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With 67 Figures

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Foreword

The author committed himself to the writing of this book soon after he started teaching a graduate course on linkage analysis and synthesis at the Universidad Nacional Autónoma de México (UNAM), in 1973. At that time he found that a great deal of knowledge on the subject, that had already been accumulated, was rather widespread and not as yet fully systematised. One exception was the work of B. Roth, of Stanford University, which already showed outstanding unity, though appearing only in the form of scientific papers in different journals. Moreover, the rate at which new results were presented either in specialised journals or at conferences all over the world, made necessary a recording of the most relevant contributions.

On the other hand, some methods of linkage synthesis, like the one of Denavit and Hartenberg (See Ch. 4), were finding a wide acceptance. It was the impression of the author, however, that the rationale behind that method was being left aside by many a researcher. Surprisingly, he found that virtually everybody was taking for granted, without giving the least explanation, that the matrix product, pertaining to a coordinate transformation from axes labelled 1 to those labelled n, should follow an order that is the inverse of the usual one. That is to say, whereas the logical representation of a coordinate transformation from axes 1 to 3, passing through those labelled 2, demands that the individual matrices $\underset{\sim}{T}_{12}$ and $\underset{\sim}{T}_{23}$ be multiplied in the order $\underset{\sim}{T}_{23}\underset{\sim}{T}_{12}$, the application of the method of Denavit and Hartenberg demands that they be placed in the inverse order, i.e. $\underset{\sim}{T}_{12}\underset{\sim}{T}_{23}$. It is explained in Chapter 4 why this is so, making use of results derived in Chapter 1. In this respect, the author departs from the common practice. In fact, while the transformations involving an affine transformation, i.e. a coordinate transformation, are usually represented by 4 x 4 matrices containing information about both the rotation and the translation, the author separates them into a matrix containing the rotation of axes and a vector containing their translation. The reason why this is done is far more than a matter of taste. As a matter of fact, it is not always necessary to carry out operations on both the rotation and the translation parts of the transformation, as is the case in dealing with spherical linkages. One more fundamental reason why the author departs from that practice is the following: in order to comprise both the rotation and the translation of axes in one single matrix, one has to define arbitrarily arrays that are not really vectors, for they contain a constant component. From the beginning, in Chapter 1, it is explained that only linear transformations are representable by matrices. Later on, in Chapter 2, it is shown that a rigid-body motion, in general, is a nonlinear transformation. This transformation is linear only if the motion is about a fixed point, which is also rigorously proven.

All through, the author has attempted to establish the rationale behind the methods of analysis, synthesis and optimisation of linkages. In this respect, Chapter 2 is crucial. In fact, it lays the foundations of the kinematics of rigid bodies in an axiomatic way, thus attempting to follow the trend of rational mechanics lead by Truesdell[1]. This Chapter in turn, is based upon Chapter 1, which outlines the facts of linear algebra, of extrema of functions and of numerical methods of solving algebraic linear and nonlinear systems, that are resorted to throughout the book. Regarding the numerical solution of equations, all possible cases are handled, i.e. algorithms are outlined that solve the said system, whether linear or nonlinear, when this is either underdetermined, determined or overdetermined. Flow diagrams illustrating the said algorithms and computer subprograms implementing them are included.

The philosophy of the book is to regard the linkages as systems capable of being modelled, analysed, synthesised, identified and optimised. Thus the methods and philosophy introduced here can be extended from linkages, i.e. closed kinematic chains, to robots and manipulators, i.e. open kinematic chains.

Back to the first paragraph, whereas early in the seventies the need to write a book on the theory and applications of the kinematics of mechanical

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1. Truesdell C., "The Classical Field Theories", in Flügge S., ed., Encyclopedia of Physics, Springer-Verlag, Berlin, 1960

systems was dramatic, presently this need has been fulfilled to a great extent by the publishing of some books in the last years. Within these, one that must be mentioned in the first place is that by Bottema and Roth[2], then the one by Duffy[3] and that by Suh and Radcliffe[4], just to mention a couple of the recently published contributions to the specialised literature in the English language. The author, nevertheless, has continued with the publication of this book because it is his feeling that he has contributed with a new point of view of the subject from the very foundations of the theory to the methods for application to the analysis and synthesis of mechanisms. This contribution was given a unified treatment, thus allowing the applications to be based upon the fundamentals of the theory laid down in the first two chapters.

Although this book evolved from the work done by the author in the course of the last eight years at the Graduate Division of the Faculty of Engineering-UNAM, a substantial part of it was completed during a sabbatical leave spent by him at the Laboratory of Machine Tools of the Aachen Institute of Technology, in 1979, under a research fellowship of the Alexander von Humboldt Foundation, to whom deep thanks are due.

The book could have not been completed without the encouragement received from several colleagues, among whom special thanks go to Profs. Bernard Roth of Stanford University, Günther Dittrich of Aachen Institute of Technology, Hiram Albala of Technion-Israel Institute of Technology and Justo Nieto of Valencia (Spain) Polytechnic University. The support given by Prof. Manfred Weck of the Laboratory of Machine Tools, Aachen, during the sabbatical leave of the author is very highly acknowledged. The discussions held with Dr. Jacques M. Hervé, Head of the Laboratory of Industrial Mechanics- Central School of Arts and Manufactures of Paris, France, contributed highly to the completion of Chapter 3.

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2 Bottema O. and Roth B., Theoretical Kinematics, North-Holland Publishing, Co., Amsterdam, 1979.

3 Duffy J., Analysis of Mechánisms and Robot Manipulators, Wiley-Interscience, Sommerset, N.J., 1980.

4 Such C. - H. and Radchiffe C.W., Kinematics and Mechanisms Design, John Wiley & Sons, Inc., N.Y., 1978.

The students of the author who, to a great extent are responsible for the writing of this book, are herewith deeply thanked. Special thanks are due to the former graduate students of the author, Messrs. Carlos López, Cándido Palacios and Angel Rojas, who are responsible for a great deal of the computer programming included here. Mrs. Carmen González Cruz and Miss Angelina Arellano typed the first versions of this work, whereas Mrs. Juana Olvera did the final draft. Their patience and very professional work is highly acknowledged.
Last, but by no means least, the support of the administration of the Faculty of Engineering-UNAM, and particularly of its Graduate Division, deserves a very special mention. Indeed, it provided the author with all the means required to complete this task.
To extend on more names of persons or institutions who somehow contributed to the completion of this book would give rise to an endless list, for which reason the author apologises for unavoidable omissions that he is forced to make.

Paris, January 1982

Jorge Angeles

Contents

1. Mathematical Preliminaries

1.0 INTRODUCTION. Some relevant mathematical results are collected in this chapter. These results find a wide application within the realm of analysis, synthesis and optimization of mechanisms. Often, rigorous proofs are not provided; however a reference list is given at the end of the chapter, where the interested reader can find the required details.

1.1. VECTOR SPACE, LINEAR DEPENDENCE AND BASIS OF A VECTOR SPACE.

A vector space, also called a linear space, over a field F [1.1]* , is a set V of objects, called vectors, having the following properties:

a) To each pair $\{\underset{\sim}{x}, \underset{\sim}{y}\}$ of vectors from the set, there corresponds one (and only one) vector, denoted $\underset{\sim}{x} + \underset{\sim}{y}$, also from V, called "the addition of $\underset{\sim}{x}$ and $\underset{\sim}{y}$" such that

   i) This addition is commutative, i.e.

$$\underset{\sim}{x} + \underset{\sim}{y} = \underset{\sim}{y} + \underset{\sim}{x}$$

   ii) It is associative, i.e., for any element $\underset{\sim}{z}$ of V,

$$\underset{\sim}{x} + (\underset{\sim}{y} + \underset{\sim}{z}) = (\underset{\sim}{x} + \underset{\sim}{y}) + \underset{\sim}{z}$$

   iii) There exists in V a unique vector $\underset{\sim}{0}$, called "the zero of V", such that, for any $\underset{\sim}{x} \in V$,

$$\underset{\sim}{x} + \underset{\sim}{0} = \underset{\sim}{x}$$

   iv) To each vector $\underset{\sim}{x} \in V$, there corresponds a unique vector $-\underset{\sim}{x}$, also in V, such that

$$\underset{\sim}{x} + (-\underset{\sim}{x}) = \underset{\sim}{0}$$

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* Numbers in brackets designate references at the end of each chapter.

b) To each pair $\{\alpha, \underset{\sim}{x}\}$, where $\alpha \in F$ (usually called "a scalar") and $\underset{\sim}{x} \in V$, there corresponds one vector $\alpha\underset{\sim}{x} \in V$, called "the product of the scalar α times $\underset{\sim}{x}$", such that:

i) This product is <u>associative</u>, i.e. for any $\beta \in F$,

$$\alpha(\beta\underset{\sim}{x}) = (\alpha\beta)\underset{\sim}{x}$$

ii) For the identity 1 of F (with respect to multiplication) the following holds

$$1\underset{\sim}{x} = \underset{\sim}{x}$$

c) The product of a scalar times a vector is <u>distributive</u>, i.e.

i) $\alpha(\underset{\sim}{x} + \underset{\sim}{y}) = \alpha\underset{\sim}{x} + \alpha\underset{\sim}{y}$

ii) $(\alpha + \beta)\underset{\sim}{x} = \alpha\underset{\sim}{x} + \beta\underset{\sim}{x}$

<u>Example 1.1.1</u>. The set of triads of real numbers (x,y,z) constitute a vector space. To prove this, define two such triads, namely (x_1,y_1,z_1) and (x_2,y_2,z_2) and show that their addition is also one such triad and it is commutative as well. To prove associativity, define one third triad, (x_3,y_3,x_3), and so on.

<u>Example 1.1.2</u> The set of all polynomials of a real variable, t, of degree less than or equal to n, for $0 \le t \le 1$, constitute a vector space over the field of real numbers.

<u>Example 1.1.3</u> The set of tetrads of the form (x,y,z,1) <u>do not</u> constitute a vector space (Why?)

Given the set of vectors $\{\underset{\sim}{x}_1, \underset{\sim}{x}_2, \ldots, \underset{\sim}{x}_n\} \subset V$ and the set of scalars $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset F$ not necessarily distinct, a <u>linear combination</u> of the n vectors is the vector defined as

$$\underset{\sim}{c} = \alpha_1 \underset{\sim}{x}_1 + \alpha_2 \underset{\sim}{x}_2 + \ldots + \alpha_n \underset{\sim}{x}_n$$

The said set of vectors is linearly independent (ℓ. i.) if $\tilde{c}$ equals zero implies that all α's are zero as well. Otherwise, the set is said to be linearly dependent (ℓ. d.)

Example 1.1.4 The set containing only one nonzero vector, $\{\tilde{x}\}$, is ℓ.i.

Example 1.1.5 The set containing only two vectors, one of which is the origin, $\{\tilde{x},\tilde{0}\}$, is ℓ.d.

The set of vectors $\{\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_n\} \subset V$ spans V if and only if every vector $\tilde{v} \in V$ can be expressed as a linear combination of the vectors of the set.

A set of vectors $B = \{x_1,x_2,\ldots,x_n\} \subset V$ is a basis for V if and only if:

i) B is linearly independent, and

ii) B spans V

All bases of a given space V contain the same number of vectors. Thus, if B is a basis for V, the number n of elements of B is the dimension of V (abreviated: n=dim V)

Example 1.1.6 In 3-dimensional Euclidean space the unit vectors $\{\tilde{i}, \tilde{j}\}$ lying parallel to the X and Y coordinate axes span the vectors in the X-Y plane, but do not span the vectors in the physical three-dimensional space.

Exercise 1.1.1 Prove that the set B given above is a basis for V if and only if each vector in V can be expressed as a unique linear combination of the elements of B.

1.2 LINEAR TRANSFORMATION AND ITS MATRIX REPRESENTATION

Henceforth, only finite-dimensional vector spaces will be dealt with and, when necessary, the dimension of the space will be indicated as an exponent of the space, i.e., V^n means dim V=n.

A transformation $\tilde{T}$, from an m-dimensional vector space U, into an n-dimensional vector space V is a rule which establishes a correspondence between an element of U and a unique element of V. It is represented as:

$$T: U^m \rightarrow V^n \tag{1.2.1}$$

If $u \in U^m$ and $v \in V^n$ are such that $T: u \rightarrow v$, the said correspondence may also be denoted as

$$v = T(u) \tag{1.2.3a}$$

T is linear if and only if, for any u, u_1 and $u_2 \in U$, and $\alpha \in F$,

i) $T(u_1 + u_2) = T(u_1) + T(u_2)$ and (1.2.3b)

ii) $T(\alpha u) = \alpha T(u)$ (1.2.3c)

Space U^m over which T is defined is called the "domain" of T, whereas the subspace of V^n containing vectors v for which eq. (1.2.3a) holds is called the "range" of T. A subspace of a given vector space V is a subset of V and is in turn a vector space, whose dimension is less than or equal to that of V

Exercise 1.2.1 Show that the range of a given linear transformation of a vector space U into a vector space V constitutes a subspace, i.e. it satisfies properties a) to c) of Section 1.1.

For a given $u \in U$, vector v, as defined by (1.2.2) is called the "image of u under T", or, simply, the "image of u" if T is selfunderstood.

An example of a linear transformation is an orthogonal projection onto a plane. Notice that this projection is a transformation of the three-dimensional Euclidean space onto a two-dimensional space (the plane). The domain of T in this case is the physical 3-dimensional space, while its range is the projection plane.

If T, as defined in (1.2.1), is such that all of V contains v's such that (1.2.2) is satisfied (for some u's), T is said to be "onto". If T is such that, for all distinct $\underset{\sim}{u}_1$ and $\underset{\sim}{u}_2$, $\underset{\sim}{T}(\underset{\sim}{u}_1)$ and $\underset{\sim}{T}(\underset{\sim}{u}_2)$ are also distinct, $\underset{\sim}{T}$ is said to be one-to-one. If $\underset{\sim}{T}$ is onto and one-to-one, it is said to be <u>invertible</u>.

If $\underset{\sim}{T}$ is invertible, to each $\underset{\sim}{v} \in V$ there corresponds a unique $\underset{\sim}{u} \in U$ such that $\underset{\sim}{v} = \underset{\sim}{T}(\underset{\sim}{u})$, so one can define a mapping $\underset{\sim}{T}^{-1}: V \rightarrow U$ such that

$$\underset{\sim}{u} = \underset{\sim}{T}^{-1}(\underset{\sim}{v}) \qquad (1.2.4)$$

$\underset{\sim}{T}^{-1}$ is called the "<u>inverse</u>" of $\underset{\sim}{T}$.

<u>Exercise 1.2.2</u> Let $\underset{\sim}{P}$ be the projection of the three-dimensional Euclidean space onto a plane, say, the X-Y plane. Thus, $\underset{\sim}{v} = \underset{\sim}{P}(\underset{\sim}{u})$ is such that the vector with components (x, y, z), is mapped into the vector with components (x, y, 0).

i) Is $\underset{\sim}{P}$ a linear transformation?

ii) Is $\underset{\sim}{P}$ onto?, one-to-one?, invertible?

A very important fact concerning linear transformations of finite dimensional vector spaces is contained in the following result:

Let $\underset{\sim}{L}$ be a linear transformation from U^m into V^n. Let B_u and B_v be bases for U^m and V^n, respectively. Then clearly, for each $\underset{\sim}{u}_i \in B_u$ its image $\underset{\sim}{L}(\underset{\sim}{u}_i) \in V$ can be expressed as a linear combination of the $\underset{\sim}{v}_k$'s in B_v. Thus

$$L(\underset{\sim}{u}_i) = \alpha_{1i}\underset{\sim}{v}_1 + \alpha_{2i}\underset{\sim}{v}_2 + \ldots + \alpha_{ni}\underset{\sim}{v}_n . \qquad (1.2.5)$$

Consequently, to represent the images of the m vectors of B_u, mn scalars like those appearing in (1.2.5) are required. These scalars can be arranged in the following manner:

$$[\underset{\sim}{A}] = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \ldots \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \ldots \alpha_{2m} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \alpha_{n1} & \alpha_{n2} & \ldots \alpha_{nm} \end{pmatrix} \qquad (1.2.6)$$

where the brackets enclosing $\underset{\sim}{A}$ are meant to denote a matrix, i.e. an array of numbers, rather than an abstract linear transformation.
$[\underset{\sim}{A}]$ is called "The matrix of $\underset{\sim}{L}$ referred to B_u and B_v" . This result is summarized in the following:

DEFINITION 1.2.1 *The i th column of the matrix representation of $\underset{\sim}{L}$, referred to B_u and B_v, contains the scalar coefficients α_{ji} of the representation (in terms of B_v) of the image of the i th vector of B_u*

Example 1.2.1 What is the representation of the reflexion $\underset{\sim}{R}$ of the 3-dimensional Euclidean space E^3 into itself, with respect to one plane, say the X-Y plane, referred to unit vectors parallel to the X,Y,Z axes?.

Solution: Let $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$, be unit vectors parallel to the X, Y and Z axes, respectively. Clearly,

$$\underset{\sim}{R}(\underset{\sim}{i}) = \underset{\sim}{i}$$
$$\underset{\sim}{R}(\underset{\sim}{j}) = \underset{\sim}{j}$$
$$\underset{\sim}{R}(\underset{\sim}{k}) = -\underset{\sim}{k}$$

Thus, the components of the images of $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ under $\underset{\sim}{R}$ are:

$$\left(\underset{\sim}{R}(\underset{\sim}{i})\right) = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \left(\underset{\sim}{R}(\underset{\sim}{j})\right) = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \left(\underset{\sim}{R}(\underset{\sim}{k})\right) = \begin{pmatrix}0\\0\\-1\end{pmatrix}$$

Hence, the matrix representation of $\underset{\sim}{R}$, denoted by $[\underset{\sim}{R}]$, is

$$(\underset{\sim}{R}) = \begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{pmatrix} \qquad (1.2.7)$$

Notice that, in this case, U = V and so, it is not necessary to use two different bases for U and V. Thus, $(\underset{\sim}{R})$, as given by (1.2.7), is the matrix representation of the reflection $\underset{\sim}{R}$ under consideration, referred to the basis $\{\underset{\sim}{i}, \underset{\sim}{j}, \underset{\sim}{k}\}$.

1.3 RANGE AND NULL SPACE OF A LINEAR TRANSFORMATION

As stated in Section 1.2, the set of vectors $v \in V$ for which there is at least one $u \in U$ such that $v = L(u)$, as pointed out in Sect. 4.2., is called "the range of L" and is represented as $R(L)$, i.e. $R(L) = (v=L(u): u \in U)$.

The set of vectors $u_0 \in U$ for which $L(u_0) = 0 \in V$ is called "the null space of L" and is represented as $N(L)$, i.e. $N(L) = \{u_0 : L(u_0)=0\}$.

It is a simple matter to show that $R(L)$ and $N(L)$ are subspaces of V and U, respectively*.

The dimensions of dom(L), R(L) and N(L) are not independent, but they are related (see (1.2)):

$$\dim \mathrm{dom}(L)=\dim R(L) + \dim N(L) \qquad (1.3.1)$$

Example 1.3.1 In considering the projection of Exercise 1.2.1, U is E^3 and thus R(P) is the X-Y plane, N(P) is the Z axis, hence of dimension 1. The X-Y plane is two-dimensional and dom(L) is three-dimensional, hence (1.3.1) holds.

Exercise 1.3.1 Describe the range and the null space of the reflection of Example 1.2.1 and verify that eq. (1.3.1) holds true.

1.4 EIGENVALUES AND EIGENVECTORS OF A LINEAR TRANSFORMATION

Let L be a linear transformation of V into itself (such an L is called an "endomorphism"). In general, the image $L(v)$ of an element v of V is linearly independent with v, but if it happens that a nonzero vector v and its image under L are linearly dependent, i.e. if

$$L(v) = \lambda v \qquad (1.4.1)$$

* The proof of this statement can be found in any of the books listed in the reference at the end of this chapter.