精 算 科 学 系 列 Volume 1

精算模型

—— 寿险和年金（英文版）

Actuarial Model: Life Insurance and Annuity

朱彦云

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Preface

This book presents the actuarial model as a combination of cash flows, time value of money, and individuals' future lifetimes. It covers life insurance and life annuities and how to set premiums and reserves for those products. The topics are closely related to the Society of Actuaries (SOA) course MLC requirements. Some examples and exercise problems come from past SOA course 3/course M/course MLC exams.

Guide to the Chapters

Chapter 1 covers interest, time value of money, and annuities with guaranteed payments.

$\bar{a}_{\overline{n}|}$ is the present value of an n-year annuity, which guarantees to pay 1 unit per year continuously over a period of n years.

$\ddot{a}_{\overline{n}|}$ is the present value of an annuity which guarantees to pay 1 unit at the beginning of each year over a period of n years.

Chapter 2 covers an individual's future lifetime $T(x)$, the probability functions, and life tables.

$K(x)$ is the integer part of $T(x)$.

$(T(x) \wedge n) = \min(T(x), n)$

Chapter 3 covers life insurance which pays benefits when the individual dies within the term of the policy. The present values of those benefits are functions of $T(x)$ or $K(x)$.

$Z = v^{T(x)}$ is the present value of a whole life insurance which pays 1 unit of benefit at the moment of the individual's death.

$Z = v^{K(x)+1}$ is the present value of a whole life insurance which pays 1 unit of benefit at the end of the year of the individual's death.

Chapter 4 covers life annuities which offer periodic payments while the individual is alive. The present values of those life annuities are functions of $T(x)$ or $K(x)$.

$Z = \bar{a}_{\overline{T(x)}|}$ is the present value of a continuous whole life annuity which pays 1 unit per year continuously while the individual is alive.

$Z = \ddot{a}_{\overline{K(x)+1}|}$ is the present value of an annual whole life annuity which pays 1 unit at the beginning of each year while the individual is alive.

Chapter 5 covers insurance premiums for life insurance and life annuities. The premiums are charged to cover benefit payments and expenses. Under the equivalence principle, the premiums are determined by:

EPV(premiums)=EPV(benefits) + EPV(expenses)

where "EPV" means "expected present value".

Chapter 6 covers insurance reserves for life insurance and life annuities. For an insurance policy that is still active, under the equivalence principle, the policy reserve at time t is:

${}_tV$ = EPV(benefits) + EPV(expenses) − EPV(premiums)

for cash flows from time t onward.

Chapter 7 covers the joint-life status (xy) and the last-survivor status $(\overline{xy})$, and life insurance and life annuities on these two new entities.

T_{xy} is the minimum of $(T(x), T(y))$ and the future lifetime of the joint-life status (xy).

$T_{\overline{xy}}$ is the maximum of $(T(x), T(y))$ and the future lifetime of the last-survivor status $(\overline{xy})$.

$v^{T_{xy}}$ is the present value of a continuous whole life insurance which pays 1 unit of benefit at the moment of the first death.

$v^{T_{\overline{xy}}}$ is the present value of a continuous whole life insurance which pays 1 unit of benefit at the moment of the second death.

Chapter 8 covers the multiple-decrement model where an the individual will dies from the risk that happens first.

T_j is the time when risk j happens, $j = 1, \cdots, m$.

$T(x)$ is the individual's future lifetime and equal to the minimum of $(T_1, \cdots, T_m)$.

Acknowledgement

This book has benefited greatly from *Actuarial Mathematics*, the great work by Newton L. Bowers, JR., Hans U. Gerber, James C. Hickman, Donald A. Jones, and Cecil J. Nesbitt. If this book brings any contribution to the actuarial modeling process, it is because it stands upon the shoulders of those giants.

I would like to thank Mr. Lixin Qin for inviting me to write this book, Dr. Shing-Tung Yau for publishing it through *International Press*, and *Higher Education Press* for publishing it in China.

I also like to thank Dr. Gexin Yu for helping me on using LaTex software in preparing this book's manuscript and editors from *International Press* and *Higher Education Press* for their tireless and excellent support.

And I thank the Society of Actuaries (SOA) for their generous permission to use the SOA past exam problems in this book, which makes it a good reference for students preparing the SOA MLC exam.

Contents

Chapter 1

Interest and Annuity-Certain

1.1 Introduction

\$1 million today can bring back more than \$1 million years later if invested wisely. Interest is a share of these investment proceeds.

This chapter covers the following topics:

- Simple and compound interests
- Accumulated values and discounted values
- Annuities-certain

1.2 Interest

Interest is the price paid for borrowing money. It depends on the the amount of the original capital and the length of the borrowing period. With an annual interval,

$$
\begin{aligned}
I_k &= A(k+1) - A(k) = \text{interest earned between time } k \text{ and } k+1, \quad k \geqslant 0 \\
i_k^e &= \frac{A(k+1) - A(k)}{A(k)} = \frac{I_k}{A(k)} = \text{the annual effective interest rate between } k \text{ and } k+1 \\
A(k) &= \text{the accumulated value of } A(0) \text{ at time } k, \quad k \geqslant 0 \\
A(0) &= \text{the initial principal at time } 0.
\end{aligned}
$$

1.2.1 Simple Interest

With simple interest, only the original principal $A(0)$ earns interest during each period. Let i_k be the annual interest rate between time k and $k+1$, then

$$A(k+1) = A(k) + A(0)i_k, \quad k \geqslant 0 \tag{1.1}$$

$$i_k^e = \frac{A(k+1) - A(k)}{A(k)} = \frac{i_k}{1 + i_0 + \cdots + i_{k-1}} \tag{1.2}$$

Example 1.2.1. *Given an initial principal of* \$1000 *and the following annual interest rates:*

$$i_0 = 5\%, \quad i_1 = 5\%, \quad i_2 = 6\%, \quad i_3 = 5\%, \quad i_4 = 4\%,$$

find the accumulated values and the annual effective interest rates from year 1 *to year* 4 *with single interest.*

Solutions:

$$A(0) = 1000, \quad i_0 = 0.05, \quad i_1 = 0.05, \quad i_2 = 0.06, \quad i_3 = 0.06, \quad i_4 = 0.04$$
$$A(k+1) = A(k) + A(0)i_k, \quad i_k^e = \frac{A(k+1) - A(k)}{A(k)}, \quad k = 0, 1, 2, 3$$

Year k	i_k	$A(k)$	i_k^e
0	0.05	1,000	0.05
1	0.05	1,050	0.04762
2	0.06	1,100	0.05455
3	0.06	1,160	0.051724
4	0.04	1,220	0.03279

When interest rate is constant ($i_k = i$),

$$A(k) = A(0)(1 + k\,i) \tag{1.3}$$

$$i_k^e = \frac{i}{1 + k\,i} \tag{1.4}$$

Example 1.2.2. *Given an initial principal of* \$1000 *and a constant annual simple interest rate of* 5%*, find the accumulated values and the annual effective interest rates from year* 1 *to year* 4.

Solutions:

$$A(k+1) = A(k) + A(0)i, \quad i_k^e = \frac{i}{1 + k\,i}, \quad k = 0, 1, 2, 3$$

Year k	$i_k = i$	$A(k)$	i_k^e
0	0.05	1,000	0.05
1	0.05	1,050	0.04762
2	0.05	1,100	0.04545
3	0.05	1,150	0.04348
4	0.05	1,200	0.04167

1.2.2 Compound Interest

With compound interest, it is the accumulated asset at the beginning of each period $A(k)$ that earns interest between time k and $k+1$.

$$A(k+1) = A(k)(1 + i_k), \quad k \geqslant 0 \tag{1.5}$$

$$i_k^e = i_k \tag{1.6}$$

Thus with compound interest, the annual effective rate is the same as the annual compound interest rate.

Example 1.2.3. *Given an initial principal of* \$1000 *and an annual interest rate of* 5% *during the first year and* 10% *during the second year, find the accumulated values at the end of year two.*

Solutions:

$$A(1) = A(0)(1 + i_0) = 1000 \times (1 + 0.05) = 1,050$$
$$A(2) = A(1)(1 + i_1) = 1050 \times (1 + 0.10) = 1,155$$

When interest rate is constant ($i_k = i$),

$$A(k) = A(0)(1 + i)^k \tag{1.7}$$

Example 1.2.4. *Given an initial principal of* \$1000 *and a constant annual effective interest rate of* 5%, *calcualte the accumulated values at the end of year* 4 *under the compound interest rule.*

Solutions:

$$A(4) = A(0)1.05^4 = 1226$$

1.2.3 Interest Convertible m-thly

With m-thly compound interest, interests are added m times per year. For example, if $m = 12$, interests are added at the end of each month. With constant interest rate,

$$A(k) = A(0)\left(1 + \frac{i^{(m)}}{m}\right)^{mk} \tag{1.8}$$

$$i_k^e = i = \left(1 + \frac{i^{(m)}}{m}\right)^m - 1 \tag{1.9}$$

with

$i^{(m)}$ = the nominal annual interest rate convertible m-thly

$\frac{i^{(m)}}{m}$ = the effective m-thly compound interest rate per m-th period

Example 1.2.5. *You are given:*

(i) *the initial principal is* \$1000.

(ii) *the nominal annual interest rate convertible monthly is* 5%.

Calculate the annual effective interest rate and the accumulated value at the end of year 4.

Solutions:

$$\frac{i^{(12)}}{12} = \frac{0.05}{12} = 0.004167$$

$$i_k^e = (1 + \frac{i^{(12)}}{12})^{12} - 1 = 0.05116$$

$$A(4) = A(0)\left(1 + \frac{i^{(12)}}{12}\right)^{12\times 4} = 1221$$

1.2.4 Force of Interest

When $m \to \infty$, interests are added continuously. Let δ_t be the force of interest at time t,

$$A(t) = A(0)e^{\int_0^t \delta_s\, ds}, \quad t \geqslant 0 \tag{1.10}$$

Example 1.2.6. *SOA course* 2 *November* 2001 *problem* 1.

Ernie makes deposits of 100 *at time* 0, *and* X *at time* 3. *The fund grows at a force of interest* $\delta_t = \frac{t^2}{100}, t > 0$.

The amount of interest earned from time 3 to time 6 is X.
Calculate X.

Solutions: At time 3, the asset accumulated from 100 of time 0:

$$A(3) = 100e^{\int_0^3 \delta_t\, dt} = 100e^{\int_0^3 \frac{t^2}{100}\, dt} = 100e^{0.09}$$

$$A(6) = (A(3) + X)e^{\int_3^6 \delta_t\, dt} = (100e^{0.09} + X)e^{\int_3^6 \frac{t^2}{100}\, dt}$$

$$= (100e^{0.09} + X)e^{0.63}$$

$$I(3,6) = A(6) - (A(3) + X) = (100e^{0.09} + X)(e^{0.63} - 1) = X$$

$$\Rightarrow \quad X = 784.59$$

When force of interest is constant $\delta_t = \delta$,

$$A(t) = A(0)e^{\delta t} \tag{1.11}$$

$$i_k^e = i = e^{\delta} - 1 \tag{1.12}$$

Example 1.2.7. *You are given:*

(i) *The initial principal is* \$1000.

(ii) *The force of interest is constant at* 5%.

Calculate the annual effective rate of interest and the accumulated values at the end of year 4.

Solutions:

$$i = e^{\delta} - 1 = e^{0.05} - 1 = 0.05127$$

$$A(4) = A(0)e^{\delta\, 4} = 1000e^{0.20} = 1221.40$$

1.2.5 Relationship among Interest Rates

With compound interest and the same amount of initial principal $A(0)$, the accumulated value at time k should be the same to be arbitrage free.

$$(1+i) = \left(1+\frac{i^{(m)}}{m}\right)^m = e^\delta \tag{1.13}$$

and

$$i \geqslant i^{(m)} \geqslant \delta \tag{1.14}$$

When $m \to \infty$,

$$i^{(m)} = \delta \tag{1.15}$$

Example 1.2.8. *Given $i = 0.05$, calculate the force of interest rate, and the nominal annual interest rates convertible half-yearly, quarterly, and monthly.*

Solutions:

$$i^{(2)} = 2(1+0.05)^{\frac{1}{2}} = 0.04939, \qquad i^{(4)} = 4(1+0.05)^{\frac{1}{4}} = 0.04909$$

$$i^{(12)} = 12(1+0.05)^{\frac{1}{12}} = 0.04889, \qquad \delta = \ln(1+0.05) = 0.04879$$

Since compound interest principle is compatible with the arbitrage-free requirement, it is to be used through this book, unless specified otherwise.

1.2.6 The Accumulation Factor

The accumulation factor at time t is defined as the ratio of $A(t)$ and $A(0)$:

$$a(t) = \frac{A(t)}{A(0)} = e^{\int_0^t \delta_s\, ds} \tag{1.16}$$

When interest rates are constant,

$$a(t) = e^{\delta\, t} = (1+i)^t \tag{1.17}$$

Example 1.2.9. *SOA course* 2 *May* 2000 *problem* 37.

A customer is offered an investment where interest is calculated according to the following force of interest:

$$\delta_t = \begin{cases} 0.02t, & 0 \leqslant t \leqslant 3 \\ 0.045, & 3 < t \end{cases}$$

The customer invests 1000 *at time* t=0.
What nominal rate of interest, compounded quarterly, is earned over the first four-year period?

Solutions:

$$A(4) = A(0)e^{\int_0^4 \delta_t dt} = 1000e^{\int_0^3 0.02t dt + \int_3^4 0.045 dt} = 1000e^{0.135}$$

$$a(4) = \frac{A(4)}{A(0)} = e^{0.135} = (1+i)^4 = \left(1+\frac{i^{(4)}}{4}\right)^{4\times 4} = e^{0.135} \quad \Rightarrow i^{(4)} = 0.0339$$

1.2.7 The Discount Factor

The discount factor at time t is the inverse of the accumulation factor at time t,

$$v(t) = \frac{1}{a(t)} = \frac{A(0)}{A(t)} = e^{-\int_0^t \delta_s\, ds} \tag{1.18}$$

When interest rates are constant,

$$v(t) = v^t = e^{-\delta\, t} = (1+i)^{-t} \tag{1.19}$$

The discount interests are paid at the beginning of each period — annually or m-thly. Let d be the annual rate of discount, then

$$d = i\, v = 1 - v \tag{1.20}$$

Let $d^{(m)}$ be the nominal annual discount rate of return convertible m-thly, then

$$d^{(m)} = i^{(m)}\, v^{\frac{1}{m}} = m\,(1 - v^{\frac{1}{m}}) \tag{1.21}$$

Example 1.2.10. *SOA course FM May* 2005 *problem* 19.

Calculate the nominal interest rate of discount convertible monthly that is equivalent to a nominal rate of interest of 18.9% *per year convertible monthly.*

Solutions:

$$1 - \frac{d^{(12)}}{12} = \left(1 + \frac{i^{(12)}}{12}\right)^{-1} \quad \Rightarrow \quad d^{(12)} = 12 \times \left[1 - \left(1 + \frac{0.189}{12}\right)^{-1}\right] = 0.18607$$

1.3 Annuities-Certain

An annuity-certain guarantees a series of payments at equal intervals. The payments can be:

- level or non-level
- annually, monthly, or continuously
- at the beginning or at the end of each period
- over a limited number of years or forever

The present value of an annuity is the sum of the discounted values of all the cash flows from that annuity. In this book, the annual interest rates are assumed to be constant and $v = (1+i)^{-1} = e^{-\delta}$, unless specified otherwise.

1.3.1 Annual Annuities-Certain

An n-year annuity-immediate offers 1 unit of payment at the end of each year for n years, see Figure 1.1. The present value of this annuity is

$$a_{\overline{n}|} = v^1 + v^2 + \ldots + v^n = \sum_{k=1}^{n} v^k = \frac{v - v^{n+1}}{1 - v} = \frac{1 - v^n}{i} \tag{1.22}$$

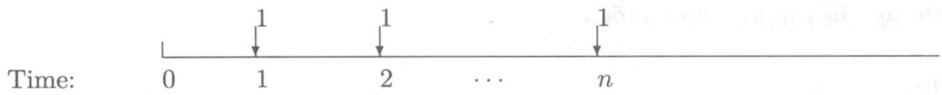

Figure 1.1: Cash flows of n-year annuity-immediate

An n-year annuity-due offers 1 unit of payment at the beginning of each year for n years, see Figure 1.2. The present value of this annuity is

$$\ddot{a}_{\overline{n}|} = v^0 + v^1 + \ldots + v^{n-1} = \sum_{k=0}^{n-1} v^k = \frac{1 - v^n}{d} = (1+i)\, a_{\overline{n}|} \tag{1.23}$$

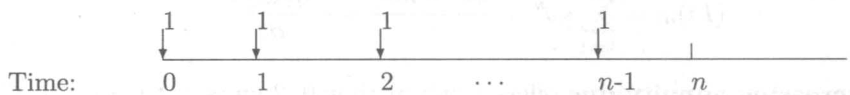

Figure 1.2: Cash flows of n-year annuity-due

An m-year deferred n-year annuity-immediate offers 1 unit of payments between time $m+1$ and time $m+n$, see Figure 1.3. The present value of this annuity is

$$_{m|}\, a_{\overline{n}|} = v^{m+1} + v^{m+2} + \ldots + v^{m+n} = \sum_{k=m+1}^{m+n} v^k = v^m \frac{1 - v^n}{i} = v^m\, a_{\overline{n}|} \tag{1.24}$$

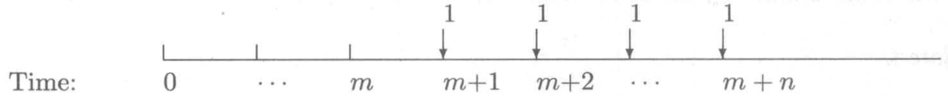

Figure 1.3: Cash flows of m-year deferred n-year annuity-immediate

An m-year deferred n-year annuity-due offers 1 unit of payments between time m and time $m+n-1$, see Figure 1.4. The present value of this annuity is

$$_{m|}\, \ddot{a}_{\overline{n}|} = v^m + v^{m+1} + \ldots + v^{m+n-1} = \sum_{k=m}^{m+n-1} v^k = v^m \frac{1 - v^n}{d} = v^m \cdot \ddot{a}_{\overline{n}|} = (1+i) \cdot {}_{m|}\, a_{\overline{n}|} \tag{1.25}$$

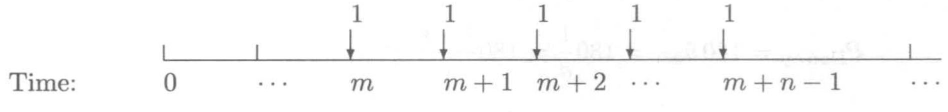

Figure 1.4: Cash flows of m-year deferred n-year annuity-due

Example 1.3.1. *Given the annual effective interest rate* $i = 5\%$.
Calculate a_5 , $\ddot{a}_5$, ${}_{10|}\, a_{\overline{5|}}$, *and* ${}_{10|}\, \ddot{a}_{\overline{5|}}$.

Solutions:

$$v = (1.05)^{-1} = 0.95238, \quad d = 1 - v = 0.04762$$
$$a_{\overline{5|}} = \frac{1 - v^5}{i} = 4.3295, \quad \ddot{a}_{\overline{5|}} = (1+i)\, a_{\overline{5|}} = 4.5460$$
$${}_{10|}\, a_{\overline{5|}} = v^{10} \cdot a_{\overline{5|}} = 2.6579, \quad {}_{10|}\ddot{a}_{\overline{5|}} = (1+i) \cdot {}_{10|}\, a_{\overline{5|}} = 2.7908$$

An n-year increasing annuity-immediate offers 1 unit at time 1, 2 units at time 2, $\cdots$, till n units at time n. The present value of this annuity is

$$(Ia)_{\overline{n|}} = \sum_{k=1}^{n} k v^k = \frac{\ddot{a}_{\overline{n|}} - n v^n}{i} = -\frac{\partial (\, a_{\overline{n|}})}{\partial \delta} \tag{1.26}$$

An n-year increasing annuity-due offers 1 unit at time 0, 2 units at time 1, $\cdots$, till n units at time $n-1$. The present value of this annuity is

$$(I\ddot{a})_{\overline{n|}} = \sum_{k=0}^{n-1} (k+1) v^k = \frac{\ddot{a}_{\overline{n|}} - n v^n}{d} = (1+i)\,(Ia)_{\overline{n|}} \tag{1.27}$$

Example 1.3.2. *SOA course* 2 *November* 2000 *problem* 20.
Sandy purchases a perpetuity-immediate that makes annual payments. The first payment is 100, *and each payment thereafter increases by* 10.
Danny purchases a perpetuity-due which makes annual payments of 180.
Using the same annual effective interest rate, $i > 0$, *the present value of both perpetuities are equal.*
Calculate i.

Solutions:

$$d = i\, v = \frac{i}{1+i}$$

The present value of Sandy's perpetuity-immediate,

$$P_{Sandy} = 100\, a_{\overline{\infty|}} + v \cdot 10 \cdot (Ia)_{\overline{\infty|}} = 100\frac{1}{i} + v \cdot 10 \cdot \frac{1}{id} = 100\frac{1}{i} + 10\frac{1}{i^2}$$

The present value of Danny's perpetuity-due,

$$P_{Danny} = 180\, \ddot{a}_{\overline{\infty|}} = 180\frac{1}{d} = 180\frac{1+i}{i}$$

$$P_{Sandy} = P_{Danny} \quad \Rightarrow \quad 100\frac{1}{i} + 10\frac{1}{i^2} = 180\frac{1+i}{i} \Rightarrow \quad i = 0.1017$$