Graduate Texts in Mathematics

R. Michael Range

Holomorphic Functions and Integral Representations in Several Complex Variables

多复变中的全纯函数和积分表示

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To my family

Sandrina,

Ofelia, Marisa, and Roberto

Preface to the Second Printing

I am pleased that Springer-Verlag has decided to reprint this book, giving me the opportunity to include a number of corrections. The changes in this printing are mainly limited to the correction of typographical and other minor errors which I have become aware of since 1986. A couple of changes should be mentioned explicitly. The definition of *completely singular* in Chapter II.2 has been modified slightly, the proof of Theorem II.2.3 has been changed accordingly, and Appendix C has been rewritten. I would like to thank J. Bruna, K. Burke, J. Fleron, W. Rudin, W. Stoll and E. Straube for their helpful comments.

Soon after publication of the first edition, E. Martinelli brought to my attention some papers by F. Severi and G. Fichera, which play a fundamental role in the development of the so-called *CR* extension theorem. This result—a strengthening of the celebrated Hartogs Extension Theorem—involves the characterization of boundary values of holomorphic functions by the *tangential* Cauchy-Riemann equations, i.e., by intrinsic conditions on the boundary. Unfortunately, much of the literature written in English related to this topic had been unaware of the work of Severi and Fichera, and thus it presents an inaccurate account of its history. In view of the new evidence and of the great interest which the Hartogs and *CR* extension phenomena have generated over many decades, I have completely revised and expanded the relevant sections of the Notes to Chapter IV, and I have added several new references to the bibliography. It is my pleasure to thank R. Gunning, L. Hörmander, J.J. Kohn, E. Martinelli, and H. Rossi for their assistance in setting the record straight.

Albany, New York
January 1998

R. Michael Range

Preface to the First Printing

The subject of this book is Complex Analysis in Several Variables. This text begins at an elementary level with standard local results, followed by a thorough discussion of the various fundamental concepts of "complex convexity" related to the remarkable extension properties of holomorphic functions in more than one variable. It then continues with a comprehensive introduction to integral representations, and concludes with complete proofs of substantial global results on domains of holomorphy and on strictly pseudoconvex domains in $\mathbb{C}^n$, including, for example, C. Fefferman's famous Mapping Theorem.

The most important new feature of this book is the systematic inclusion of many of the developments of the last 20 years which centered around integral representations and estimates for the Cauchy–Riemann equations. In particular, integral representations are the principal tool used to develop the global theory, in contrast to many earlier books on the subject which involved methods from commutative algebra and sheaf theory, and/or partial differential equations. I believe that this approach offers several advantages: (1) it uses the several variable version of tools familiar to the analyst in one complex variable, and therefore helps to bridge the often perceived gap between complex analysis in one and in several variables; (2) it leads quite directly to deep global results without introducing a lot of new machinery; and (3) concrete integral representations lend themselves to estimations, therefore opening the door to applications not accessible by the earlier methods.

The Contents and the opening paragraphs of each chapter will give the reader more detailed information about the material in this book.

A few historical comments might help to put matters in perspective. Already by the middle of the 19th century, B. Riemann had recognized that the description of all complex structures on a given compact surface involved complex *multidimensional* "moduli spaces." Before the end of the century, K. Weierstrass, H. Poincaré, and P. Cousin had laid the foundation of the local theory and generalized important global results about holomorphic functions from regions in the complex plane to product domains in $\mathbb{C}^2$ or in $\mathbb{C}^n$. In 1906, F. Hartogs discovered domains in $\mathbb{C}^2$ with the property that all functions holomorphic on it necessarily extend holomorphically to a strictly larger domain, and it rapidly became clear that an understanding of this new phenomenon—which does not appear in one complex variable—would be a central problem in multidimensional function theory. But in spite of major contributions by Hartogs, E.E. Levi, K. Reinhardt, S. Bergman, H. Behnke, H. Cartan, P. Thullen, A. Weil, and others, the principal global problems were still unsolved by the mid 1930s. Then K. Oka introduced some brilliant new ideas, and from 1936 to 1942 he systematically solved these problems one after the other. However, Oka's work had much more far-reaching implications. In 1940, H. Cartan began to investigate certain algebraic notions implicit in Oka's work, and in the years thereafter, he and Oka, independently, began to widen and deepen the algebraic foundations of the theory, building upon K. Weierstrass' Preparation Theorem. By the time the ideas of Cartan and Oka became widely known in the early 1950s, they had been reformulated by Cartan and J.P. Serre in the language of sheaves. During the 1950s and early 1960s, these new methods and tools were used with great success by Cartan, Serre, H. Grauert, R. Remmert, and many others in building the foundation for the general theory of "complex spaces," i.e., the appropriate higher dimensional analogues of Riemann surfaces. The phenomenal progress made in those years simply overshadowed the more constructive methods present in Oka's work up to 1942, and to the outsider, Several Complex Variables seemed to have become a new abstract theory which had little in common with classical complex analysis.

The solution of the $\bar{\partial}$-Neumann problem by J.J. Kohn in 1963 and the publication in 1966 of L. Hörmander's book in which Several Complex Variables was presented from the point of view of the theory of partial differential equations, signaled the beginning of a reapproachment between Several Complex Variables and Analysis. Around 1968–69, G.M. Henkin and E. Ramirez—in his dissertation written under H. Grauert—introduced Cauchy-type integral formulas on strictly pseudoconvex domains. These formulas, and their application shortly thereafter by Grauert/Lieb and Henkin to solving the Cauchy–Riemann equations with supremum norm estimates, set the stage for the solution of "hard analysis" problems during the 1970s. At the same time, these developments led to a renewed and rapidly increasing interest in Several Complex Variables by analysts with widely differing backgrounds.

First plans to write a book on Several Complex Variables reflecting these latest developments originated in the late 1970s, but they took concrete form only in 1982 after it was discovered how to carry out relevant global constructions directly by means of integral representations, thus avoiding the need to introduce other tools at an early stage in the development of the theory. This emphasis on integral representations, however, does not at all mean that coherent analytic sheaves and methods from partial differential equations are no longer needed in Several Complex Variables. On the contrary, these methods are and will remain indispensable. Therefore, this book contains a long motivational discussion of the theory of coherent analytic sheaves as well as numerous references to other topics, including the theory of the $\bar{\partial}$-Neumann problem, in order to encourage the reader to deepen his or her knowledge of Several Complex Variables. On the other hand, the methods presented here allow a rather direct approach to substantial global results in $\mathbb{C}^n$ and to applications and problems at the present frontier of knowledge, which should be made accessible to the interested reader without requiring much additional technical baggage. Furthermore, the fact that integral representations have led to the solution of major problems which were previously inaccessible would suggest that these methods, too, have earned a lasting place in complex analysis in several variables.

In order to limit the size of this book, many important topics—for which fortunately excellent references are available—had to be omitted. In particular, the systematic development of global results is limited to regions in $\mathbb{C}^n$. Of course, Stein manifolds are introduced and mentioned in several places, but even though it is possible to extend the approach via integral representations to that level of generality, not much would be gained to compensate for the additional technical complications this would entail. Moreover, it is my view that the reader who has reached a level at which Stein manifolds (or Stein spaces) become important should in any case systematically learn the relevant methods from partial differential equations and coherent analytic sheaves by studying the appropriate references.

I have tried to trace the original sources of the major ideas and results presented in this book in extensive Notes at the end of each chapter and, occasionally, in comments within the text. But it is almost impossible to do the same for many Lemmas and Theorems of more special type and for the numerous variants of classical arguments which have evolved over the years thanks to the contributions of many mathematicians. Under no circumstances does the lack of a specific attribution of a result imply that the result is due to the author. Still, the expert in the field will perhaps notice here and there some simplifications in known proofs, and novelties in the organization of the material. The Bibliography reflects a similar philosophy: it is not intended to provide a complete encyclopedic listing of all articles and books written on topics related to this book. I believe, however, that it does adequately document the material discussed here, and I offer my sincerest apologies for any omissions or errors of judgment in this regard. In addition, I have included a perhaps somewhat random selection of quite recent articles for the sole purpose of guiding the reader to places in the literature from where he or she may begin to explore specific topics in more detail, and also find the way back to other (earlier) contributions on such topics. Altogether, the references in the Bibliography, along with all the references quoted in them, should give a fairly complete picture of the literature on the topics in Several Complex Variables which are discussed in this book.

We all know that one learns best by doing. Consequently, I have included numerous exercises. Rather than writing "another book" hidden in the exercises, I have mainly included problems which test and reinforce the understanding of the material discussed in the text. Occasionally the reader is asked to provide missing steps of proofs; these are always of a routine nature. A few of the exercises are quite a bit more challenging. I have not identified them in any special way, since part of the learning process involves being able to distinguish the easy problems from the more difficult ones.

The prerequisites for reading this book are: (1) A solid knowledge of calculus in several (real) variables, including Taylor's Theorem, Implicit Function Theorem, substitution formula for integrals, etc. The calculus of differential forms, which should really be part of such a preparation, but too often is missing, is discussed systematically, though somewhat compactly, in Chapter III. (2) Basic complex analysis in one variable. (3) Lebesgue measure in $\mathbb{R}^n$, and the elementary theory of Hilbert and Banach spaces as it is needed for an understanding of L^p spaces and of the orthogonal projection onto a closed subspace of L^2. (4) The elements of point set topology and algebra. Beyond this, we also make crucial use of the Fredholm alternative for perturbations of the identity by compact operators in Banach spaces. This result is usually covered in a first course in Functional Analysis, and precise references are given.

Before beginning the study of this book, the reader should consult the Suggestions for the Reader and the chart showing the interdependence of the chapters, on pp. xvii–xix.

It gives me great pleasure to express my gratitude to the three persons who have had the most significant and lasting impact on my training as a mathematician. First, I want to mention H. Grauert. His lectures on Several Complex Variables, which I was privileged to hear while a student at the University of Göttingen, introduced me to the subject and provided the stimulus to study it further. His early support and his continued interest in my mathematical development, even after I left Göttingen in 1968, is deeply appreciated. I discussed my plans for this book with him in 1982, and his encouragement contributed to getting the project started. Once I came to the United States, I was fortunate to study under T.W. Gamelin at UCLA. He introduced me to the Theory of Function Algebras, a fertile ground for applying the new tools of integral representations which were becoming known around that time, and he took interest in my work and supervised my dissertation. Finally, I want to mention Y.T. Siu. It was a great experience for me—while a "green" Gibbs Instructor at Yale University—to have been able to continue learning from him and to collaborate with him.

Regarding this book, I am greatly indebted to my friend and collaborator on recent research projects, Ingo Lieb. He read drafts of virtually the whole

book, discussed many aspects of it with me, and made numerous helpful suggestions. W. Rudin expressed early interest and support, and he carefully read drafts of some chapters, making useful suggestions and catching a number of typos. S. Bell, J. Ryczaj, and J. Wermer also read portions of the manuscript and provided valuable feedback. Students at SUNY at Albany patiently listened to preliminary versions of parts of this book; their interest and reactions have been a positive stimulus. My colleague R. O'Neil showed me how to prove the real analysis result in Appendix C.

I thank JoAnna Aveyard, Marilyn Bisgrove, and Ellen Harrington for typing portions of the manuscript. Special thanks are due to Mary Blanchard, who typed the remaining parts and completed the difficult job of incorporating all the final revisions and corrections. B. Tomaszewski helped with the proofreading. The Department of Mathematics and Statistics of the State University of New York at Albany partially supported the preparation of the manuscript.

I would also like to acknowledge the National Science Foundation for supporting my research over many years. Several of the results incorporated in this book are by-products of projects supported by the N.S.F.

Finally, I want to express my deepest appreciation to my family, who, for the past few years, had to share me with this project. Without the constant encouragement and understanding of my wife Sandrina, it would have been difficult to bring this work to completion. My children's repeated questioning if I would ever finish this book, and the fact that early this past summer my 6-year-old son Roberto started his own "book" and proudly finished it in one month, gave me the necessary final push.

R. Michael Range

Suggestions for the Reader

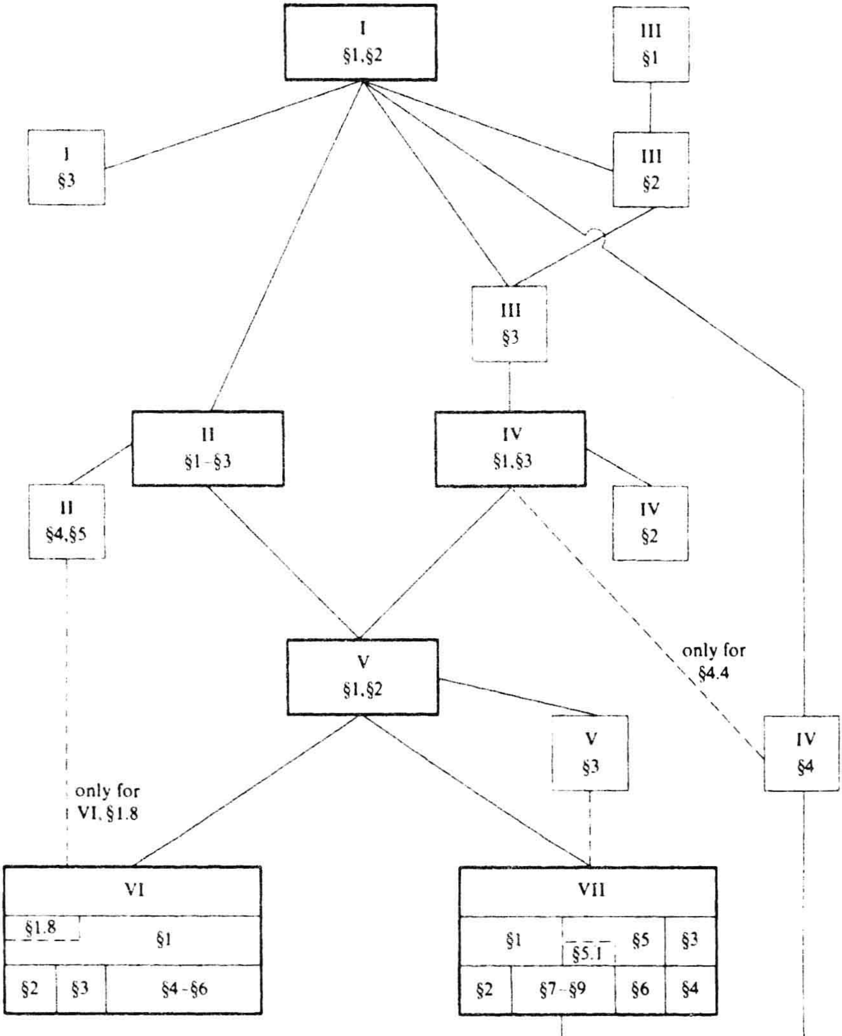
This book may be used in many ways as a text for courses and seminars, or for independent study, depending on interest, background, and time limitations. The following are just intended as a few suggestions. The reader should refer to the chart on page xix showing the interdependence of chapters in order to visualize matters more clearly.

(1) The obvious suggestion is to cover the entire book. Typically this will require more than two semesters. If time is a factor, certain sections may be omitted: natural candidates are §3 in Chapter I, §4, §5 in II, §2 in IV, §2, §3, §6 in VI, and, if necessary, parts of VII.
(2) Another possibility is a first course in Several Complex Variables, to be followed by a course which will emphasize the general theory, i.e., complex spaces, sheaves, etc. Such an introductory course could include I, §2.1, §2.7–§2.10, and §3 in II, III as needed, §1, §3 in IV, §1, §2 in V, and VI.
(3) A first course in Several Complex Variables which emphasizes recent developments on analytic questions, in preparation for studying the relevant research literature on weakly (or strictly) pseudoconvex domains, could be based on the following selection: §1, §2 in I, §1–3 in II, III as needed, §1, §3, §4 in IV, V, and VII. This could be done comfortably in a year course.
(4) The more advanced reader who is familiar with the elements of Several Complex Variables, and who primarily wants to learn about integral representations and some of their applications, may concentrate on Chapters IV (I advise reading §3 in III beforehand!), V, and VII.
(5) Finally, I have found the following selection of topics quite effective for a one-semester introduction to Several Complex Variables for students with limited technical background in several (real) variables: §1 and §2.1–§2.5 in I, §1–§3 in II, §1 (without 1.8), §4, §5, and §6 (if time) in VI. In order to handle

Chapter VI, one simply states without proof the vanishing theorem $H^1_{\bar\partial}(K) = 0$, i.e., the solvability of the Cauchy–Riemann equations in neighborhoods of K, for a compact pseudoconvex compactum K in $\mathbb{C}^n$. In case $n = 1$, this result is easily proved by reducing it to the case where the given (0, 1)-form $f d\bar{z}$ has compact support. This procedure, of course, does not work in general because multiplication by a cutoff function destroys the necessary integrability condition in case $n > 1$. Assuming $H^1_{\bar\partial}(K) = 0$, it is easy to solve the Levi problem (cf. §1.4 in V), and one can then proceed directly with Chapter VI. Notice that only the vanishing of $H^1_{\bar\partial}$ is required in Chapter VI, so all discussions involving $(0, q)$ forms for $q > 1$ can be omitted! In such a course it is also natural to present a proof of the Hartogs Extension Theorem based on the (elementary) solution of $\bar\partial$ with *compact supports* (see Exercise E.2.4 and E.2.5 in IV for an outline, or consult Hörmander's book [Hör 2]).

Within each chapter Theorems, Lemmas, Remarks, etc., are numbered in one sequence by double numbers; for example, Lemma 2.1 refers to the first such statement in §2 in that same chapter. A parallel sequence identifies formulas which are referred to sometime later on; e.g., (4.3) refers to the third numbered formula in §4. References to Theorems, formulas, etc., in a *different* chapter are augmented by the Roman numeral identifying that chapter.

Interdependence of the Chapters



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