Claude Chevalley

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CLAUDE CHEVALLEY

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THEORY OF LIE GROUPS

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To

ELIE CARTAN

AND

HERMANN WEYL

INTRODUCTION

Expository books on the theory of Lie groups generally confine themselves to the local aspect of the theory. This limitation was probably necessary as long as general topology was not yet sufficiently well elaborated to provide a solid base for a theory in the large. These days are now passed, and we have thought that it would be useful to have a systematic treatment of the theory from a global point of view. The present volume introduces the main basic principles which govern the theory of Lie groups.

A Lie group is at the same time a group, a topological space and a manifold: it has therefore three kinds of "structures," which are interrelated with each other. The elementary properties of abstract groups are by now sufficiently well known to the general mathematical public to make it unnecessary for such a book as this one to contain a purely group-theoretic chapter. The theory of topological groups, however, has been included and is treated in Chapter II. The greatest part of this chapter is concerned with the theory of covering spaces and groups, which is developed independently from the theory of paths. Chapter III is concerned with the theory of (analytic) manifolds (independently of the notion of group). Our definition of a manifold is inspired by the definition of a Riemann surface given by H. Weyl in his book "Die Idee der Riemannschen Flache"; it has, compared with the definition by overlapping system of coordinates, the advantage of being intrinsic. The theory of involutive systems of differential equations on a manifold is treated not only from the local point of view but also in the large. In order to achieve this, a definition of the submanifolds of a manifold is given according to which a submanifold is not necessarily a topological subspace of the manifold in which it is imbedded.

The notions of topological group and manifold are combined together in Chapter IV to give the notions of analytic group and Lie group. An analytic group is a topological group which is given *a priori* as a manifold; a Lie group (at least when it is connected) is a topological group which can be endowed with a structure of manifold in such a way that it becomes an analytic group. It is shown that, if this is possible, the manifold-structure in question is uniquely determined, so that connected Lie groups and analytic groups are in reality the same things defined in different ways. We shall see however in the second volume that the difference becomes a real one when *complex* analytic groups are considered instead of the real ones which are treated here.

Chapter V contains an exposition of the theory of exterior differential forms of Cartan which plays an essential role in the general theory of Lie groups, as well in its topological as in its differential geometric aspects. This theory leads in particular to the construction of the invariant integral on a Lie group. In spite of the fact that this invariant integration can be defined on arbitrary locally compact groups, we have thought that it is more in the spirit of a treatise on Lie groups to derive it from the existence of left invariant differential forms.

Chapter VI is concerned with the general properties of *compact* Lie groups. The fundamental fact is of course contained in the statement of Peter-Weyl's theorem which guarantees the existence of faithful linear representations. We have also included a proof of the generalization by Tannaka of the Pontrjagin duality theorem. A slight modification of the original proof of Tannaka shows that a compact Lie group may be considered as the set of real points of an algebraic variety in a complex affine space, the whole variety being itself a Lie group on which complex coordinates can be introduced.

The second volume of this book, now in preparation, will be mainly concerned with the theory and classification of semi-simple Lie Groups.

In preparing this book, I have received many valuable suggestions from several of my friends, in particular from Warren Ambrose, Gerhardt Hochschild, Deane Montgomery and Hsiao Fu Tuan. I was helped in reading the proofs by John Coleman and Norman Hamilton. I have also received precious advice from Professor H. Weyl and Professor S. Lefschetz. To all of them I am glad to express here my deep gratitude.

C. C.

THEORY OF LIE GROUPS

Some Notations Used in This Book

I. We denote by ϕ the empty set, by $\{a\}$ the set composed of the single element a.

If f is a mapping of a set A into a set B, and if X is a sub-set of B, we denote by $\overset{-1}{f}(X)$ the set of the elements $a \epsilon A$ such that $f(a) \epsilon X$. If g is a mapping of B into a third set C, we denote by $g \circ f$ the mapping which assigns to every $a \epsilon A$ the element $g(f(a))$.

We use the signs $\cup$, $\cap$ to represent respectively the intersection and the union of sets. If E_α is a collection of sets, the index α running over a set A, we denote by $\bigcup_{\alpha \epsilon A} E_\alpha$ the union of all sets E_α and by $\bigcap_{\alpha \epsilon A} E_\alpha$ their intersection. We denote by δ_{ij} the Kronecker symbol, equal to 1 if $i = j$ and to 0 if $i \neq j$.

II. If G is a group, we call "neutral element" the element ϵ of G such that $\epsilon\sigma = \sigma$ for every $\sigma \epsilon G$.

We say that a sub-group H of G is "distinguished" if the conditions $\sigma \epsilon G$, $\tau \epsilon H$ imply $\tau\sigma\tau^{-1} \epsilon H$.

If $\sigma = (a_{ij})$ represents a matrix, the symbol $\boxed{\sigma} = \boxed{a_{ij}}$ stands for the determinant of the matrix; $\delta p \sigma$ stands for the trace of the matrix.

If $\mathfrak{M}$, $\mathfrak{N}$ are vector spaces over the same field K, we call *product* of $\mathfrak{M}$ and $\mathfrak{N}$, and denote by $\mathfrak{M} \times \mathfrak{N}$, the set of the pairs $(\mathbf{e}, \mathbf{f})$ with $\mathbf{e} \epsilon \mathfrak{M}$, $\mathbf{f} \epsilon \mathfrak{N}$, this set being turned in a vector space by the conventions

$$(\mathbf{e}, \mathbf{f}) + (\mathbf{e}', \mathbf{f}') = (\mathbf{e} + \mathbf{e}', \mathbf{f} + \mathbf{f}')$$
$$a(\mathbf{e}, \mathbf{f}) = (a\mathbf{e}, a\mathbf{f}) \qquad \text{for } a \epsilon K.$$

III. *Topology.* We call topological spaces only the spaces in which Hausdorf separation axiom is satisfied.

A neighbourhood of a point p in space $\mathfrak{B}$ is understood to be a set N such that there exists an open set U such that $p \varepsilon U \subset N$; N need not be open itself.

The adherence $\bar{A}$ of a set A in a topological space is the set of those points p such that every neighbourhood of p meets A. Every point of $\bar{A}$ is said to be adherent to A. We shall make use of the possibility of defining the topology in a space by the operation $A \to \bar{A}$ of adherence (cf. Alexandroff-Hopf, *Topologie*, Kap. 1).

Intervals. If a and b are real numbers such that $a \leqslant b$, we denote by $]a, b[$ the open interval of extremities a and b. We set $]a, b] =]a, b[\cup \{b\}$, $[a, b[=]a, b[\cup \{a\}$, $[a, b] =]a, b[\cup \{a\} \cup \{b\}$.

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CHAPTER I

The Classical Linear Groups

Summary. Chapter I introduces the classical linear groups whose study is one of the main objects of Lie group theory. The unitary and orthogonal groups are defined in §I, together with a series of other groups. Their fundamental property of being compact is established.

Section II is concerned with the study of the exponential of a matrix. The property for a matrix of being orthogonal or unitary is defined by a system of non-linear relationships between its coefficients; the exponential mapping gives a parametric representation of the set of unitary (or orthogonal) matrices by matrices whose coefficients satisfy *linear* relations (Cf. Proposition 5, §II, p. 8). The reader may observe that the spaces M^{s}, M^{sh}, M^{S}, M^{R} which are introduced on p. 8 all contain $YX - XY$ whenever they contain X and Y. Although we could have given here an elementary explanation of this fact, we have not done so, on account of the fact that the full importance of this result can only be grasped much later (in Chapter IV). In the cases of the orthogonal and unitary group, the linearization can also be accomplished by the Cayley parametrization (which we have not introduced); however, the exponential mapping is more advantageous from our point of view because it preserves some properties of the ordinary exponential function (Cf. Proposition 3, §IV, p. 13).

Sections III and IV are preliminary to the result which will be proved in Section V (Proposition 1, p. 14). Hermitian matrices are defined in terms of the unitary geometry in a complex vector space (unitary geometry is defined by the notion of hermitian product of two vectors, just as euclidean geometry can be defined in terms of the scalar product). Proposition 2, §III, p. 10 shows that the unitary matrices are the isometric transformations of a unitary geometry.

The proposition which asserts that the full linear group can be decomposed topologically into the product of the unitary group and the space of positive definite hermitian matrices (Proposition 1, §V, p. 14) is the prototype of the theorems which allow us to derive topological properties of general Lie groups from the properties of compact groups. A similar decomposition is given for the complex orthogonal group (Proposition 2, §V, p. 15).

Sections VI and VII are preliminary to the definition of the symplectic groups. The symplectic group is defined to be the group of isometric transformations of a symplectic geometry (Definition 1, §VII, p. 20). In §IX, we construct a representation of $Sp(n)$ by complex matrices of degree $2n$. The consideration of the conditions which the matrices of this representation must satisfy leads to the introduction of a new group, the complex symplectic group $Sp(n, C)$. It can be seen easily that $Sp(n, C)$ stands in the same relation to $Sp(n)$ as $GL(n, C)$ to $U(n)$ or as $O(n, C)$ to $O(n)$. A proposition of the type of Proposition 1, §V, p. 14 could be derived without much difficulty for $Sp(n, C)$. However, we have not found it necessary to state this

proposition, which is contained as a special case of a theorem proved later (Corollary to Theorem 5, Chapter VI, §XII, p. 211).

§I. THE FULL LINEAR GROUP AND SOME OF ITS SUBGROUPS

The n-dimensional complex cartesian space C^n may be considered as a vector space of dimension n over the field C of complex numbers. Let $\mathbf{e}_i$ be the element of C^n whose i-th coordinate is 1 and whose other coordinates are 0. The elements $\mathbf{e}_1, \cdots, \mathbf{e}_n$ form a base of C^n over C.

A linear endomorphism α of C^n is determined when the elements $\alpha\mathbf{e}_i = \Sigma_{j=1}^n a_{ji}\mathbf{e}_j$ are given. There corresponds to this endomorphism a matrix (a_{ij}) of degree n; we shall denote this matrix by the same letter α as the endomorphism itself. Conversely, to any matrix of degree n with complex coefficients, there corresponds an endomorphism of C^n.

Let α and β be two endomorphisms of C^n, and let (a_{ij}) and (b_{ij}) be the corresponding matrices. Then $\alpha \circ \beta$ is again an endomorphism, whose matrix (c_{ij}) is the product of the matrices (a_{ij}) and (b_{ij}); i.e.

$$c_{ij} = \Sigma_{k=1}^n a_{ik}b_{kj} \tag{1}$$

We shall denote by $\mathfrak{M}_n(C)$ the set of all matrices of degree n with coefficients in C. If $(a_{ij})\varepsilon\mathfrak{M}_n(C)$, we set $b_{i+(j-1)n} = a_{ij}$ and we associate with the matrix (a_{ij}) the point of coordinates $b_1, \cdots, b_{n^2}$ in C^{n^2}. We obtain in this way a one-to-one correspondence between $\mathfrak{M}_n(C)$ and C^{n^2}. Since C^{n^2} is a topological space, we can define a topology in $\mathfrak{M}_n(C)$ by the requirement that our correspondence shall be a homeomorphism between $\mathfrak{M}_n(C)$ and C^{n^2}.

Let $\mathfrak{E}$ be any topological space, and let φ be a mapping of $\mathfrak{E}$ into $\mathfrak{M}_n(C)$. If $t\varepsilon\mathfrak{E}$, let $a_{ij}(t)$ be the coefficients of the matrix $\varphi(t)$. It is clear that φ will be continuous if and only if each function $a_{ij}(t)$ is continuous.

It follows immediately from this remark and from the formulas (1) that the product $\sigma\tau$ of two matrices σ, τ is a continuous function of the pair (σ, τ), considered as a point of the space $\mathfrak{M}_n(C) \times \mathfrak{M}_n(C)$.

If $\alpha = (a_{ij})$, we shall denote by ${}^t\alpha$ the *transpose* of α, i.e. the matrix (a'_{ij}), with $a'_{ij} = a_{ji}$. We shall denote by $\bar{\alpha}$ the complex conjugate of α, i.e. the matrix $\bar{\alpha} = (\bar{a}_{ij})$. It is clear that the mappings $\alpha \rightarrow {}^t\alpha$, $\alpha \rightarrow \bar{\alpha}$ are homeomorphisms of order 2 of $\mathfrak{M}_n(C)$ with itself. If α and β are any two matrices, we have

$${}^t(\alpha\beta) = {}^t\beta\,{}^t\alpha \qquad \overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$$

A matrix σ will be called *regular* if it has an inverse, i.e., if there exists a matrix σ^{-1} such that $\sigma\sigma^{-1} = \sigma^{-1}\sigma = \epsilon$, where ϵ is the unit matrix of degree n. A necessary and sufficient condition for a matrix σ to be regular is that its determinant $\boxed{\sigma}$ be $\neq 0$.

If an endomorphism σ of C^n maps C^n onto itself (and not onto some subspace of lower dimension), the corresponding matrix σ is regular and σ has a reciprocal endomorphism σ^{-1}.

If σ is a regular matrix, we have

$$^t(\sigma^{-1}) = (^t\sigma)^{-1} \qquad \bar{\sigma}^{-1} = \overline{(\sigma^{-1})}$$

If σ and τ are regular matrices, $\sigma\tau$ is also regular and we have

$$(\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1}$$

It follows that the regular matrices form a group with respect to the operation of multiplication.

Definition 1. *The group of all regular matrices of degree n with complex coefficients is called the general linear group. We shall denote it by $GL(n, C)$.*

Since the determinant of a matrix is obviously a continuous function of the matrix, $GL(n, C)$ is an open subset of $\mathfrak{M}_n(C)$. We may consider the elements of $GL(n, C)$ as points of a topological space, which is a subspace of $\mathfrak{M}_n(C)$.

If $\sigma = (a_{ij})$ is a regular matrix, the coefficients b_{ij} of σ^{-1} are given by expressions of the form

$$b_{ij} = A_{ij}\boxed{\sigma}^{-1}$$

where the A_{ij}'s are polynomials in the coefficients of σ. It follows that the mapping $\sigma \to \sigma^{-1}$ of $GL(n, C)$ onto itself is continuous. Since this mapping coincides with its reciprocal mapping, it is a homeomorphism of order 2 of $GL(n, C)$ with itself.

The mappings $\sigma \to \bar{\sigma}$ and $\sigma \to {}^t\sigma$ are homeomorphisms of $GL(n, C)$ with itself. The first but not the second is also an automorphism of the group $GL(n, C)$.

If $\sigma \in GL(n, C)$, we shall denote by σ^* the matrix defined by the formula

$$\sigma^* = {}^t\sigma^{-1}$$

We have

$$(\sigma\tau)^* = \sigma^*\tau^* \qquad (\sigma^*)^{-1} = (\sigma^{-1})^*$$

Hence, the mapping $\sigma \to \sigma^*$ is a homeomorphism and an automorphism of order 2 of $GL(n, C)$.

Definition 2. *A matrix σ is said to be orthogonal if $\sigma = \bar{\sigma} = \sigma^*$. The set of all orthogonal matrices of degree n will be denoted by $O(n)$. If only $\sigma = \sigma^*$, σ is said to be complex orthogonal; the set of these matrices will be denoted by $O(n, C)$. If only $\bar{\sigma} = \sigma^*$, σ is said to be unitary. The set of all unitary matrices will be denoted by $U(n)$.*

Since the mappings $\sigma \rightarrow \bar{\sigma}$ and $\sigma \rightarrow \sigma^*$ are continuous, the sets $O(n)$, $O(n, C)$ and $U(n)$ are closed subsets of $GL(n, C)$. Because these mappings are automorphisms, $O(n)$, $O(n, C)$ and $U(n)$ are subgroups of $GL(n, C)$. We have clearly

$$O(n) = O(n, C) \cap U(n).$$

Definition 3. *We shall say that the matrix σ is real if its coefficients are real,* i.e. *if $\sigma = \bar{\sigma}$. The set of all real matrices of degree n will be denoted by $\mathfrak{M}_n(R)$. The set $\mathfrak{M}_n(R) \cap GL(n, C)$ will be denoted by $GL(n, R)$.*

Therefore, we have also

$$O(n) = GL(n, R) \cap O(n, C)$$

The determinant of the product of two matrices being the product of the determinants of these matrices, it follows that the matrices of determinant 1 form a subgroup of $GL(n, C)$.

Definition 4. *The group of all matrices of determinant* 1 *in $GL(n, C)$ is called the special linear group. This group is denoted by $SL(n, C)$. We set $SL(n, R) = SL(n, C) \cap GL(n, R)$; $SO(n) = SL(n, C) \cap O(n)$; $SU(n) = SL(n, C) \cap U(n)$.*

It is clear that $SL(n, C)$, $SL(n, R)$, $SO(n)$, $SU(n)$ are subgroups and closed subsets of $GL(n, C)$. They may be considered as subspaces of $GL(n, C)$.

Theorem 1. *The spaces $U(n)$, $O(n)$, $SU(n)$ and $SO(n)$ are compact.*

Since $O(n)$, $SU(n)$ and $SO(n)$ are closed subsets of $U(n)$, it is sufficient to prove that $U(n)$ is compact. A matrix σ is unitary if and only if ${}^t\sigma\bar{\sigma} = \epsilon$, where ϵ is the unit matrix (in fact, this condition implies that σ is regular and that $\sigma^* = \bar{\sigma}$). If $\sigma = (a_{ij})$, the equation ${}^t\sigma\bar{\sigma} = \epsilon$ is equivalent to the conditions

$$\Sigma_j a_{ji}\bar{a}_{jk} = \delta_{ik}$$

Since the left sides of these equations are continuous functions of σ, $U(n)$ is not only a closed subset of $GL(n, C)$ but also of $\mathfrak{M}_n(C)$. Moreover, the conditions $\Sigma_j a_{ji}\bar{a}_{ji} = 1$ imply $|a_{ij}| \leqslant 1 (1 \leqslant i, j \leqslant n)$. It follows that the coefficients of a matrix $\sigma \varepsilon U(n)$ are bounded. If we take into account the homeomorphism established between $\mathfrak{M}_n(C)$

and C^{n^2}, we see that $U(n)$ is homeomorphic to a closed bounded subset of C^{n^2}. Theorem 1 is thereby proved.

§II. The Exponential of a Matrix

Let α be any matrix of degree n, and let μ be an upper bound for the absolute values of the coefficients $x_{ij}(\alpha)$ of α. Let $x_{ij}^{(p)}(\alpha)$ be the coefficients of $\alpha^p (0 \leqslant p < \infty$; we set $\alpha^0 = \epsilon =$ the unit matrix). We assert that $|x_{ij}^{(p)}(\alpha)| \leqslant (n\mu)^p$. This is true for $p = 0$. Assume that our inequality holds for some integer $p \geqslant 0$; then

$$|x_{ij}^{(p+1)}(\alpha)| = \left|\Sigma_k x_{ik}^{(p)}(\alpha) x_{kj}(\alpha)\right| \leqslant n(n\mu)^p\mu = (n\mu)^{p+1}$$

which proves that the inequality holds for $p + 1$.

It follows that each of the n^2 series $\Sigma_{p=0}^{\infty} \frac{1}{p!} x_{ij}^{(p)}(\alpha)$ converges uniformly on the set of all α such that $|x_{ij}(\alpha)| \leqslant \mu$. In other words, the series $\epsilon + \frac{\alpha}{1} + \frac{\alpha^2}{2!} + \cdots + \frac{\alpha^p}{p!} + \cdots$ is always convergent, and uniformly so when α remains in a bounded region of the set $\mathfrak{M}_n(C)$.

Definition 1. *We denote by* exp α *the sum of the series* $\Sigma_0^{\infty} \frac{1}{p!} \alpha^p$.

The function exp α is thus defined and continuous on $\mathfrak{M}_n(C)$ and maps $\mathfrak{M}_n(C)$ into itself.

Proposition 1. *If σ is a regular matrix of degree n, then*

$$\exp(\sigma\alpha\sigma^{-1}) = \sigma(\exp \alpha)\sigma^{-1}$$

In fact, we have $\sigma\alpha^p\sigma^{-1} = (\sigma\alpha\sigma^{-1})^p$, and hence $\exp(\sigma\alpha\sigma^{-1}) = \Sigma_0^{\infty} \frac{1}{p!}(\sigma\alpha\sigma^{-1})^p = \Sigma_0^{\infty}\sigma\left(\frac{1}{p!}\alpha^p\right)\sigma^{-1} = \sigma\left(\Sigma_0^{\infty}\frac{1}{p!}\alpha^p\right)\sigma^{-1} = \sigma(\exp \alpha)\sigma^{-1}$.

Proposition 2. *If $\lambda_1, \cdots, \lambda_n$ are the characteristic roots of α, each occurring a number of times equal to its multiplicity, the characteristic roots of* exp α *are* exp $\lambda_1, \cdots,$ exp λ_n.

We shall prove this by induction on n. It is obvious for $n = 1$, because then α is a complex number. Now, assume that $n > 1$ and that the proposition holds for matrices of degree $n - 1$.

Let λ_1 be a characteristic root of α; then there is an element $\mathbf{a} \neq 0$ in C^n such that $\alpha\mathbf{a} = \lambda_1\mathbf{a}$. Let $\mathbf{e}_1$ be the point whose coordinates are $1, 0, \cdots, 0$. Because $\mathbf{a} \neq 0$, there exists a regular matrix σ such that $\sigma\mathbf{a} = \mathbf{e}_1$. Then $\sigma\alpha\sigma^{-1}\mathbf{e}_1 = \lambda\mathbf{e}_1$; in other words,

$$\sigma\alpha\sigma^{-1} = \begin{pmatrix} \lambda_1 & * \cdots\cdots * \\ 0 & \\ \cdot & (\tilde{\alpha}) \\ \cdot & \\ 0 & \end{pmatrix}$$

where the *'s indicate complex numbers and $\tilde{\alpha}$ is a matrix of degree $n - 1$. We have

$$\sigma\alpha^p\sigma^{-1} = \begin{pmatrix} \lambda_1^p & * \cdots\cdots * \\ 0 & \\ \cdot & (\tilde{\alpha}^p) \\ \cdot & \\ \cdot & \\ 0 & \end{pmatrix}$$

and therefore

$$\exp\,(\sigma\alpha\sigma^{-1}) = \begin{pmatrix} \exp\lambda_1 & * \cdots\cdots * \\ 0 & \\ \cdot & \\ \cdot & (\exp\tilde{\alpha}) \\ \cdot & \\ \cdot & \\ 0 & \end{pmatrix}$$

If $\lambda_2, \cdots, \lambda_n$ are the characteristic roots of $\tilde{\alpha}$, those of α, which are the same as those of $\sigma\alpha\sigma^{-1}$, are $\lambda_1, \lambda_2, \cdots, \lambda_n$. The proposition being true for matrices of degree $n - 1$, it follows that the characteristic roots of $\exp\tilde{\alpha}$ are $\exp\lambda_2, \cdots, \exp\lambda_n$, and those of $\exp(\sigma\alpha\sigma^{-1})$ are $\exp\lambda_1, \exp\lambda_2, \cdots, \exp\lambda_n$. But these are also the characteristic roots of $\sigma(\exp\alpha)\sigma^{-1}$ (Cf. Proposition 1) and hence of $\exp\alpha$. Proposition 2 is thereby proved.

Corollary 1. *The determinant of the matrix* $\exp\alpha$ *is* $\exp \mathrm{Sp}\,\alpha$.

This follows at once from the facts that the trace and the determinant of a matrix are respectively the sum and the product of the characteristic roots.

Corollary 2. *The exponential of any matrix is a regular matrix.*

Proposition 3. *If* α *and* β *are permutable matrices* (i.e. *if* $\alpha\beta = \beta\alpha$) *then* $\exp(\alpha + \beta) = (\exp\alpha)(\exp\beta)$.

Since α and β are permutable, we can expand $(\alpha + \beta)^p$ by the binomial formula:

$$\frac{1}{P!}(\alpha + \beta)^P = \Sigma_0^P \frac{\alpha^k}{k!}\frac{\beta^{p-k}}{(p-k)!}$$