D. Mumford J. Fogarty F. Kirwan

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Geometric Invariant Theory

Third Enlarged Edition

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Preface to third edition

Just over ten years have passed since the publication of the second edition of this book. In these ten years there have been many developments relating to geometric invariant theory. So many, in fact, that although the new references in this edition outnumber the references in the second edition by a very large margin they form only a selection of the work done in this area in the last decade[1].

This edition of the book has been extended to take account of one of these developments, one which was just hinted at in the second edition[2]. A close and very fruitful relationship has been discovered between geometric invariant theory for quasi-projective complex varieties and the moment map in symplectic geometry, and a chapter has been added describing this relationship and some of its applications. In an infinite-dimensional setting the moment map links geometric invariant theory and Yang-Mills theory, which has of course been the focus of much attention among mathematicians over the last fifteen years.

In style this extra chapter is closer to the appendices added in the second edition than to the original text. In particular no proofs are given where satisfactory references exist.

On the many other exciting developments related to geometric invariant theory since the publication of the second edition I regret that I lack sufficient expertise to do more than add some relevant publications to the list of references. Among these are several survey articles and books on invariant theory, moduli problems and related areas published in the last ten years, including [119], [349], [486], [489], [494], [508], [529], [545], [565], [579], [598], [602], [608], [613], [648], [686–688], [694], [698], [743], [769], [792], [800], [806], [809], [837], [845], [859], [868], [873], [879], [895], [899], and [908].

The reader should be warned that this edition is something of a collage. Chapters 1–7 were written twenty eight years ago. Chapter 8 is new. The Appendices were written about twelve years ago, but references to recent work in each area have been added in footnotes at the

[1] Indeed they form a rather haphazard selection, I fear, and my apologies are due to the many authors whose works have been omitted.

[2] See Appendix 2C.

beginning of each subsection. The other authors and I hope that it will still be useful.

Finally I would like to express my thanks to all those who have given me comments and advice, and in particular to David Mumford and Simon Donaldson for their help.

Oxford, December, 1992 FRANCES KIRWAN

Preface to second edition

In the 16 years since this book was published, there has been an explosion of activity in algebraic geometry. In particular, there has been great progress in invariant theory and the theory of moduli. Reprinting this monograph gave us the opportunity of making various revisions. The first edition was written primilarly as a research monograph on a geometric way to formulate invariant theory and its applications to the theory of moduli. In this edition we have left the original text essentially intact, but have added appendices, which sketch the progress on the topics treated in the text, mostly without proofs, but with discussions and references to all the original papers.

In the preface to the 1st edition, it was explained that most of the invariant theoretic results were proven only in char. 0 and that therefore the applications to the construction of moduli spaces were either valid only in char. 0, or else depended on the particular invariant theoretic results in Ch. 4 which could be established by elementary methods in all characteristics. However, a conjecture was made which would extend all the invariant theory to char. p. Fortunately, W. HABOUSH [128] has proved this conjecture, hence the book is now more straightforward and instead of giving one construction of the moduli space $\mathcal{M}_g$ of curves of genus g over $\boldsymbol{Q}$ by means of invariant theory, and one construction of $\mathcal{M}_g$ and $\mathcal{A}_{g,1}$ (the moduli space of principally polarized abelian varieties) over $\boldsymbol{Z}$ by means of the covariant of points of finite order and Torelli's theorem, we actually give or sketch the following constructions of these spaces:

1) $\mathcal{M}_g$ and $\mathcal{A}_g$ are constructed by proving the stability of the Chow forms of both curves and abelian varieties (Ch. 4, § 6; Appendix 3B)

2) $\mathcal{M}_g$ and $\mathcal{A}_g$ are independently constructed by covariants of finite sets of points (Appendix 7C; Ch. 7, § 3)

3) $\mathcal{A}_g$ is constructed by an explicit embedding by theta constants (Appendix 7B).

All of this is valid over $\boldsymbol{Z}$ except that the covariant approach to $\mathcal{M}_g$ uses higher Weierstrass points and is valid only over $\boldsymbol{Q}$ (unless one can prove their finiteness for high enough multiples $|nK|$, in char. p: see Appendix 7C).

This preface gives us the opportunity to draw attention to some basic open questions in invariant theory and moduli theory. 3 questions raised in the 1st edition have been answered:

a) the geometric reductivity of reductive groups has been proven by HABOUSH, op. cit.,

b) the existence of canonical destabilizing flags for unstable points has been proven by KEMPF [171] and ROUSSEAU [285],

c) the stability of Chow forms and Hilbert points of pluricanonically embedded surfaces of general type has been proven by GIESEKER [116].

Pursuing the ideas in c), leads one to ask:

d) which polarized elliptic and $K3$-surfaces have stable Chow forms?

and what is more difficult probably:

e) can one compactify the moduli spaces of smooth surfaces by allowing suitable singular polarized surfaces which are still „asymptotically stable", i.e., have stable Chow forms when embedded by any complete linear system which is a sufficiently large multiple of the polarization?

In a more classical direction, now that the reasons for the existence of $\mathcal{M}_g$ and $\mathcal{A}_g$ are so well understood, the time seems ripe to try to understand their geometry more deeply, e.g.

f) Can one calculate, or bound, some birational invariants* of $\mathcal{M}_g$ or $\mathcal{A}_g$? Investigate the cohomology ring and chow ring of $\mathcal{M}_g$ or $\mathcal{A}_g$.
and

g) Find explicit Siegel modular forms vanishing on the Jacobian locus or cutting out this locus, and relate the various known special properties of Jacobians.

For those who want to learn something of the subject of moduli, we want to say what they will *not* find here and where more background on these topics may be found. The subject of moduli divides at present into 3 broad areas: *deformation theory*, *geometric invariant theory*, and *the theory of period maps*. Deformation theory deals with local questions: infinitesimal deformations of a variety, or analytic germs of deformations. Period maps deal with the construction of moduli of Hodge structures and the construction of moduli spaces of varèties by attaching a family of Hodge structures to a family of vareties. Both of these subjects are discussed only briefly in this monograph (see Appendix 5B). Unfortunately, deformation theory has not received a systematic treatment by anyone: a general introduction to the theory of moduli as a whole including deformation theory is given in SESHADRI [304], deformations of *singularities* are treated in ARTIN [45]. The origins of the algebraic

* For large g, $\mathcal{M}_g$ and $\mathcal{A}_g$ are varieties of general type: FREITAG [108], [109], HARRIS–MUMFORD [337], HARRIS [603], MUMFORD [755]. See new references to Appendix 5D.

treatment of the subject are in GROTHENDIECK [13], exp. 195 and SCHLESSINGER, LICHTENBAUM [289], [185]. The theory of period maps is largely the creation of GRIFFITHS [121], and a survey of the theory can be found in GRIFFITHS-SCHMID [124]. Expository or part expository/part research articles on geometric invariant theory proper and related questions of moduli have been written by DIEUDONNÉ-CARREL [85], GIESEKER [119], NEWSTEAD [247], and MUMFORD [213], [218], [220]. We hope this monograph will help to make the subject accessible.

Writing these appendices has brought home to us very clearly how many people have been involved in invariant theory and moduli problems. It has been exciting to try to express coherently all their results and their interconnections. We are sure, however, that some people have been overlooked. For this we can only offer the hackneyed excuse that even together we had only four hands and two heads.

D. MUMFORD
J. FOGARTY
Cambridge, Mass.
November, 1981

Preface to first edition

The purpose of this book is to study two related problems: when does an orbit space of an algebraic scheme acted on by an algebraic group exist? And to construct moduli schemes for various types of algebraic objects. The second problem appears to be, in essence, a special and highly non-trivial case of the first problem. From an Italian point of view, the crux of both problems is in passing from a birational to a biregular point of view. To construct both orbit spaces and moduli "generically" are simple exercises. The problem is whether, within the set of all models of the resulting birational class, there is one model whose geometric points classify the set of orbits in some action, or the set of algebraic objects in some moduli problem. In both cases, it is quite possible that some orbits, or some objects are so exceptional, or, as we shall say, are ***not stable***, so that they must be left out of the model. The difficulty is to pin down the meaning of stability in a given case. One of the most intriguing unsolved problems, in this regard, is that of the moduli of non-singular polarized surfaces. Which such surfaces are not stable, in the sense that there is no moduli scheme for them and their deformations? This property is very delicate.

One of my principles has been not to worry too much about the difference between characteristic O, and finite characteristics. A large part of this book is, therefore, devoted to a theory developed only in characteristic O.* I am convinced, however, that it is almost entirely valid in all characteristics. What is necessary is to find some property of semi-simple algebraic groups in all characteristics which takes the place of the full reducibility of representations which is valid only in characteristic O. I conjecture, in fact, that if a semi-simple algebraic group G is represented in a vector space V, and if V_0 is an invariant subspace of codimension 1, then *for some* ν the invariant subspace of codimension 1

$$V_0 \cdot S^{p^\nu-1}(V) \subset S^{p^\nu}(V)$$

* The hypothesis of characteristic O is disguised in the assumption that the group which is acting is reductive (by reductive we always mean that all its representations are completely reducible). But, in characteristic p, only relatively unimportant groups are reductive (cf. [28]), so the theory is uninteresting.

is complemented by an invariant 1-dimensional subspace.** Nonetheless, this is unknown, and one consequence is that this book is divided fairly sharply into two halves. Although both parts are closely analogous, they are logically independent. One half consists in Chapters 1, 2, 4 and 5 which deal essentially only with characteristic zero, and yield a construction of the moduli scheme for curves over $\boldsymbol{Q}$. The, other half consists in Chapters 3, 6, and 7, which deal with the ,,arithmetic case", i.e., over Spec ($\boldsymbol{Z}$), and yield a construction of the moduli schemes for curves over $\boldsymbol{Z}$. From a standpoint of content, however, Chapters 1, 2, 3 and 4 deal with orbit space problems, while Chapters 5, 6 and 7 deal with moduli.

This book is written entirely in the language of schemes. Of course, the results, for most purposes, could have been stated and proven in a classical language. However, it seems to me that algebraic geometry fulfills only in the language of schemes that essential requirement of all contemporary mathematics: to state its definitions and theorems in their natural abstract and formal setting in which they can be considered independently of geometric intuition. Moreover, it seems to me incorrect to assume that any geometric intuition is lost thereby: for example, the underlying variety in an algebraic scheme is rediscovered, and perhaps better understood through the concept of geometric points. As another example, the theory of schemes has made it possible, in a very intuitive way, to finally dispose of that famous embarrassment to the Italian school: the lack of an algebraic proof of the completeness of the characteristic linear system of suitable complete continuous systems on a surface in characteristic O (cf. [40, 18, 33 and 36]).

It is my pleasure to acknowledge at this point the great encouragement and stimulation which *I* have received from OSCAR ZARISKI, JOHN TATE, and ALEXANDER GROTHENDIECK. In addition, *I* want to give credit to the many mathematicians from whom *I* have taken a great deal. This book is primarily an original monograph, but secondarily an exposition of a whole topic, so *I* have taken the liberty of including anybody else's results when relevant. *I* am particularly conscious of my indebtedness to GROTHENDIECK, HILBERT, and NAGATA. It is impossible to enumerate all the sources from which *I* have borrowed, but this is a partial list:

Ch. 1 and 2 owe a great deal to D. HILBERT [14],

§ 1.2. was developed independently by C. CHEVALLEY, N. IWAHORI, and M. NAGATA,

§ 2.2. is largely a theory of J. TITS,

Ch. 3 was worked out by J. TATE and myself,

** Here $S^k V$ stands for the k^{th} symmetric power of V.

§ 4.3 includes an example of M. NAGATA,

§ 4.5 is a theorem of B. KOSTANT.

In Ch. 5 and 7, the whole approach to moduli via functors is due to A. GROTHENDIECK,

§ 5.3 and 5.4 follow suggestions of A. GROTHENDIECK,

Ch. 6 is almost entirely the work of A. GROTHENDIECK.

Finally a word about references: the tremendous contributions made by GROTHENDIECK to both the technique and the substance of algebraic geometry have not always been paralleled by their publication in permanently available form. In particular, for many of his results, we have only the barest outlines of proofs, as presented in the Bourbaki Seminar (reprinted in [13]). Nonetheless, since all the results which *I* want to use have been presented in detail in seminars at Harvard and will be published before too long by GROTHENDIECK, there seems no harm in making full use of them. For the convenience of the reader, the results which are only to be found in [13], and some others for which no good reference is available, are reproduced in Ch. 0, § 5. We have not reproduced, however, the results which we need from the semi-published Seminar Notes "Séminaire géométrie algébrique, IHES, 1960—61" since full proofs appear there. The results in exposés 3 and 8 of these notes are among the most vital tools which we use, and a familiarity with them is essential in order to read Chapters 6 and 7.

Harvard University, March, 1965

DAVID MUMFORD

Contents

Chapter 0

Preliminaries

We list first some notations and conventions which we will follow:

(1) A "pre-scheme X/S" means a morphism from the pre-scheme X to the pre-scheme S. If $S = \text{Spec}(R)$, we shall abbreviate this to "a pre-scheme X/R".

(2) An S-valued point of a pre-scheme X means a morphism from S to X. If $S = \text{Spec}(R)$, we shall abbreviate this to "an R-valued point" of X. If, moreover, R is an algebraically closed field, such a point will be referred to as a "geometric point" of X.

(3) Given pre-schemes X/k, Y/k, where k is a field fixed in some discussion, then all morphisms $f: X \to Y$ will be understood to be k-morphisms; and $X \underset{\text{Spec}(k)}{\times} Y$ will be be abbreviated to $X \times Y$. Moreover, $\bar{k}$ will stand for an algebraic closure of k, and we shall abbreviate $X \times \text{Spec}(\bar{k})$ to $\bar{X}$. In this case, an algebraic pre-scheme* X/k will be called a pre-*variety* if $\bar{X}$ is irreducible and reduced. Finally, given a k-rational point $x \in X$, the image point, as reduced subscheme of X, will be denoted $\{x\}$.

(4) If Z is a closed subscheme of X, or a cycle on X, then supp (Z) will denote the closed subset of X which is the support of Z.

(5) The symbols $\boldsymbol{A}^n$, and $\boldsymbol{P}_n$ will denote affine n-space and projective n-space *over* $\mathbf{Z}$, (i.e. Proj $\mathbf{Z}[X_0, \ldots, X_n]$) unless, in a particular chapter, all considerations are over a ground field k, in which case they will denote affine n-space and projective n-space over k. $PGL(n)$ will be the projective group acting on $\boldsymbol{P}_n$, $GL(n)$ the general linear group acting on $\boldsymbol{A}^n$, and $\boldsymbol{G}_m = GL(1)$. The same conventions on the base scheme hold for these group schemes.

(6) 1_X is the identity morphism from X to X. p_1 and p_2 are the projections from $X \underset{S}{\times} Y$ to X and Y; p_{12}, etc. are the projections from $X \underset{S}{\times} Y \underset{S}{\times} Z$ to $X \underset{S}{\times} Y$, etc. If $f: X_1 \to X_2$ and $g: Y_1 \to Y_2$ are S-morphisms, then $f \times g: X_1 \underset{S}{\times} Y_1 \to X_2 \underset{S}{\times} Y_2$ is the product. If $f: X \to Y_1$ and

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* i.e. a scheme of finite type over Spec(k).

$g : X \to Y_2$ are S-morphisms, then $(f, g) : X \to Y_1 \times_S Y_2$ is the induced S-morphism.

(7) If X is a pre-scheme, and $x \in X$, then $\kappa(x)$ will denote the residue field of $\mathcal{O}_{x,X}$. If X is a scheme over a field k, X_k will denote the set of points $x \in X$ such that $k \cong \kappa(x)$.

§ 1. Definitions

Definition 0.1. A group pre-scheme G/S is a morphism $\pi : G \to S$ of pre-schemes, plus S-morphisms $\mu : G \times_S G \to G$, $\beta : G \to G$, $e : S \to G$ satisfying the usual identities:

(a) *Associativity:*

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{1_G \times \mu} & G \times_S G \\ \Big\downarrow {\scriptstyle \mu \times 1_G} & & \Big\downarrow {\scriptstyle \mu} \\ G \times_S G & \xrightarrow{\quad \mu \quad} & G \end{array}$$

commutes.

(b) *Law of inverse:* The compositions

$$G \xrightarrow{\Delta} G \times_S G \underset{\beta \times 1_G}{\overset{1_G \times \beta}{\rightrightarrows}} G \times_S G \xrightarrow{\mu} G$$

both equal $e \circ \pi$ (here Δ is the diagonal).

(c) *Law of identity:* The compositions

$$\begin{array}{ccccc} & & S \times_S G & & \\ & \cong\!\!\nearrow & & \searrow^{e \times 1_G} & \\ G & & & & G \times_S G \xrightarrow{\mu} G \\ & \cong\!\!\searrow & & \nearrow_{1_G \times e} & \\ & & G \times_S S & & \end{array}$$

both equal 1_G.

Definition 0.2. An *algebraic group* G over a field k, is a group pre-scheme G/k which is an algebraic *scheme*, *smooth* over k.

Definition 0.3. A group pre-scheme G/S *acts* or *operates* on a pre-scheme X/S if an S-morphism $\sigma : G \times_S X \to X$ is given, such that:

(a)
$$\begin{array}{ccc} G \times_S G \times_S X & \xrightarrow{1_G \times \sigma} & G \times_S X \\ \Big\downarrow {\scriptstyle \mu \times 1_X} & & \Big\downarrow {\scriptstyle \sigma} \\ G \times_S X & \xrightarrow{\quad \sigma \quad} & X \end{array}$$

commutes (where μ is the group law for G).