xiaonan Wu

Contents

Introduction

Many problems in science and engineering are described by partial differential equations on unbounded domains, and must be solved numerically. The flow around an airfoil (see Fig. 0-1), stress analysis of a dam with an infinite foundation (see Fig. 0-2), flow in a long pipe (see Fig. 0-3), and wave propagation in the space (sound wave, elastic wave, electric magnetic wave, etc.) are typical examples. For these problems, the main difficulty is the unboundedness of the domain. Normal numerical methods, such as the finite difference and finite element methods, cannot be applied directly to these problems. One way to solve the problem is to introduce an artificial boundary, and divide the physical domain into two parts: the bounded computational domain and the remaining unbounded domain. The artificial boundary becomes the boundary (or a part of the boundary) of the computational domain. If we can find the boundary condition on the artificial boundary satisfied by the solution of the original problem, then we can reduce the original problem to a boundary value problem on the bounded computational domain, and solve it numerically. In early literature, the boundary condition at infinity is usually applied directly on the artificial boundary. The Dirichlet boundary condition (or Neumann boundary condition) is the commonly used boundary condition. In general, this boundary condition is not the exact boundary condition satisfied by the solution of the original problem,

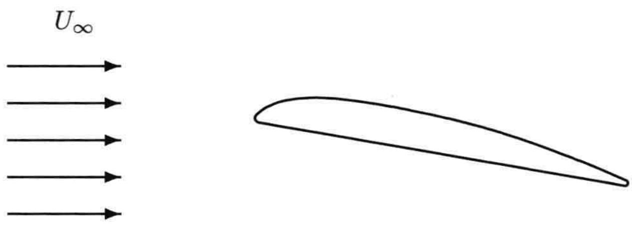

Fig. 0-1 Flow around an airfoil

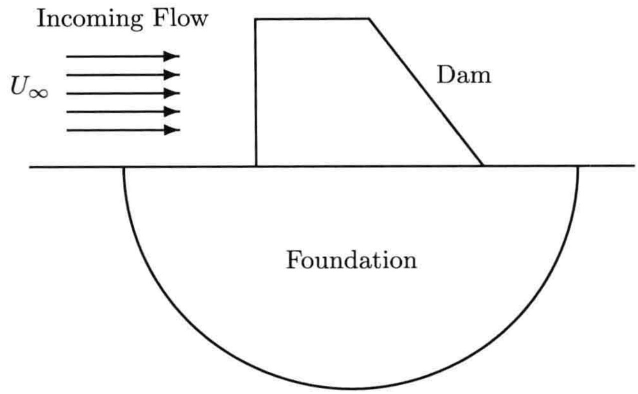

Fig. 0-2 Stress analysis of a dam with an infinite foundation

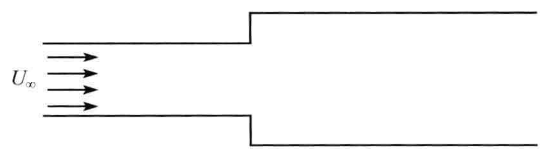

Fig. 0-3 Flow in a long pipe

it is only a rough approximation to the exact boundary condition. If an accurate numerical solution is required, then the computational domain must be large enough to ensure the accuracy. However, this again increases the computational work and memory requirement. Therefore, finding the exact boundary condition or constructing a highly accurate approximate boundary condition on the artificial boundary becomes an essential problem in solving partial differential equations on unbounded domains.

In the past three decades, this problem has attracted the attention of many mathematicians and engineers, and much has been done for various problems with artificial boundaries in different shapes by using various methods and techniques. Gradually, the artificial boundary method has become an important and efficient numerical method in solving partial differential equations on unbounded domains. For a given problem, the main step in the artificial boundary method is to construct a 'suitable' boundary condition on the artificial boundary satisfied by the solution of the original problem, and then reduce the original problem to a boundary value or initial boundary value problem on the bounded computational domain (we will simply call this the reduced problem henceforth.). What kind of ABC is 'suitable'? It should satisfy the following basic requirements:

(1) The reduced problem is well-posed, i.e., the reduced problem has a unique solution and the solution depends continuously on the boundary value (and initial value).

(2) Restricted to the bounded computational domain, the solution of the reduced problem is the same as the solution of the original problem, or it is a good approximation of the solution of the original problem.

(3) The bounded computational domain should be as small as possible, in order to reduce the computational work and memory requirement.

(4) The reduced problem on the bounded computational domain should be easily solved numerically.

Considerable research has been done along this direction. Engquist and Majda (1977) used rational approximation to the wave equation and obtained the absorbing boundary condition. Han and Ying (1980) studied the numerical solution of the exterior problem of the two-dimensional (2-D) Laplace equation, introduced a circle as the artificial boundary, and obtained the exact boundary condition on the artificial boundary by using the Hilbert transform for the integral equation, which is the Steklov-Poincaré mapping (also called Dirichlet to Neumann mapping) satisfied by the solution of the original problem on the artificial boundary. Feng (1980), Feng and Yu (1983) discussed the numerical solution of the interior and exterior problem of the 2-D Laplace and Helmholtz equations, and obtained the exact boundary condition on the artificial boundary (in terms of an integral with strong singularity, also called Natural Boundary Method) by using the Green function. Goldstein (1982) studied the numerical solution of some Helmholtz type equations on unbounded domains, and obtained the ABC for the reduced problem. Feng (1984) studied the exterior problem of the Helmholtz equation, and obtained approximate asymptotic ABCs, which are a group of local ABCs. Han and Wu (1985-A, 1985-B, 1992) obtained an exact ABC in the series form for the exterior problem of the 2-D Laplace and elastic equations by using Fourier series. They obtained a series of approximate ABCs by truncating the infinite series naturally, which are global (non-local) ABCs. These approximate ABCs do not contain integrals with strong singularities, and can be naturally coupled with the finite element method. Yu (1985) also obtained the exact ABC in the series form for the exterior problem of the 2-D Laplace equation. At the same time, Han and Wu (1985-A), Yu (1985) also obtained a series of local ABCs for the exterior problem of the 2-D Laplace equation. Hagstrom and Keller (1986, 1987) discussed the numerical solution of partial differential equation on an infinite cylinder. The infinite cylinder was truncated into a finite one, and the ABC on the cross section was obtained. Furthermore, the ABC was applied to solving the nonlinear partial differential equation on the infinite cylinder (by linearizing the nonlinear partial differential equation on the exterior domain of the finite cylinder). Halpern and Schatzman (1989) studied the 2-D steady Oseen system with small viscosity, and obtained a group of boundary conditions on the straight line artificial boundary. Nataf (1989) analyzed the steady Oseen system, and obtained the open boundary condition on the straight line artificial boundary. Keller and Givoli (1989) studied the exterior problem of the Laplace and Helmholtz equations, and obtained the global (non-local) ABC in series form.

In the late 90s of the last century, the study of the artificial boundary method reached a fast-expanding stage, and soon encompassed many important problems in science and technology governed by partial differential equations. For example, exact boundary conditions are obtained for the two and three-dimensional (3-D) Navier (linear elastic) equations, Stokes equations, and Oseen equations on the artificial boundaries, and absorbing boundary conditions were obtained for the sound wave equation, the Klein-Gordon equation, the Schrödinger equation, elastic wave equations, and electric magnetic wave equations. In recent years, the focus has been on the numerical solution of nonlinear partial differential equations on unbounded domains. For some nonlinear partial differential equations, such as the Burgers equation, the Kardar-Parisi-Zhang (K-P-Z) equation for describing the growth of thin film, the one-dimensional (1-D) cubic nonlinear Schrödinger equation, and the Korteweg-de Vries equation, exact boundary conditions have been obtained on the artificial boundaries (Han, Wu and Xu (2006); Xu, Han, and Wu (2006); Zheng (2006-A); Zheng (2006-B)). In the meantime, various approximate nonlinear ABCs have been obtained for several nonlinear wave propagation problems (for example, Szetlel (2006-A, 2006-B); Xu and Han (2006); Xu, Han, and Wu (2007)). The techniques and methods in devising and constructing the ABCs tend to vary and mature, and their applications have been seen in many fields of science and technologies.

The ABCs can be divided into two categories: the explicit ABCs and the implicit ABCs. The explicit boundary conditions can be further divided into global ABCs, local ABCs, and discrete ABCs. The global ABCs are usually given by integrals on the artificial boundary involving the unknown function and its derivatives, and the exact ABCs for most problems are the global ABCs. In practical computations, the global ABC requires large computational work and memory, especially for the evolution equations, for which the ABC usually depends on historical data. If a long-time computation is needed, then a fast algorithm is necessary for evaluating the integral in the ABC. For the numerical solution of the elliptic problems on unbounded domains, the global ABC is a good choice, since it can be naturally coupled with the finite element method. The local ABC is usually given by an equation involving the unknown function and its derivatives. In general, the local ABC is only an approximation of the exact ABC (except for the 1-D sound wave equation, where its exact ABC is a local ABC). If high accuracy is required, then high order derivatives of the unknown function are needed in the local ABC, which introduces a new difficulty for the computation. The local ABC is often used for the numerical computation of the linear and nonlinear wave equations. Sometimes, we can discretize the original problem on the unbounded physical domain, find the condition on the artificial boundary satisfied by the discrete (numerical) solution on the unbounded domain, and then reduce the original problem to a boundary-value problem on the computational domain. We call this kind of condition on the artificial boundary satisfied by the discrete (numerical) solution as the discrete ABC. The implicit ABC is given by an implicit integral equation on the artificial boundary involving the unknown function and its derivatives. In many cases, it comes from the coupling formula of the boundary element method and the finite element method. The main advantage of the implicit ABC is that it has no requirement on the shape of the artificial boundary. In practical problems, the ABC has other names also, which are used to emphasize a certain property of the chosen artificial boundary condition. Absorbing ABC, non-reflecting ABC, open boundary condition, transparent ABC, totally absorbing ABC, nonlinear ABC, etc. are a few such examples.

As the fast development of the artificial boundary method, the mathematical basis and theoretical analysis of the artificial boundary method have been set up gradually. For the exterior problem of the second-order elliptic equation, the Navier (linear elastic) system, and the Stokes system, optimal error estimates for the numerical solution using the artificial boundary method have been obtained. Furthermore, the dependence of the error on the size of the mesh, the position of the artificial boundary, and the accuracy of the ABC are also being understood now. For some evolution equations like the heat equation and the Schrödinger equation, error estimates of the finite difference solution using the artificial boundary method have been obtained.

The contents of the book are based on research studies of the authors in the past 30 years, and the research of other experts. This book systematically introduces the numerical schemes and theoretical basis of the artificial boundary method. It can be used as a textbook for graduate students in the area of science and engineering computation, and it can also be used as a reference book for persons working in the area of science and engineering computation.

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