N. Young

An Introduction to Hilbert Space

希尔伯特空间导论

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NICHOLAS YOUNG
Department of Mathematics
Glasgow University

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FOREWORD

The basic notions of the theory of Hilbert space are current in many parts of pure and applied mathematics, and in physics, engineering and statistics. They are well worth a place in any honours mathematics course, and Chapters 1 to 8 of this book aim to present them in a way accessible to undergraduate students. A course in Hilbert space is likely to be the last analysis course for many students, and it should therefore be able to stand on its own: it should not depend for its motivation on further study of abstract analysis, but should as far as possible have a value which is apparent either on aesthetic grounds or for its scientific or practical applications. For this reason I have included more historical and background material than is customary, and have omitted some of the major theorems about Banach spaces which are traditionally taught in introductory courses on functional analysis, but which are really more appropriate to students who will be pursuing operator theory further (the closed graph, Hahn–Banach and uniform boundedness theorems). The second half of the book describes two substantial applications. One of these is standard: the Sturm–Liouville theory of eigenfunction expansions, and its role in the solution of the partial differential equations of mathematical physics by the method of separation of variables. The other (in Chapters 12 to 16) is less common, but is nevertheless ideal for a final year course. It is beautiful mathematics, it is relatively recent and visibly useful. It also entails the development of some standard operator theory along the way, and exhibits very well the connection between abstract analysis and the more classical field of complex analysis.

Although the book was written primarily for an undergraduate audience, I hope it may be found useful for graduate courses also. I firmly believe that functional analysis is best approached through a sound grounding in Hilbert space theory, and am confident that students will be better able to benefit from one of the many excellent advanced texts on functional analysis and its applications if they first master the material contained herein. Chapters 12 to 16 may also be of interest to some electrical engineers. Some recent developments, particularly in control and filter design, require familiarity with this aspect of operator theory.

Chapters 1 to 8 are based on a compulsory course of twenty lectures which I gave to third year honours students at Glasgow University, and the remainder of the book, with a few omissions, on an optional twenty lecture course for fourth year students. In forty lectures at the undergraduate level it should be possible to cover the whole book except for the Adamyan–Arov–Krein theorem and the proofs of Fatou's theorem and the existence of square roots of positive operators. Chapters 12 to 16 do not depend on Chapters 9 to 11: they can be read straight after Chapter 8.

The book presupposes introductory courses in real analysis, linear algebra and topology (metric spaces suffice). For Chapters 12 to 16, and some of the problems earlier in the book, elementary complex analysis is required. It is tacitly assumed in Chapters 9 to 11 that the reader has met differential equations before, though formal requirements are slight. I have taken pains *not* to assume knowledge of the Lebesgue integral: the reader is asked only to believe that there is a definition of integral which makes $L^2(a, b)$ complete and the continuous functions a dense subspace. However, I am obliged to admit that there are parts of Chapter 13 which will feel distinctly more comfortable to those who are familiar with Lebesgue measure.

I am grateful to Dr Frances Goldman and Dr Philip Spain for reading the text and making useful suggestions. I am also very thankful that, despite the all-conquering march of the word processor, Cambridge University Press was willing to accept manuscript, so that I do not have to thank anyone for his excellent typing.

CONTENTS

INTRODUCTION

Functional analysis is a branch of mathematics which uses the intuitions and language of geometry in the study of functions. The classes of functions with the richest geometric structure are called Hilbert spaces, and the theory of these spaces is the core around which functional analysis has developed. One can begin the story of this development with Descartes' idea of algebraicizing geometry. The device of using co-ordinates to turn geometric questions into algebraic ones was so successful, for a wide but limited range of problems, that it dominated the thinking of mathematicians for well over a century. Only slowly, under the stimulus of mathematical physics, did the perception dawn that the correspondence between algebra and geometry could also be made to operate effectively in the reverse direction. It can be useful to represent a point in space by a triple of numbers, but it can also be advantageous, in dealing with triples of numbers, to think of them as the co-ordinates of points in space. This might be termed the geometrization of algebra: it enables new concepts and techniques to be derived from our intuition for the space we live in. It is regrettable that this intuition is limited to three spatial dimensions, but mathematicians have not allowed this circumstance to prevent them from using geometric terminology in handling n-tuples of numbers when $n > 3$. In the context of $\mathbb{R}^n$ one routinely speaks of points, spheres, hyperplanes and subspaces. Though such language comes to seem very natural to us, it still depends on analogy, and we must have recourse to algebra and analysis to verify that our analogies are valid and to determine which analogies are useful.

Once the geometric habit of mind was established in relation to $\mathbb{R}^n$ it was natural to extend it to other common objects of mathematics which enjoy a similar linear structure, such as functions and infinite sequences of real numbers. This is a bolder leap into the unknown, and we must expect that our intuition for physical space will prove a shakier guide than it was for $\mathbb{R}^n$. Indeed, the task of sorting out the right basic concepts in the geometry of infinite-dimensional spaces preoccupied leading analysts for some decades around the turn of the century. Thereafter the geometric viewpoint proved its worth, and came to provide the backdrop for the greater part of modern work in differential and integral equations, quantum mechanics and other disciplines to which mathematics is applied.

The study of differential and integral equations arising in physics was one of the main impulses to the emergence of functional analysis. A precursor of the subject can be seen in attempts by several mathematicians to treat such equations as limits in some sense of finite systems of equations. This approach had fair success, particularly in the hands of Hilbert, and it still has plenty of life in the domain of numerical analysis. Suppose, for example, one wishes to solve the integral equation

$$\int_0^1 K(x, y) f(y)\,\mathrm{d}y = g(x).$$

Here K and g are known continuous functions on $[0, 1] \times [0, 1]$ and $[0, 1]$ respectively, and one is looking for a continuous solution f. It seems natural to approximate this system by the finite system

$$\sum_{j=0}^{n-1} K\left(\frac{i}{n}, \frac{j}{n}\right) f_{jn} \cdot \frac{1}{n} = g\left(\frac{i}{n}\right),$$

$i = 0, 1, \ldots, n-1$. Assuming that this system of n linear equations in the n unknowns $f_{0n}, \ldots, f_{n-1,n}$ has a unique solution, one might expect that, for large n, f_{jn} ought to be close to $f(j/n)$, at least under further conditions on K and g.

Hilbert was by no means the first to use this device. Fourier himself was led to introduce Fourier series in a rather similar way. In studying the conduction of heat he encountered the differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

subject to certain boundary conditions. By the method of the separation of variables he derived the solution

$$V(x, y) = \sum_{m=1}^{\infty} a_m \mathrm{e}^{-(2m-1)x} \cos(2m-1)y,$$

where the coefficients a_m are determined by the infinite system of linear equations

$$\sum_{1}^{\infty} a_m = 1,$$

$$\sum_{1}^{\infty} (2m-1)^2 a_m = 0,$$

$$\sum_{1}^{\infty} (2m-1)^4 a_m = 0,$$

$$\dots$$

Fourier handled these by taking the first k equations and truncating them to k terms. This gives a $k \times k$ system which has a solution $a_1^{(k)}, \dots, a_k^{(k)}$. On letting $k \to \infty$ Fourier obtained the desired solution of the infinite system.

Although this trick often worked, it has its dangers. Consider the infinite system

$$\begin{aligned} x_1 + x_2 + x_3 + \cdots &= 1, \\ x_2 + x_3 + \cdots &= 1, \\ x_3 + \cdots &= 1, \\ \dots \end{aligned}$$

No choice of the x_j will satisfy this system, yet Fourier's limiting procedure would yield the apparent solution $x_j = 0$ for all j.

By virtue of powerful technique and a perception of what was important, Hilbert was able to make great contributions using this idea. Nevertheless, mathematicians came to regard the method as inadequate. It is clumsy and notationally complicated. The procedure of passage to the limit is difficult, and, indeed, it has been asserted that Hilbert did not always accomplish it correctly (see Reid, 1970). He himself did not arrive at the modern geometric viewpoint: Hilbert never used 'Hilbert space'. It was other mathematicians, particularly Erhardt Schmidt and Frigyes Riesz, who reflected on his results and discovered the right conceptual framework for them. Thereby they created a simpler, more elegant and more powerful theory. In this one does not try to reduce essentially infinite-dimensional questions to finite-dimensional geometry and then 'let $n \to \infty$': instead one develops the geometry of the objects of analysis as they naturally occur, using the familiar finite-dimensional geometry rather as a guide and analogy.

1

Inner product spaces

Some important metric notions such as length, angle and the energy of physical systems can be expressed in terms of the *inner product* (x, y) of vectors $x, y \in \mathbb{C}^n$. This is defined by

$$(x, y) = \sum_{i=1}^{n} x_i \bar{y}_i, \tag{1.1}$$

where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and $\bar{y}_i$ is the complex conjugate of y_i. We wish to construct an infinite-dimensional version of this inner product. The most obvious attempt is to consider the space $\mathbb{C}^{\mathbb{N}}$ of all complex sequences indexed by $\mathbb{N}$. This is a complex vector space in a natural way, but it is not clear how we can extend the notion of inner product to it. If we replace the finite sum in (1.1) by an infinite one then the series will fail to converge for many pairs of sequences. We therefore restrict attention to a subspace of $\mathbb{C}^{\mathbb{N}}$.

1.1 Definition ℓ^2 denotes the vector space over $\mathbb{C}$ of all complex sequences $x = (x_n)_{n=1}^{\infty}$ which are square summable, that is, satisfy

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty,$$

with componentwise addition and scalar multiplication, and with inner product given by

$$(x, y) = \sum_{n=1}^{\infty} x_n \bar{y}_n, \tag{1.2}$$

where $x = (x_n)$, $y = (y_n)$. □

'Componentwise' means the following: if $x = (x_n)$, $y = (y_n) \in \ell^2$ and $\lambda \in \mathbb{C}$ then

$$x + y = (x_n + y_n)_{n=1}^{\infty},$$
$$\lambda x = (\lambda x_n)_{n=1}^{\infty}.$$

Let us check that this definition of inner product does make sense. Using the Cauchy–Schwarz inequality we find, for $k \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{k} |x_n \bar{y}_n| &= \sum_{n=1}^{k} |x_n||y_n| \\ &\leqslant \left\{\sum_{n=1}^{k} |x_n|^2\right\}^{1/2} \left\{\sum_{n=1}^{k} |y_n|^2\right\}^{1/2} \\ &\leqslant \left\{\sum_{n=1}^{\infty} |x_n|^2\right\}^{1/2} \left\{\sum_{n=1}^{\infty} |y_n|^2\right\}^{1/2}.\end{aligned}$$

If (x_n) and (y_n) are square summable sequences then the latter expression is a finite number independent of k. Thus the series (1.2) converges absolutely, and so (x, y) is defined by (1.2) as a complex number for any $x, y \in \ell^2$.

It is obvious that ℓ^2 is closed under scalar multiplication but less so that it is closed under addition: we defer the proof of this to Exercise 1.12 below.

Let us make precise what it means to say that $\mathbb{C}^n$ and ℓ^2 are spaces with an inner product.

1.2 Definition An *inner product* (or *scalar product*) on a complex vector space V is a mapping

$$(\cdot\,,\,\cdot): V \times V \to \mathbb{C}$$

such that, for all $x, y, z \in V$ and all $\lambda \in \mathbb{C}$,

(i) $(x, y) = (y, x)^-$;

(ii) $(\lambda x, y) = \lambda(x, y)$;

(iii) $(x + y, z) = (x, z) + (y, z)$;

(iv) $(x, x) > 0 \qquad$ when $x \neq 0$.

An inner product space (or *pre-Hilbert space*) is a pair $(V, (\cdot\,,\,\cdot))$ where V is a complex vector space and $(\cdot\,,\,\cdot)$ is an inner product on V. □

It is routine to check that the formulae (1.1) and (1.2) do define inner products on $\mathbb{C}^n$ and ℓ^2 in the sense of Definition 1.2. There are many other inner product spaces which arise in analysis, most of them having inner products defined in terms of integrals.

1.3 Exercise Show that the formula

$$(f, g) = \int_0^1 f(t)\overline{g(t)}\, dt$$

defines an inner product on the complex vector space $C[0, 1]$ of all continuous $\mathbb{C}$-valued functions on $[0, 1]$, with pointwise addition and scalar multiplication. □

1.4 Exercise Show that the formula

$$(A, B) = \text{trace}(B^* A)$$

defines an inner product on the space $\mathbb{C}^{m \times n}$ of $m \times n$ complex matrices, where $m, n \in \mathbb{N}$ and B^* denotes the conjugate transpose of B. □

The conditions (ii) and (iii) in the definition of inner product are often summarized by the statement that $(\cdot, \cdot)$ is linear in the first argument. It follows from the definition that it is also *conjugate linear* in the second argument: this means that it satisfies (i) and (ii) of the following.

1.5 Theorem For any x, y, z in an inner product space V and any $\lambda \in \mathbb{C}$,

(i) $(x, y + z) = (x, y) + (x, z)$;

(ii) $(x, \lambda y) = \bar{\lambda}(x, y)$;

(iii) $(x, 0) = 0 = (0, x)$;

(iv) if $(x, z) = (y, z)$ for all $z \in V$ then $x = y$.

Proof. (i) Using Definition 1.2(i) and (iii) we have

$$\begin{aligned}(x, y + z) &= (y + z, x)^- \\ &= [(y, x) + (z, x)]^- \\ &= (y, x)^- + (z, x)^- \\ &= (x, y) + (x, z).\end{aligned}$$

The proof of (ii) is similar. To prove (iii) put $\lambda = 0$ in (ii).

(iv) If $(x, z) = (y, z)$ then

$$\begin{aligned}0 &= (x, z) + (-1)(y, z) \\ &= (x, z) + (-y, z) = (x - y, z).\end{aligned}$$

If this holds for all $z \in V$ then in particular it holds when $z = x - y$; thus $(x - y, x - y) = 0$. By 1.2(iv) it follows that $x - y = 0$. □

1.1 Inner product spaces as metric spaces

In the familiar case of $\mathbb{R}^3$ the magnitude $|\boldsymbol{u}|$ of a vector $\boldsymbol{u}$ is equal to $(\boldsymbol{u}, \boldsymbol{u})^{1/2}$, and the Euclidean distance between points with position vectors $\boldsymbol{u}, \boldsymbol{v}$ is $|\boldsymbol{u} - \boldsymbol{v}|$. We copy this to introduce a natural metric in an inner product space.

1.6 Definition The *norm* of a vector x in an inner product space is defined to be $(x, x)^{1/2}$. It is written $\|x\|$. □

Thus, for $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ we have

$$\|x\| = (|x_1|^2 + \cdots + |x_n|^2)^{1/2},$$

while for $f \in C[0,1]$, with the inner product described in Exercise 1.3,

$$\|f\| = \left\{\int_0^1 |f(t)|^2 \, dt\right\}^{1/2}.$$

1.7 *Exercise* Let $x = (1/n)_{n=1}^{\infty} \in \ell^2$. Show that $\|x\| = \pi/\sqrt{6}$. What is $\|I_n\|$ where $I_n \in \mathbb{C}^{n \times n}$ is the identity matrix and the inner product of Exercise 1.4 is used? □

1.8 *Theorem* For any x in an inner product space V and any $\lambda \in \mathbb{C}$

(i) $\|x\| \geqslant 0$; $\|x\| = 0$ if and only if $x = 0$;

(ii) $\|\lambda x\| = |\lambda| \|x\|$.

Proof. (ii)

$$\|\lambda x\| = (\lambda x, \lambda x)^{1/2} = \{\lambda\bar{\lambda}(x,x)\}^{1/2}$$
$$= |\lambda| \|x\|.$$ □

One knows that in $\mathbb{R}^3$ (x,y) is $\|x\| \|y\|$ times the cosine of an angle, from which it follows that $|(x,y)| \leqslant \|x\| \|y\|$. This relation continues to hold in a general inner product space.

1.9 *Theorem* For x, y in an inner product space V,

$$|(x,y)| \leqslant \|x\| \|y\|, \tag{1.3}$$

with equality if and only if x and y are linearly dependent.

(1.3) is known as the *Cauchy–Schwarz inequality*.

Proof. Suppose first that x and y are linearly dependent – say $x = \lambda y$ where $\lambda \in \mathbb{C}$. Then both sides of (1.3) equal $|\lambda| \|y\|^2$, and so (1.3) holds with equality.

Now suppose that x and y are linearly independent: we must show that (1.3) holds with strict inequality. For any $\lambda \in \mathbb{C}$, $x + \lambda y \neq 0$ and therefore

$$\begin{aligned} 0 &< (x + \lambda y, x + \lambda y) \\ &= (x, x + \lambda y) + (\lambda y, x + \lambda y) \\ &= (x,x) + (x, \lambda y) + (\lambda y, x) + (\lambda y, \lambda y) \\ &= \|x\|^2 + \bar{\lambda}(x,y) + \lambda(x,y)^- + |\lambda|^2 \|y\|^2 \\ &= \|x\|^2 + 2\,\mathrm{Re}\{\bar{\lambda}(x,y)\} + |\lambda|^2 \|y\|^2. \end{aligned}$$

Pick a complex number u of unit modulus such that $\bar{u}(x,y) = |(x,y)|$. On putting $\lambda = tu$ we deduce that, for any $t \in \mathbb{R}$,

$$0 < \|x\|^2 + 2|(x,y)|t + \|y\|^2 t^2.$$