

Leonid B. Koralov
Yakov G. Sinai

Theory of Probability and Random Processes

Second Edition

概率论和随机过程 第2版

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Preface

This book is primarily based on a one-year course that has been taught for a number of years at Princeton University to advanced undergraduate and graduate students. During the last year a similar course has also been taught at the University of Maryland.

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Leonid Korolov
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Part I

Probability Theory

Random Variables and Their Distributions

1.1 Spaces of Elementary Outcomes, σ -Algebras, and Measures

The first object encountered in probability theory is the space of elementary outcomes. It is simply a non-empty set, usually denoted by Ω , whose elements $\omega \in \Omega$ are called elementary outcomes. Here are several simple examples.

Example. Take a finite set $X = \{x^1, \dots, x^r\}$ and the set Ω consisting of sequences $\omega = (\omega_1, \dots, \omega_n)$ of length $n \geq 1$, where $\omega_i \in X$ for each $1 \leq i \leq n$. In applications, ω is a result of n statistical experiments, while ω_i is the result of the i -th experiment. It is clear that $|\Omega| = r^n$, where $|\Omega|$ denotes the number of elements in the finite set Ω . If $X = \{0, 1\}$, then each ω is a sequence of length n made of zeros and ones. Such a space Ω can be used to model the result of n consecutive tosses of a coin. If $X = \{1, 2, 3, 4, 5, 6\}$, then Ω can be viewed as the space of outcomes for n rolls of a die.

Example. A generalization of the previous example can be obtained as follows. Let X be a finite or countable set, and I be a finite set. Then $\Omega = X^I$ is the space of all functions from I to X .

If $X = \{0, 1\}$ and $I \subset \mathbb{Z}^d$ is a finite set, then each $\omega \in \Omega$ is a configuration of zeros and ones on a bounded subset of d -dimensional lattice. Such spaces appear in statistical physics, percolation theory, etc.

Example. Consider a lottery game where one tries to guess n distinct numbers and the order in which they will appear out of a pool of r numbers (with $n \leq r$). In order to model this game, define $X = \{1, \dots, r\}$. Let Ω consist of sequences $\omega = (\omega_1, \dots, \omega_n)$ of length n such that $\omega_i \in X, \omega_i \neq \omega_j$ for $i \neq j$, and $X = \{1, \dots, r\}$. It is easy to show that $|\Omega| = r!/(r-n)!$.

Later in this section we shall define the notion of a probability measure, or simply probability. It is a function which ascribes real numbers between zero

and one to certain (but not necessarily all!) subsets $A \subseteq \Omega$. If Ω is interpreted as the space of possible outcomes of an experiment, then the probability of A may be interpreted as the likelihood that the outcome of the experiment belongs to A . Before we introduce the notion of probability we need to discuss the classes of sets on which it will be defined.

Definition 1.1. A collection \mathcal{G} of subsets of Ω is called an algebra if it has the following three properties.

1. $\Omega \in \mathcal{G}$.
2. $C \in \mathcal{G}$ implies that $\Omega \setminus C \in \mathcal{G}$.
3. $C_1, \dots, C_n \in \mathcal{G}$ implies that $\bigcup_{i=1}^n C_i \in \mathcal{G}$.

Example. Given a set of elementary outcomes Ω , let \mathcal{G} contain two elements: the empty set and the entire set Ω , that is $\mathcal{G} = \{\emptyset, \Omega\}$. Define $\bar{\mathcal{G}}$ as the collection of all the subsets of Ω . It is clear that both \mathcal{G} and $\bar{\mathcal{G}}$ satisfy the definition of an algebra. Let us show that if Ω is finite, then the algebra $\bar{\mathcal{G}}$ contains $2^{|\Omega|}$ elements.

Take any $C \subseteq \Omega$ and introduce the function $\chi_C(\omega)$ on Ω :

$$\chi_C(\omega) = \begin{cases} 1 & \text{if } \omega \in C, \\ 0 & \text{otherwise,} \end{cases}$$

which is called the indicator of C . It is clear that any function on Ω taking values zero and one is an indicator function of some set and determines this set uniquely. Namely, the set consists of those ω , where the function is equal to one. The number of distinct functions from Ω to the set $\{0, 1\}$ is equal to $2^{|\Omega|}$.

Lemma 1.2. Let Ω be a space of elementary outcomes, and \mathcal{G} be an algebra. Then

1. The empty set is an element of \mathcal{G} .
2. If $C_1, \dots, C_n \in \mathcal{G}$, then $\bigcap_{i=1}^n C_i \in \mathcal{G}$.
3. If $C_1, C_2 \in \mathcal{G}$, then $C_1 \setminus C_2 \in \mathcal{G}$.

Proof. Take $C = \Omega \in \mathcal{G}$ and apply the second property of Definition 1.1 to obtain that $\emptyset \in \mathcal{G}$. To prove the second statement, we note that

$$\Omega \setminus \bigcap_{i=1}^n C_i = \bigcup_{i=1}^n (\Omega \setminus C_i) \in \mathcal{G}.$$

Consequently, $\bigcap_{i=1}^n C_i \in \mathcal{G}$. For the third statement, we write

$$C_1 \setminus C_2 = \Omega \setminus ((\Omega \setminus C_1) \cup C_2) \in \mathcal{G}.$$

□

Lemma 1.3. *If an algebra \mathcal{G} is finite, then there exist non-empty sets $B_1, \dots, B_m \in \mathcal{G}$ such that*

1. $B_i \cap B_j = \emptyset$ if $i \neq j$.
2. $\Omega = \bigcup_{i=1}^m B_m$.
3. *For any set $C \in \mathcal{G}$ there is a set $I \subseteq \{1, \dots, m\}$ such that $C = \bigcup_{i \in I} B_i$ (with the convention that $C = \emptyset$ if $I = \emptyset$).*

Remark 1.4. The collection of sets $B_i, i = 1, \dots, m$, defines a partition of Ω . Thus, finite algebras are generated by finite partitions.

Remark 1.5. Any finite algebra \mathcal{G} has 2^m elements for some integer $m \in \mathbb{N}$. Indeed, by Lemma 1.3, there is a one-to-one correspondence between \mathcal{G} and the collection of subsets of the set $\{1, \dots, m\}$.

Proof of Lemma 1.3. Let us number all the elements of \mathcal{G} in an arbitrary way:

$$\mathcal{G} = \{C_1, \dots, C_s\}.$$

For any set $C \in \mathcal{G}$, let

$$C^1 = C, \quad C^{-1} = \Omega \setminus C.$$

Consider a sequence $b = (b_1, \dots, b_s)$ such that each b_i is either $+1$ or -1 and set

$$B^b = \bigcap_{i=1}^s C_i^{b_i}.$$

From the definition of an algebra and Lemma 1.2 it follows that $B^b \in \mathcal{G}$. Furthermore, since

$$C_i = \bigcup_{b: b_i=1} B^b,$$

any element C_i of \mathcal{G} can be obtained as a union of some of the B^b . If $b' \neq b''$, then $B^{b'} \cap B^{b''} = \emptyset$. Indeed, $b' \neq b''$ means that $b'_i \neq b''_i$ for some i , say $b'_i = 1, b''_i = -1$. In the expression for $B^{b'}$ we find $C_i^1 = C_i$, so $B^{b'} \subseteq C_i$. In the expression for $B^{b''}$ we find $C_i^{-1} = \Omega \setminus C_i$, so $B^{b''} \subseteq \Omega \setminus C_i$. Therefore, all B^b are pair-wise disjoint. We can now take as B_i those B^b which are not empty. \square

Definition 1.6. *A collection \mathcal{F} of subsets of Ω is called a σ -algebra if \mathcal{F} is an algebra which is closed under countable unions, that is $C_i \in \mathcal{F}, i \geq 1$, implies that $\bigcup_{i=1}^{\infty} C_i \in \mathcal{F}$. The elements of \mathcal{F} are called measurable sets, or events.*

As above, the simplest examples of a σ -algebra are the trivial σ -algebra, $\mathcal{F} = \{\emptyset, \Omega\}$, and the σ -algebra $\overline{\mathcal{F}}$ which consists of all the subsets of Ω .

Definition 1.7. *A measurable space is a pair (Ω, \mathcal{F}) , where Ω is a space of elementary outcomes and \mathcal{F} is a σ -algebra of subsets of Ω .*

Remark 1.8. A space of elementary outcomes is said to be discrete if it has a finite or countable number of elements. Whenever we consider a measurable space (Ω, \mathcal{F}) with a discrete space Ω , we shall assume that \mathcal{F} consists of all the subsets of Ω .

The following lemma can be proved in the same way as Lemma 1.2.

Lemma 1.9. *Let (Ω, \mathcal{F}) be a measurable space. If $C_i \in \mathcal{F}$, $i \geq 1$, then $\bigcap_{i=1}^{\infty} C_i \in \mathcal{F}$.*

It may seem that there is little difference between the concepts of an algebra and a σ -algebra. However, such an appearance is deceptive. As we shall see, any interesting theory (such as measure theory or probability theory) requires the notion of a σ -algebra.

Definition 1.10. *Let (Ω, \mathcal{F}) be a measurable space. A function $\xi : \Omega \rightarrow \mathbb{R}$ is said to be \mathcal{F} -measurable (or simply measurable) if $\{\omega : a \leq \xi(\omega) < b\} \in \mathcal{F}$ for each $a, b \in \mathbb{R}$.*

Below we shall see that linear combinations and products of measurable functions are again measurable functions. If Ω is discrete, then any real-valued function on Ω is a measurable function, since \mathcal{F} contains all the subsets of Ω .

In order to understand the concept of measurability better, consider the case where \mathcal{F} is finite. Lemma 1.3 implies that \mathcal{F} corresponds to a finite partition of Ω into subsets B_1, \dots, B_m , and each $C \in \mathcal{F}$ is a union of some of the B_i .

Theorem 1.11. *If ξ is \mathcal{F} -measurable, then it takes a constant value on each element of the partition B_i , $1 \leq i \leq m$.*

Proof. Suppose that ξ takes at least two values, a and b , with $a < b$ on the set B_j for some $1 \leq j \leq m$. The set $\{\omega : a \leq \xi(\omega) < (a+b)/2\}$ must contain at least one point from B_j , yet it does not contain the entire set B_j . Thus it can not be represented as a union of some of the B_i , which contradicts the \mathcal{F} -measurability of the set. \square

Definition 1.12. *Let (Ω, \mathcal{F}) be a measurable space. A function $\mu : \mathcal{F} \rightarrow [0, \infty)$ is called a finite non-negative measure if*

$$\mu\left(\bigcup_{i=1}^{\infty} C_i\right) = \sum_{i=1}^{\infty} \mu(C_i)$$

whenever $C_i \in \mathcal{F}$, $i \geq 1$, are such that $C_i \cap C_j = \emptyset$ for $i \neq j$.

The property expressed in Definition 1.12 is called the countable additivity (or the σ -additivity) of the measure.

Remark 1.13. Most often we shall omit the words finite and non-negative, and simply refer to μ as a measure. Thus, a measure is a σ -additive function on \mathcal{F} with values in \mathbb{R}^+ . In contrast, σ -finite and signed measures, to be introduced in Chapter 3, take values in $\mathbb{R}^+ \cup \{+\infty\}$ and \mathbb{R} , respectively.

Definition 1.14. Let g be a binary function on Ω with values 1 (true) and 0 (false). It is said that g is true almost everywhere if there is an event C with $\mu(C) = \mu(\Omega)$ such that $g(\omega) = 1$ for all $\omega \in C$.

Definition 1.15. A measure P on a measurable space (Ω, \mathcal{F}) is called a probability measure or a probability distribution if $P(\Omega) = 1$.

Definition 1.16. A probability space is a triplet (Ω, \mathcal{F}, P) , where (Ω, \mathcal{F}) is a measurable space and P is a probability measure. If $C \in \mathcal{F}$, then the number $P(C)$ is called the probability of C .

Definition 1.17. A measurable function defined on a probability space is called a random variable.

Remark 1.18. When P is a probability measure, the term “almost surely” is often used instead of “almost everywhere”.

Remark 1.19. Let us replace the σ -additivity condition in Definition 1.12 by the following: if $C_i \in \mathcal{F}$ for $1 \leq i \leq n$, where n is finite, and $C_i \cap C_j = \emptyset$ for $i \neq j$, then

$$\mu\left(\bigcup_{i=1}^n C_i\right) = \sum_{i=1}^n \mu(C_i).$$

This condition leads to the notion of a finitely additive function, instead of a measure. Notice that finite additivity implies superadditivity for infinite sequences of sets. Namely,

$$\mu\left(\bigcup_{i=1}^{\infty} C_i\right) \geq \sum_{i=1}^{\infty} \mu(C_i)$$

if the sets C_i are disjoint. Indeed, otherwise we could find a sufficiently large n such that

$$\mu\left(\bigcup_{i=1}^{\infty} C_i\right) < \sum_{i=1}^n \mu(C_i),$$

which would violate the finite additivity.

Let Ω be discrete. Then $p(\omega) = P(\{\omega\})$ is the probability of the elementary outcome ω . It follows from the definition of the probability measure that

1. $p(\omega) \geq 0$.
2. $\sum_{\omega \in \Omega} p(\omega) = 1$.