NONLINEAR
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SCIENCE

Vladimir V. Uchaikin

Fractional Derivatives for Physicists and Engineers

Volume II Applications

物理及工程中的分数维微积分

第 II 卷 应用

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7.1 Tautochrone problem

7.1.1 Non-relativistic case

We start reviewing applications of fractional calculus with the following mechanical problem.

A classical, non-relativistic, point particle of mass m with potential energy $U = mgy$ begins to slide without friction along a curve running through a vertical plane $x-y$ to the origin and reaches it at time τ (Fig. 7.1). The problem is to find the function $\tau(h)$ that specifies the total time of descent from an initial height h. A special case of the problem when $\tau(h) = \text{const}$ is called the *tautochrone problem* (from Greek prefixes *tauto* meaning "same" and *chrono* "time").

The principle of conservation energy says

$$\frac{m}{2}\left(\frac{ds}{dt}\right)^2 = mg(h-y),$$

where s is the length of the curved segment between the origin and a current position, and $h-y$ is the descent of the particle during time t. Hence

$$dt = -\frac{ds}{\sqrt{2g(h-y)}}.$$

After integrating, we directly arrive at the equation

$$\tau(h) = \int_0^h \frac{ds/dy}{\sqrt{2g(h-y)}}dy = \sqrt{\frac{\pi}{2g}}\,{}_0^{1/2}\mathsf{D}_h s(h)$$

called *Abel's integral equation*, solution of which has opened a way to the land of fractional equations (Abel, 1881).

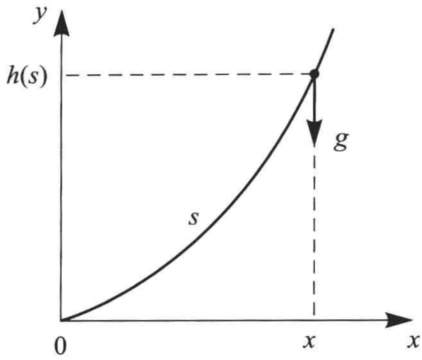

Fig. 7.1 Abel's mechanical problem.

Muñoz and Fernandéz-Anaya (2010) apply the fractional approach to investigation of the properties of the tautochrone and brachistochrone curves by introducing a family of curves complying with relations where the time of descent is proportional to a fractional power of the height difference.

7.1.2 Relativistic case

The relativistic counterpart of this problem has been studied by Kamath (1992). The methods of fractional calculus are shown to be more useful in the derivation of the exact relativistic tautochrone. Relativistic kinematics yields the equation of conservation energy

$$mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} + Q,$$

where Q is the energy lost by the particle from the gravitation field as it is released from a height h and is given by

$$Q = mc^2\left\{1-\exp[g(h-y)/c^2]\right\}$$

(Goldstein and Bender, 1986). From two these equations, we find

$$v(y) = c\sqrt{1-\exp[2g(y-h)/c^2]}.$$

The time of fall is

$$T = \int_0^T dt = -\int_h^0 \frac{ds}{v(y)} = \int_0^h \frac{s'(y)dy}{c\sqrt{1-\exp[2g(y-h)/c^2]}},$$

with $s'(y) = ds/dy$ being the arclength along the path joining the initial $(x_0, y_0 = h)$ and final $(0,0)$ end points. By rewriting this equation as

$$cT = i\int_0^h \frac{e^{-gy/c^2}s'(y)dy}{\sqrt{e^{-2gh/c^2}-e^{-2gy/c^2}}} = -\frac{ic^2}{2g}\int_0^h \frac{(-2g/c^2)e^{-gy/c^2}}{\sqrt{e^{-2gh/c^2}-e^{2gy/c^2}}}s'(y)dy,$$

one arrives at the fractional equation for the function $\eta(h) = s'(h)e^{gh/c^2}$ determining the sought curve:

$$c\sqrt{\pi}\,{}_0\mathsf{D}_{\mu(h)}^{-1/2}\eta(h) = 2iTg,$$

with $\mu(h) = e^{2gh/c^2} - 1$. Converting the equation to

$${}_0\mathsf{D}_{\mu(h)}^{1/2}\,{}_0\mathsf{D}_{\mu(h)}^{-1/2}\eta(h) = {}_0\mathsf{D}_{\mu(h)}^{1/2}(2iTg/c\sqrt{\pi}),$$

and using the composition rules, the author reduces it to the form

$$\sqrt{\pi}e^{gh/c^2}s'(h) = \frac{2iTg}{c\sqrt{\pi}}\frac{d}{d\mu(h)}\int_0^h \frac{-2ge^{-2gy/c^2}dy}{c^2\sqrt{e^{-2gh/c^2}-e^{-2gy/c^2}}}$$

$$= 2\frac{2iTg}{c\sqrt{\pi}}\frac{d[\mu(h)]^{1/2}}{d\mu(h)} = \frac{2iTg}{c\sqrt{\pi}}[\mu(h)]^{-1/2}.$$

Solution of this equation leads to the following parametric representation of the sought tautochrone:

$$e^{2gy/c^2} = 1 + \left(\frac{2Tg}{\pi c}\right)^2\cos^2\theta,$$

$$\frac{gx}{c^2} = \theta - \frac{\pi}{2} + a\left(\frac{\pi}{2} - \arctan\left(\frac{1}{a}\tan\theta\right)\right)$$

with

$$a = 1 + \left(\frac{2Tg}{\pi c}\right)^2.$$

The non-relativistic tautochrone problem was generalized to an arbitrary potential $U(y)$ as well (Gómez and Marquina, 2008). In this case

$$dt = -\frac{ds}{\sqrt{(2/m)U(h-y)}}$$

and

$$T = -\int_{y_0}^{y}\frac{s'(y)dy}{\sqrt{(2/m)[U(y_0)-U(y)]}}. \qquad (7.1)$$

Making $z = U(y)$, one can write

$$\frac{ds}{dy}dy = \frac{ds}{dz}dz$$

with

$$dz = U'dy,$$

so then Eq. (7.1) takes the form

$$T = -\sqrt{\frac{m}{2}} \int_{z_0}^{z} \frac{s'(z)dz}{\sqrt{z_0 - z}}.$$

Solution of this equation

$$x = \int_0^y \sqrt{\frac{2T^2}{m\pi^2}\frac{U'^2}{U} - 1}\, dy$$

was found in (Flores and Osler, 1999).

7.2 Inverse problems

7.2.1 Finding potential from a period-energy dependence

Let us look at Sect. 12 of "Mechanics" (Landau and Lifshitz, 1981) devoted to finding of potential energy $U(x)$ from the oscillation period T given as a function of the total energy E, $T = T(E)$ (Fig. 7.2). Starting as before from the principle of energy conservation, they obtain the integral equation

$$T(E) = 2\sqrt{2m} \int_0^{X(E)} \frac{dx}{\sqrt{E - U(x)}}, \tag{7.2}$$

where $X(E)$ is a root of equation $U(x) = E$ (for simplicity let the potential be an even function, monotonically increasing with moving from the origin). Passage to integration variable U leads to the fractional differential equation

$$T(E) = 2\sqrt{2m} \int_0^E \frac{dX(U)}{dU} \frac{dU}{\sqrt{E - U}} = 2\sqrt{2m\pi}\ {}_0^{1/2}\mathsf{D}_E X(E),$$

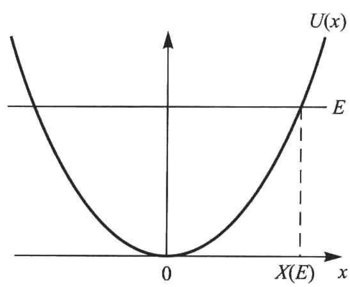

Fig. 7.2 Illustration to Eq. (7.2).

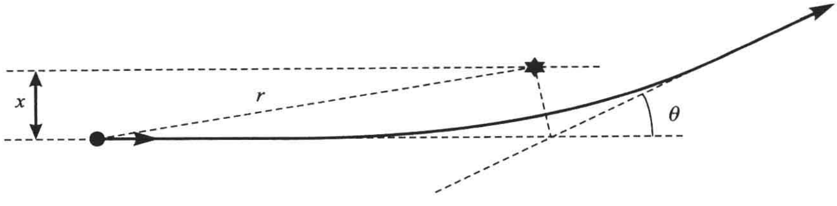

Fig. 7.3 Illustration to Eq. (7.3).

the integration of which yields Eq. (12.2) of the mentioned "Mechanics":

$$X(U) = \frac{1}{2\pi\sqrt{2m}} \int_0^U \frac{T(E)dE}{\sqrt{U-E}}.$$

It belongs to the class of *inverse problems*[1] formulated as finding properties of a system by observation of its motion. Just as Newton came to differential calculus considering a primal mechanical problem, Abel found a way to fractional differential calculus solving some special inverse problem.

7.2.2 Finding potential from scattering data

One more inverse mechanical problem is to find the interaction potential $U(r)$ from experimental data on scattering of particles with a given energy E. It is related to solving some integral equation (see (Buck, 1974) and Sect. 11.6 of book (Pavlenko, 2002)). The simplest statement of this problem is assuming the potential function $U(r)$ to be monotonically increasing and vanishing at infinity find it form the connection between the scattering angle θ and the impact parameter x is expressed by formula (Landau and Lifshitz, 1981) (Fig 7.3)

$$\theta(x) = \frac{x}{E} \int_x^\infty \frac{dU}{dr} \frac{dr}{\sqrt{r^2-x^2}}. \tag{7.3}$$

The formula represents a right-hand Gerasimov-Caputo semiderivative with respect to x^2:

$$\Theta(x) \equiv \frac{\theta(x)}{x} = \frac{1}{E} \int_{x^2}^\infty \frac{dU}{dr^2} \frac{dr^2}{\sqrt{r^2-x^2}} = \frac{\sqrt{\pi}}{E} \, {}_{x^2}^{1/2}\mathsf{D}_\infty U(x^2).$$

Inverting this relation gives the result

$$U(r) = \frac{E}{\sqrt{\pi}} \, {}_{r^2}\mathsf{I}_\infty^{1/2} \Theta(r^2) = \frac{2E}{\pi} \int_r^\infty \frac{\theta(x)dx}{\sqrt{x^2-r^2}}, \tag{7.4}$$

[1] The term "inverse problem" was introduced by Soviet-Armenian astrophysicist Victor Ambartsumian. His paper on the inverse Sturm-Liouville problem (1929) was found by Swedish mathematicians at the end of the Second World War and accepted as a foundation of a new discipline.

coincident with Eq. (11.16) in the book (Pavlenko, 2002). Finding $x(\theta)$ from the equation

$$\frac{1}{2}\frac{dx^2}{d\cos\theta} = \left[\frac{d\sigma(\theta)}{d\Omega}\right]_{\text{exper}},$$

inverting the obtained function and substituting the result $\theta(x) = \theta_{\text{exper}}(x)$ into Eq. (7.4), we come to the desired function $U(r)$.

7.2.3 Stellar systems

The construction of self-consistent models for stellar systems is of great interest in astrophysics. The most straightforward way to built such models is to start with an assumed potential defining the mass density ρ and the families of stars orbits with the distribution function f. The integral relation connecting f and ρ is known as the *self-gravitation equation*. Pedraza et al. (2008) generalized the classical methods involving the fractional derivatives. We sketch here this approach.

According to Jean's theorem, the phase space distribution function $f(\mathbf{r},\mathbf{v})$ is a function only of the isolating integrals of motion that are conserved in each orbit. For spherical symmetry these are the energy E and the angular momentum L_z.[2] Assume that Φ is the gravitational potential and define a relative potential $\Psi = -\Phi + \Phi_0$ and a relative energy $\varepsilon = -E + \Phi_0$, in such a way that the system has only stars with energy $\varepsilon > 0$. An axisymmetric system admits two isolated integrals: z-component of the angular momentum about the z-axis, $L_z = Rv_\varphi$ and $\varepsilon = -E + \Phi_0$. For a steady-state axisymmetric stellar system, the even part of the distribution function with respect to L_z, f_+, related to the mass density as

$$\rho(R,\Psi) = \frac{4\pi}{R}\int_0^{\Psi}\int_0^{R\sqrt{2(\Psi-\varepsilon)}} f_+(\varepsilon, L_z)dL_z d\varepsilon. \tag{7.5}$$

The authors seek the solution of the integral equation in the form

$$f_+(\varepsilon, L_z) = \sum_n L_z^{2\alpha_n} h_n(\varepsilon).$$

Inserting it into Eq. (7.5) and integrating with respect to L_z they obtain

$$\rho(R,\Psi) = \sum_n R^{2\alpha_n}\tilde{\rho}_n(\Psi), \quad \alpha_n > -1/2, \tag{7.6}$$

where

$$\tilde{\rho}_n(\Psi) = \frac{4\pi 2^{\alpha_n+1/2}}{2\alpha_n+1}\int_0^{\Psi} h_n(\varepsilon)(\Psi-\varepsilon)^{\alpha_n+1/2}d\varepsilon.$$

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[2] As indicated in the cited work, as far back as Eddington showed that it is possible to obtain such distribution functions by first expressing the density as a function of the potential, and then solving an Abel integral equation.

This integral equation can easily be inverted by means of fractional derivatives method,

$$h_n(\varepsilon) = \frac{{}_0\mathrm{D}_{\Psi}^{\alpha_n+3/2}\tilde{\rho}_n(\Psi)\Big|_{\Psi=0}}{\pi 2^{\alpha_n+3/2}\Gamma(\alpha_n+1/2)},$$

so the result is

$$f_+(\varepsilon, L_z) = \sum_n \frac{{}_0\mathrm{D}_{\Psi}^{\alpha_n+3/2}\tilde{\rho}_n(\Psi)\Big|_{\Psi=0}}{\pi 2^{\alpha_n+3/2}\Gamma(\alpha_n+1/2)} L_z^{2\alpha_n}.$$

Here $\tilde{\rho}_n(\Psi)$ is assumed to be known because of Eq. (7.6).

7.3 Motion through a viscous fluid

7.3.1 Entrainment of fluid by a moving wall

We will consider motion of a body in an incompressible viscous Newton's fluid. In absence of external bulk forces, the unsteady flow of such a fluid is governed by the Navier-Stockes (N-S) equations system

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j^2},$$

where u_i, $i = 1,2,3$, are velocity components along x_i-coordinate axis respectively, ρ is the density, p is the pressure, and ν denotes the kinematic viscosity. This system should be supplemented with the incompressibility condition

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0.$$

For specificity of solution, this system must be accompanied by initial and boundary conditions.

Let us apply the N-S equations to the problem of entrainment of fluid by an unbounded (or of large size) plate moving in xOz-plane along x-axis (Fig. 7.4). The plate is at rest until $t = 0$ and then begins to move with a constant velocity V to positive direction of x-axis. The liquid particles velocity will contain only one nonzero component, $u_x = u$, and this component, due to unboundedness of the plate, will depend upon only z-coordinate,

$$u_x = u(z,t), \quad u_y = u_z = 0.$$

The pressure fall is zero too,