Steven G. Krantz
Harold R. Parks

THE IMPLICIT FUNCTION THEOREM

HISTORY, THEORY, AND APPLICATIONS

隐函数定理

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The Implicit Function Theorem
by Steven G. Krantz and Harold R. Parks

To the memory of Kennan Tayler Smith (1926–2000)

Preface

The implicit function theorem is, along with its close cousin the inverse function theorem, one of the most important, and one of the oldest, paradigms in modern mathematics. One can see the germ of the idea for the implicit function theorem in the writings of Isaac Newton (1642–1727), and Gottfried Leibniz's (1646–1716) work explicitly contains an instance of implicit differentiation. While Joseph Louis Lagrange (1736–1813) found a theorem that is essentially a version of the inverse function theorem, it was Augustin-Louis Cauchy (1789–1857) who approached the implicit function theorem with mathematical rigor and it is he who is generally acknowledged as the discoverer of the theorem. In Chapter 2, we will give details of the contributions of Newton, Lagrange, and Cauchy to the development of the implicit function theorem.

The form of the implicit function theorem has evolved. The theorem first was formulated in terms of complex analysis and complex power series. As interest in, and understanding of, real analysis grew, the real-variable form of the theorem emerged. First the implicit function theorem was formulated for functions of two real variables, and the hypothesis corresponding to the Jacobian matrix being nonsingular was simply that one partial derivative was nonvanishing. Finally, Ulisse Dini (1845–1918) generalized the real-variable version of the implicit function theorem to the context of functions of any number of real variables. As mathematicians understood the theorem better, alternative proofs emerged, and the associated modern techniques have allowed a wealth of generalizations of the implicit function theorem to be developed.

Today we understand the implicit function theorem to be an *ansatz*, or a way of looking at problems. There are implicit function theorems, inverse function theorems, rank theorems, and many other variants. These theorems are valid on

Euclidean spaces, manifolds, Banach spaces, and even more general settings. Roughly speaking, the implicit function theorem is a device for solving equations, and these equations can live in many different settings.

In addition, the theorem is valid in many categories. The textbook formulation of the implicit function theorem is for C^k functions. But in fact the result is true for $C^{k,\alpha}$ functions, Lipschitz functions, real analytic functions, holomorphic functions, functions in Gevrey classes, and for many other classes as well. The literature is rather opaque when it comes to these important variants, and a part of the present work will be to set the record straight.

Certainly one of the most powerful forms of the implicit function theorem is that which is attributed to John Nash (1928–) and Jürgen Moser (1928–1999). This device is actually an infinite iteration scheme of implicit function theorems. It was first used by John Nash to prove his celebrated imbedding theorem for Riemannian manifolds. Jürgen Moser isolated the technique and turned it into a powerful tool that is now part of partial differential equations, functional analysis, several complex variables, and many other fields as well. This text will culminate with a version of the Nash–Moser theorem, complete with proof.

This book is one both of theory and practice. We intend to present a great many variants of the implicit function theorem, complete with proofs. Even the important implicit function theorem for real analytic functions is rather difficult to pry out of the literature. We intend this book to be a convenient reference for all such questions, but we also intend to provide a compendium of examples and of techniques. There are applications to algebra, differential geometry, manifold theory, differential topology, functional analysis, fixed point theory, partial differential equations, and to many other branches of mathematics. One learns mathematics (in part) by watching others do it. We hope to set a suitable example for those wishing to learn the implicit function theorem.

The book should be of interest to advanced undergraduates, graduate students, and professional mathematicians. Prerequisites are few. It is not necessary that the reader be already acquainted with the implicit function theorem. Indeed, the first chapter provides motivation and examples that should make clear the form and function of the implicit function theorem. A bit of knowledge of multivariable calculus will allow the reader to tackle the elementary proofs of the implicit function theorem given in Chapter 3. Rudiments of real and functional analysis are needed for the third proof in Chapter 3 which uses the Contraction Mapping Fixed Point Principle. Some knowledge of complex analysis is required for a complete reading of the historical material—this seems to be unavoidable since the earliest rigorous work on the implicit function theorem was formulated in the context of complex variables. In many cases a willing suspension of disbelief and a bit of determination will serve as a thorough grounding in the basics.

There are many sophisticated applications of implicit function theorems, particularly the Nash–Moser theorem, in modern mathematics. The imbedding theorem for Riemannian manifolds, the imbedding theorem for CR manifolds, and the deformation theory of complex structures are just a few of them. Richard Hamilton's masterful survey paper (see the Bibliography) indicates several more applications from different parts of mathematics. While each of these is a lovely *tour de force* of modern analytical technique, it is also the case that each requires considerable technical background. In order to keep the present volume as self-contained as possible, we have decided not to include any of these modern applications; instead we have provided exclusively classical applications of the implicit function theorem. For a basic book on the subject, we have found this choice to be most propitious.

We intend this book to be a useful resource for scientists of all types. We have exerted a considerable effort to make the bibliography extensive (if not complete). Therefore topics that can only be touched on here can be amplified with further reading. Although there are no formal exercises, the extensive remarks provide grist for further thought and calculation. We trust that our exposition will imbue our readers with some of the same fascination that led to the writing of this book.

There are a number of people whom we are pleased to thank for their helpful comments and contributions: David Barrett, Michael Crandall, John P. D'Angelo, Gerald B. Folland, Judith Grabiner, Robert E. Greene, Lars Hörmander, Seth Howell, Kang-Tae Kim, Laszlo Lempert, Maurizio Letizia, Richard Rochberg, Walter Rudin, Steven Weintraub, Dean Wills, Hung-Hsi Wu. Robert Burckel cast his critical eye on every page of our manuscript and the result is a much cleaner and more accurate book. Librarian Barbara Luszczynska performed yeoman service in helping us to track down references. This book is better because of the friendly assistance of all these good people; but, of course, all remaining failings are the province of the authors.

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Introduction to the Implicit Function Theorem

1.1 Implicit Functions

To the beginning student of calculus, a function is given by an analytic expression such as

$$f(x) = x^3 + 2x^2 - x - 3, \tag{1.1}$$

$$g(y) = \sqrt{y^2 + 1}, \tag{1.2}$$

or

$$h(t) = \cos(2\pi t). \tag{1.3}$$

In fact, 250 years ago this was the approach taken by Léonard Euler (1707–1783) when he wrote (see Euler [EB 88]):

> *A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.*

Almost immediately, one finds the notion of "function as given by a formula" to be too limited for the purposes of calculus. For example, the locus of

$$y^5 + 16y - 32x^3 + 32x = 0 \tag{1.4}$$

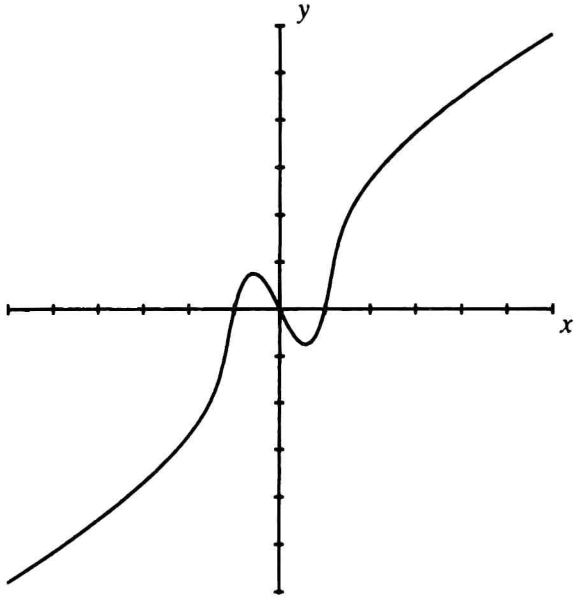

Figure 1.1. The Locus of Points Satisfying (1.4)

defines the nice subset of $\mathbf{R}^2$ that is sketched in Figure 1.1. The figure leads us to suspect that the locus is the graph of y as a function of x, but no formula for that function exists.

In contrast to the naive definition of functions as formulas, the modern, set-theoretic definition of a function is formulated in terms of the graph of the function. Precisely, a *function* with *domain* X and *codomain* or *range* Y is a subset, let us call it f, of the cartesian product

$$X \times Y = \{(x, y) : x \in X, \ y \in Y\}$$

having the properties that (i) for each $x \in X$ there is an element $(x, y) \in f$, and (ii) if $(x, y) \in f$ and $(x, \widetilde{y}) \in f$, then $y = \widetilde{y}$. In case these two properties hold, the choice of $x \in X$ determines the unique $y \in Y$ for which $(x, y) \in f$; because of this uniqueness, we find it a convenient shorthand to write

$$y = f(x)$$

to mean that $(x, y) \in f$.

Example 1.1.1 The locus defined by (1.4) has the property that, for each choice of $x \in \mathbf{R}$, there is a unique $y \in \mathbf{R}$ such that the pair (x, y) satisfies the equation. Thus there is a function, f, in the modern sense, such that the graph $y = f(x)$ is the locus of (1.4).

To confirm this assertion, we fix a value of $x \in \mathbf{R}$ and consider the left-hand side of (1.4) as a function of y alone. That is, we will examine the behavior of

$$F(y) = y^5 + 16y - 32x^3 + 32x$$

with x fixed.

Since the powers of y in $F(y)$ are odd, we have $\lim_{y\to-\infty} F(y) = -\infty$ and $\lim_{y\to+\infty} F(y) = +\infty$. Also we have

$$F'(y) = 5y^4 + 16 > 0\,,$$

so $F(y)$ is strictly increasing as y increases. By the intermediate value theorem, we see that $F(y)$ attains the value 0 for a unique value of y. That value of y is the value of $f(x)$ for the fixed value of x under consideration. □

Note that it is not clear from (1.4) by itself that y is a function of x. Only by doing the extra work in the example can we be certain that y really is uniquely defined as a function of x. Because it is not immediately clear from the defining equation that a function has been given, we say that the function is defined *implicitly* by (1.4). In contrast, when we see

$$y = f(x) \tag{1.5}$$

written, we then take it as a hypothesis that $f(x)$ is a function of x; no additional verification is required, even when in the right-hand side the function is simply a symbolic representation as in (1.5) rather than a formula as in (1.1), (1.2), and (1.3). To distinguish them from implicitly defined functions, the functions in (1.1), (1.2), (1.3), and (1.5) are called (in this book) *explicit* functions.

1.2 An Informal Version of the Implicit Function Theorem

Thinking heuristically, one usually expects that one equation in one variable

$$F(x) = c\,,$$

c a constant, will be sufficient to determine the value of x (though the existence of more than one, but only finitely many, solutions would come as no surprise).[1] When there are two variables, one expects that it will take two simultaneous equations

$$\begin{aligned} F(x, y) &= c\,, \\ G(x, y) &= d\,, \end{aligned}$$

[1] What we are doing is informally describing the notion of "degrees of freedom" that is commonly used in physics.

c and d constants, to determine the values of both x and y. In general, one expects that a system of m equations in m variables

$$
\begin{aligned}
F_1(x_1, x_2, \ldots, x_m) &= c_1, \\
F_2(x_1, x_2, \ldots, x_m) &= c_2, \\
&\vdots \\
F_m(x_1, x_2, \ldots, x_m) &= c_m,
\end{aligned}
\tag{1.6}
$$

$c_1, c_2, \ldots, c_m$ constants, will be just the right number of equations to determine the values of the variables. But of course we must beware of redundancies among the equations. That is, we must check that the system is nondegenerate—in the sense that a certain determinant does not vanish.

In case the equations in (1.6) are all linear, we can appeal to linear algebra to make our heuristic thinking precise (see any linear algebra textbook): A necessary and sufficient condition to guarantee that (1.6) has a unique solution for all values of the constants c_i is that the matrix of coefficients of the linear system has rank m.

We continue to think heuristically: If there are more variables than equations in our system of simultaneous equations, say

$$
\begin{aligned}
F_1(x_1, x_2, \ldots, x_n) &= c_1, \\
F_2(x_1, x_2, \ldots, x_n) &= c_2, \\
&\vdots \\
F_m(x_1, x_2, \ldots, x_n) &= c_m,
\end{aligned}
\tag{1.7}
$$

where the c's are still constants and where $n > m$, then we would hope to treat those $n - m$ extra variables as parameters, thereby forcing m of the variables to be implicit functions of the $n - m$ parameters. Again, in the case of linear functions, the situation is well understood: As long as the matrix of coefficients has rank m, it will be possible to express some set of m of the variables as functions of the other $n - m$ variables. Moreover, for any set of m independent columns of the matrix of coefficients of the linear system, the corresponding m variables can be expressed as functions of the other variables.

In the general case, as opposed to the linear case, the system of equations (1.7) defines a completely arbitrary subset of $\mathbf{R}^n$ (an arbitrary closed subset if the functions are continuous). Only under special conditions will (1.7) define m of the variables to be implicit functions of the other $n - m$ variables. It is the purpose of the implicit function theorem to provide us with a powerful method, or collection of methods, for insuring that we are in one of those special situations for which the heuristic argument is correct.

The implicit function theorem is grounded in differential calculus; and the bedrock of differential calculus is linear approximation. Accordingly, one works in a neighborhood of a point $(p_1, p_2, \ldots, p_n)$, where the equations in (1.7) all hold at $(p_1, p_2, \ldots, p_n)$ and where the functions in (1.7) can all be linearly approximated by their differentials. We are now in a position to state the implicit function theorem in informal terms (we shall give a more formal enunciation later):

> **(Informal) Implicit Function Theorem** *Let the functions in (1.7) be continuously differentiable. If (1.7) holds at* $(p_1, p_2, \ldots, p_n)$ *and if, when the functions in (1.7) are replaced by their linear approximations, a particular set of m variables can be expressed as functions of the other* $n - m$ *variables, then, for (1.7) itself, the same m variables can be defined to be implicit functions of the other* $n - m$ *variables in a neighborhood of* $(p_1, p_2, \ldots, p_n)$*. Additionally, the resulting implicit functions are continuously differentiable and their derivatives can be computed by implicit differentiation using the familiar method learned as part of the calculus.*

Let us look at a very simple example in which there is only one, well-understood, equation in two variables. We will treat this example in detail for the benefit of the reader who is not already comfortable with the ideas we have been discussing.

Example 1.2.1 Consider

$$x^2 + y^2 = 1 . \tag{1.8}$$

The locus of points defined by (1.8) is the circle of radius 1 centered at the origin. Of course, in a suitable neighborhood of any point $P = (p, q)$ satisfying (1.8) and for which $q \neq 0$, we can solve the equation to express y explicitly as

$$y = \pm\sqrt{1 - x^2} ,$$

where the choice of $+$ or $-$ is dictated by whether q is positive or negative. (Likewise, we could just as easily have dealt with the case in which $p \neq 0$ by solving for x as an explicit function of y.)

The usefulness of the implicit function theorem stems from the fact that we can avoid explicitly solving the equation. To take the point of view of the implicit function theorem, we linearly approximate the left-hand side of (1.8). In a neighborhood of a point $P = (p, q)$, a continuously differentiable function $F(x, y)$ is linearly approximated by

$$a\,\Delta x + b\,\Delta y + c ,$$

where a is the value of $\partial F/\partial x$ evaluated at P, Δx is the change in x made in going from $P = (p, q)$ to the point (x, y), b is the value of $\partial F/\partial y$ evaluated at P, Δy is the change in y made in going from $P = (p, q)$ to the point (x, y), and c is the value of F at P. In this example, $F(x, y) = x^2 + y^2$, the left-hand side of (1.8).

We compute

$$\frac{\partial}{\partial x}\left(x^2 + y^2\right)\Big|_{(x,y)=(p,q)} = 2p$$

and

$$\frac{\partial}{\partial y}\left(x^2+y^2\right)\bigg|_{(x,y)=(p,q)} = 2q \, .$$

Thus, in a neighborhood of the point $P = (p, q)$ which satisfies (1.8), the left-hand side of (1.8) is linearly approximated by

$$(2p)(x-p)+(2q)(y-q)+1 = 2px + 2qy - 1 \, .$$

When we replace the left-hand side of (1.8) by its linear approximation and simplify we obtain

$$px + qy = 1 \, , \tag{1.9}$$

which, of course, is the equation of the tangent line to the circle at the point P.

The implicit function theorem tells us that whenever we can solve the approximating linear equation (1.9) for y as a function of x, then the original equation (1.8) defines y implicitly as a function of x. Clearly, we can solve (1.9) for y as a function of x exactly when $q \neq 0$, so it is in this case that the implicit function theorem guarantees that (1.8) defines y as an implicit function of x. This agrees perfectly with what we found when we solved the equation explicitly. □

Remark 1.2.2 Looking at the circle, we see that it is impossible to use (1.8) to define y as a function of x in any open interval around $x = 1$ or in any open interval around $x = -1$. For other equations, an implicit function may happen to exist in a neighborhood of a point at which the implicit function theorem does not apply but, in such a case, the function may or may not be differentiable.

An example in which there are three variables and two equations will serve to illustrate the connection between linear algebra and the implicit function theorem.

Example 1.2.3 Fix $R \geq \sqrt{2}$ and consider the pair of equations

$$\begin{array}{rcl} x^2+y^2+z^2 & = & R^2 \, , \\ xy & = & 1 \end{array} \tag{1.10}$$

near the point $P = (1, 1, \rho)$, where $\rho = \sqrt{R^2 - 2}$.

We could solve the system explicitly. But it is instructive to instead take the point of view of the implicit function theorem. There are three variables and two equations, so the heuristic argument above tells us to expect two variables to be implicit functions of the third.

Computing partial derivatives and evaluating at $(1, 1, \rho)$ to linearly approximate the functions in (1.10), we obtain the equations

$$\begin{array}{rcl} x+y+\rho z & = & 2+\rho^2 \, , \\ x+y & = & 2 \, . \end{array} \tag{1.11}$$

This system of equations is the linearization of the original system. The first equation in (1.11) defines the tangent plane at P of the locus defined by the first equation in (1.10) and the second equation in (1.11) defines the tangent plane at the same point of the locus defined by the second equation in (1.10). Clearly, the two tangent planes have a non-trivial intersection because both automatically contain the point P.

The requirement that needs to be verified before the implicit function theorem can be applied is that we can solve the linear system (1.11) for two of the variables as a function of the third. Geometrically, this corresponds to showing that the intersection of the tangent planes is a line, because it is along a line in $\mathbf{R}^3$ that two of the variables can be expressed as a function of the third.

We now appeal to linear algebra. The matrix of coefficients for the linear system is

$$D = \begin{pmatrix} 1 & 1 & \rho \\ 1 & 1 & 0 \end{pmatrix}.$$

The necessary and sufficient condition for being able to solve (1.11) for two of the variables as a function of the third is that D have rank 2. Clearly, the rank of D is 2 if and only if $\rho \neq 0$. Thus, when $R > \sqrt{2}$, the implicit function theorem then guarantees that some pair of the variables can be defined implicitly in terms of the remaining variable.

On the other hand, when $\rho = 0$, or equivalently when $R = \sqrt{2}$, the rank of D is 1 and the implicit function theorem does *not* apply. Not only does the implicit function theorem not apply, but it is easy to see that $(1, 1, 0)$ and $(-1, -1, 0)$ are the only solutions of (1.10).

Assume now that $\rho \neq 0$. The implicit function theorem tells us that if we can solve the linear system (1.11) for a particular pair of the variables in terms of the third, then the original system of equations defines the same two variables as implicit functions of the third near $(1, 1, \rho)$. To determine which pairs of variables are functions of the third, we again appeal to linear algebra. Any two independent columns of D will correspond to variables in (1.11) that can be expressed as functions of the third. Thus, the implicit function theorem gives us the pair $x(y)$ and $z(y)$ satisfying (1.10), or the pair $y(x)$ and $z(x)$ satisfying (1.10).

In this example, not only does the implicit function theorem *not* allow us to assert the existence of $x(z)$ and $y(z)$ satisfying (1.10), but no such functions exist. □

1.3 The Implicit Function Theorem Paradigm

In the last section, we described the heuristic thinking behind the implicit function theorem and stated the theorem in informal terms. Even though the heuristic argument behind the result is rather simple, the implicit function theorem is a fundamental and powerful part of the foundation of modern mathematics. Originally conceived over two hundred years ago as a tool for studying celestial mechanics