Elements of Algebraic Graphs

代数图基础

by

Yanpei Liu

刘彦佩 著

University of Science and
Technology of China Press

中国科学技术大学出版社

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中国科学技术大学

校友文库

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内 容 简 介

本书以图的代数表示为起点，着重于多面形、曲面、嵌入和地图等对象，用一个统一的理论框架，揭示在更具普遍性的组合乃至代数构形中，可通过局部对称性反映全局性质．特别是通过多项式型的不变量刻画这些构形在不同拓扑、组合和代数变换下的分类．同时，也提供这些分类在算法上的实现和复杂性分析．虽然本书中的结论多以作者的前期工作为基础发展得到，但仍有一定数量的新结果．例如，关于图在给定亏格曲面上可嵌入性的识别，沿四个不同理论思路的判准就是新近得到的．在亏格为零的特殊情形下，从它们中的一个可一举导出 Euler、Whitney、MacLane 和 Lefschetz 在图的平面性方面沿不同理论路线的结果．

本书可供纯粹数学、应用数学、系统科学以及计算机科学等方面的大学生及相关教师使用，还可供相关专业研究生和数学研究人员阅读．

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Yanpei Liu

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总　　序

大学最重要的功能是向社会输送人才，培养高质量人才是高等教育发展的核心任务. 大学对于一个国家、民族乃至世界的重要性和贡献度，很大程度上是通过毕业生在社会各领域所取得的成就来体现的.

中国科学技术大学建校只有短短的五十余年，之所以迅速成为享有较高国际声誉的著名大学，主要就是因为她培养出了一大批德才兼备的优秀毕业生. 他们志向高远、基础扎实、综合素质高、创新能力强，在国内外科技、经济、教育等领域做出了杰出的贡献，为中国科大赢得了“科技英才的摇篮”的美誉.

2008 年 9 月，胡锦涛总书记为中国科大建校五十周年发来贺信，对我校办学成绩赞誉有加，明确指出：半个世纪以来，中国科学技术大学依托中国科学院，按照全院办校、所系结合的方针，弘扬红专并进、理实交融的校风，努力推进教学和科研工作的改革创新，为党和国家培养了一大批科技人才，取得了一系列具有世界先进水平的原创性科技成果，为推动我国科教事业发展和社会主义现代化建设做出了重要贡献.

为反映中国科大五十年来的人才培养成果，展示我校毕业生在科技前沿的研究中所取得的最新进展，学校在建校五十周年之际，决定编辑出版《中国科学技术大学校友文库》50 种. 选题及书稿经过多轮严格的评审和论证，入选书稿学术水平高，被列入“十一五”国家重点图书出版规划.

入选作者中，有北京初创时期的第一代学生，也有意气风发的少年班毕业生；有“两院”院士，也有中组部“千人计划”引进人才；有海内外科研院所、大专院校的教授，也有金融、IT 行业的英才；有默默奉献、矢志报国的科技将军，也有在国际前沿奋力拼搏的科研将才；有“文革”后留美学者中第一位担任美国大学系主任的青年教授，也有首批获得新中国博士学位的中年学者 …… 在母校五十周年华诞之际，他们通过著书立说的独特方式，向母校献礼，其深情厚谊，令

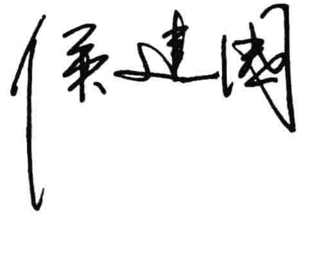
人感佩!

《文库》于 2008 年 9 月纪念建校五十周年之际陆续出版，现已出书 53 部，在学术界产生了很好的反响. 其中，《北京谱仪 II：正负电子物理》获得中国出版政府奖；中国物理学会每年面向海内外遴选 10 部“值得推荐的物理学新书”，2009 年和 2010 年，《文库》先后有 3 部专著入选；新闻出版总署总结“‘十一五’国家重点图书出版规划”科技类出版成果时，重点表彰了《文库》的 2 部著作；新华书店总店《新华书目报》也以一本书一个整版的篇幅，多期访谈《文库》作者. 此外，尚有十数种图书分别获得中国大学出版社协会、安徽省人民政府、华东地区大学出版社研究会等政府和行业协会的奖励.

这套发端于五十周年校庆之际的文库，能在两年的时间内形成现在的规模，并取得这样的成绩，凝聚了广大校友的智慧和对母校的感情. 学校决定，将《中国科学技术大学校友文库》作为广大校友集中发表创新成果的平台，长期出版. 此外，国家新闻出版总署已将该选题继续列为“十二五”国家重点图书出版规划，希望出版社认真做好编辑出版工作，打造我国高水平科技著作的品牌.

成绩属于过去，辉煌仍待新创. 中国科大的创办与发展，首要目标就是围绕国家战略需求，培养造就世界一流科学家和科技领军人才. 五十年来，我们一直遵循这一目标定位，积极探索科教紧密结合、培养创新拔尖人才的成功之路，取得了令人瞩目的成就，也受到社会各界的肯定. 在未来的发展中，我们依然要牢牢把握“育人是大学第一要务”的宗旨，在坚守优良传统的基础上，不断改革创新，进一步提高教育教学质量，努力践行严济慈老校长提出的“创寰宇学府，育天下英才”的使命.

是为序.



中国科学技术大学校长
中国科学院院士
第三世界科学院院士
2010 年 12 月

Preface

Graphs as a combinatoric topic was formed from Euler for a solution of Konigsberg Seven Bridge problem dated in 1736. Maps as a mathematical topic arose probably from the four color problem (see in Birkhoff(1913) and Ore(1967)) and the more general map coloring problem (see in Hilbert, Cohn-Vossen(1932), Ringel(1985) and Liu(1979)) in the mid of nineteenth century. I could not list even main references on them because it is well known for a large range of readers and beyond the scope of this book. Here, I only intend to present a comprehensive theory of maps and graphs as algebraic structures which has been developed mostly by myself only in recent few decades.

However, as described in the book Liu(2008), maps can be seen as from polyhedra in origin to graphs in development via abstraction. This is why algebraic graphs are much concerned with in the present stage.

In the beginning, maps in mathematics were as a topological, or geometric object even with geographical consideration in Kempe(1879). The first formal definition of a map was done by Heffter(1891) in the 19th century. However, it was not paid an attention to by mathematician until 1960 when Edmonds published a note in the AMS Notices with the dual form of Heffter's in Edmonds(1960) and Liu(1983).

Although this concept was widely used in literature as Liu(1979a; 1979b; 1994a; 1994b; 1995a), Ringel(1985; 1959; 1974), Stahl(2007; 1978), et al, its disadvantage for the nonorientable case involved does not bring with convenience for clarifying the related mathematical thinking.

Since Tutte described the nonorientability in a new way as in Tutte(1979; 1970; 1984), a number of authors begin to develop it in combinatorization of continuous objects in Little(1988), Liu(1995b; 1999; 2001; 2002), Vince(1983; 1995), et al.

The above representations are all with complication in constructing an embedding, or all distinct embeddings of a graph on a surface. However, the joint tree model of an embedding completed in recent years and initiated from the early articles at the end of seventies in the last century by the present author in Liu(1979a; 1979b) enables us to make the complication much simpler.

Because of the generality that in any asymmetric object there is some kind of local symmetry, the concepts of graphs and maps are just put in such a rule. In fact, the former is corresponding to that a group of two elements sticks on an edge and the latter is that a group of four elements sticks on an edge such that a graph without symmetry at all is in company with local symmetry. This treatment will bring more advantages for observing the structure of a graph. Of course, the latter is with restriction of the former because of the latter as a permutation and the former as a partition.

The joint tree representation of an embedding of a graph on 2-dimensional manifolds (or simply 2-manifolds), particularly surfaces (compact 2-manifolds without boundary in our case), is described in Liu(2009) for simplifying a number of results old and new.

This book contains the following subjects.

In Chapter 1, an abstract graph and its embedding on surfaces are much concerned because they are motivated to building up the theory of abstract graphs.

The second chapter is for the formal definition of abstract maps. One can see that this matter is a natural generalization of graph embedding on surfaces.

The third chapter is on the duality not only for maps themselves but also for operations on maps from one surface to another. One can see how naturally the duality is deduced from the abstract maps described in the second chapter.

The fourth chapter is on the orientability. One can see how formally the orientability is designed as a combinatorial invariant.

The fifth chapter concentrates on the classification of orientable maps. The sixth chapter is for the classification of nonorientable maps.

From the two chapters: Chapter 5 and Chapter 6, one can see how the procedure is simplified for these classifications.

The seventh chapter is on the isomorphisms of maps and provides an efficient algorithm for the justification and recognition of an isomorphism of two maps, which has been shown to be useful for determining the automorphism group of a map in the eighth chapter. Moreover, it enables us to access an automorphism of a graph much simply.

The ninth and the tenth chapters observe the number of distinct asymmetrized maps with the size as a parameter. In the former, only one vertex maps are counted by favorite formulas and in the latter, general maps are counted from differential equations. More progresses about this kind of counting are referred to read the recent book: Liu(1999) and many further articles: Baxter(2001), Bender et al(1996), Cai, Liu(2001; 1999), and Ren, Liu(2001a;

2001b; 2000), etc.

The next chapter, Chapter 11, only presents some ideas for accessing the symmetric census of maps and further, of graphs. This topic can be done on the basis of the relationship between maps and embeddings.

Chapter 12 describes in brief on genus polynomial of a graph and all its upper maps rooted and unrooted on the basis of the joint tree model. Recent progresses on this aspect are referred to read the articles: Chen, Liu(2006; 2007), Chen, Liu, Hao(2006), Hao, Liu(2004; 2008), Huang, Liu(2000; 2002), Li, Liu(2000), Mao, Liu(2004), Mao, Liu, Wei(2006), Wan, Liu(2005; 2006), Zhao, Liu(2004; 2006), etc.

Chapter 13 is on the census of maps with vertex or face partitions. Although such census involves with much complication and difficulty, because of the recent progress on a basic topic about trees via an elementary method firstly used by the author himself we are able to do a number of types of such census in very simple way. This chapter reflects on such aspects around.

Chapter 14 is on functional equations discovered in the census of a variety of maps on sphere and general surfaces. All of them have not yet been solved up to now.

The three chapters, i.e., Chapter 15–Chapter 17, are with much attention to graphs via relationship among polyhedra, embeddings and maps.

The last chapter, i.e., Chapter 18 is on surface embeddability of graphs. Four approaches are described. More notably, one of them turns out all the classic planarity theorems of Lefschetz (on double covering) in Lefschetz(1965), Whitney (on duality) in Whitney(1933) and MacLane (on cycle basis) in MacLane(1937) are much generalized and much simplified at a time.

Each chapter has a section of Notes in all of which more than 200 research problems difficult and accessible in certain extent are mentioned with some historical remarks.

Three appendices are complement to the context. One provides the clarification of the concepts of polyhedra, surfaces, embeddings, and maps and their relationship. The other two are for exhaustively calculating numerical results and listing all rooted and unrooted maps for small graphs.

Although I have been trying to design this book self contained as much as possible, some books such as Dixon, Mortimer(1996), Massey(1967) and Garey, Johnson(1979) might be helpful to those not familiar with basic knowledge of permutation groups, topology and computing complexity as background.

Since early nineties of the last century, a number of my former and present graduates were or are engaged with topics related to this book. Among them, I have to mention Dr. Y. Liu, Dr. Y. Q. Huang, Dr. J. L. Cai, Dr. D. M. Li, Dr. H. Ren, Dr. R. X. Hao, Dr. Z. X. Li, Dr. L. F. Mao, Dr. E. L. Wei, Dr.

W. L. He, Dr. L. X. Wan, Dr. Y. C. Chen, Dr. Y. Xu, Dr. W. Z. Liu, Dr. Z. L. Shao, Dr. Y. Yang, Dr. G. H. Dong, Dr. J. C. Zeng, Dr. S. X. Lv, Ms. X. M. Zhao, Mr. L. F. Li, Ms. H. Y. Wang, Ms. Z. Chai, Mr. Z. L. Zhu, et al for their successful work on this aspect.

On this occasion, I should express my heartiest appreciation of the financial support by KOSEF of Korea from the Com^{2}MaC (Combinatorial and Computational Mathematics Research Center) of the Pohang University of Science and Technology in the summer of 2001. In that period, the intention of this book was established. Moreover, I should be also appreciated to the Natural Science Foundation of China for the research development reflected in this book under its grants (60373030, 10571013 and 10871021).

Y. P. Liu
Beijing, China
May, 2012

Contents

Chapter 1

Abstract Graphs

- A graph is considered as a partition on the union of sets obtained from each element of a given set the binary group $B = \{0, 1\}$ sticks on.
- A surface, i.e., a compact 2-manifold without boundary in topology, is seen as a polygon of even edges pairwise identified.
- An embedding of a graph on a surface is represented by a joint tree of the graph. A joint tree of a graph consists of a plane extended tree with labeled cotree semiedges. Two semiedges of a cotree edge have the same label as the cotree edge with a binary index. An extended tree is compounded of a spanning tree with cotree semiedges.
- Combinatorial properties of an embedding in abstraction are particularly discussed for the formal definition of a map.

1.1 Graphs and Networks

Let X be a finite set. For any $x \in X$, the binary group $B = \{0, 1\}$ *sticks on* x to obtain $Bx = \{x(0), x(1)\}$. $x(0)$ and $x(1)$ are called the *ends* of x, or Bx. If Bx is seen as an ordered set $\langle x(0), x(1)\rangle$, then $x(0)$ and $x(1)$ are, respectively, *initial* and *terminal* ends of x. Let

$$\mathcal{X} = \sum_{x \in X} Bx, \tag{1.1}$$

i.e., the disjoint union of all Bx ($x \in X$). $\mathcal{X}$ is called the *ground set.*

A (*directed*) *pregraph* is a *partition* Par= $\{P_1, P_2, \ldots\}$ of the ground set $\mathcal{X}$, i.e.,

$$\mathcal{X} = \sum_{i \geqslant 1} P_i. \tag{1.2}$$

Bx (or $\langle x(0), x(1)\rangle$), or simply denoted by x ($x \in X$) itself is called an (*arc*) *edge* and P_i ($i \geqslant 1$) a *node* or *vertex*.

A (directed) pregraph is written as $G = (V, E)$ where V =Par and

$$E = B(X) = \{Bx|x \in X\}(= \{\langle x(0), x(1)\rangle|x \in X\}).$$

If X is a finite set, the (directed) pregraph is called *finite*; otherwise, *infinite*. In this book, (*directed*) pregraphs are all finite.

If $X = \emptyset$, then the (directed) pregraph is said to be *empty* as well.

An edge (arc) is considered to have two semiedges each of them is incident with only one end (semiarcs with directions of one from the end and the other to the end). An edge (arc) is with two ends identified is called a selfloop (di-selfloop); otherwise, a link (di-link). If t edges (arcs) have same ends (same direction) are called a multiedge (multiarc), or t-edge (t-arc).

Example 1.1 There are two directed pregraphs on $X = \{x\}$, i.e.,

$$\text{Par}_1 = \{\{x(0)\}, \{x(1)\}\}, \quad \text{Par}_2 = \{\{x(0), x(1)\}\}.$$

They are all distinct pregraphs as well as shown in Fig. 1.1.

Fig. 1.1 Directed Pregraphs of 1 edge

Further, pregraphs of size 2 are observed.

Example 1.2 On $X = \{x_1, x_2\}$, the 15 directed pregraphs are as follows:

$$\text{Par}_1 = \{\{x_1(0)\}, \{x_1(1)\}, \{x_2(0)\}, \{x_2(1)\}\},$$
$$\text{Par}_2 = \{\{x_1(0), x_1(1)\}, \{x_2(0)\}, \{x_2(1)\}\},$$
$$\text{Par}_3 = \{\{x_1(0), x_2(0)\}, \{x_1(1)\}, \{x_2(1)\}\},$$
$$\text{Par}_4 = \{\{x_1(0), x_2(1)\}, \{x_1(1)\}, \{x_2(0)\}\},$$
$$\text{Par}_5 = \{\{x_1(0)\}, \{x_1(1), x_2(0)\}, \{x_2(1)\}\},$$
$$\text{Par}_6 = \{\{x_1(0)\}, \{x_1(1), x_2(1)\}, \{x_2(0)\}\},$$
$$\text{Par}_7 = \{\{x_1(0)\}, \{x_1(1)\}, \{x_2(1), x_2(0)\}\},$$

$$\begin{aligned}
\mathrm{Par}_8 &= \{\{x_1(0), x_1(1), x_2(0)\}, \{x_2(1)\}\},\\
\mathrm{Par}_9 &= \{\{x_1(0), x_1(1), x_2(1)\}, \{x_2(0)\}\},\\
\mathrm{Par}_{10} &= \{\{x_1(0), x_2(0), x_2(1)\}, \{x_1(1)\}\},\\
\mathrm{Par}_{11} &= \{\{x_1(0)\}, \{x_1(1), x_2(0), x_2(1)\}\},\\
\mathrm{Par}_{12} &= \{\{x_1(0), x_1(1), x_2(0), x_2(1)\}\},\\
\mathrm{Par}_{13} &= \{\{x_1(0), x_1(1)\}, \{x_2(0), x_2(1)\}\},\\
\mathrm{Par}_{14} &= \{\{x_1(0), x_2(0)\}, \{x_1(1), x_2(1)\}\},\\
\mathrm{Par}_{15} &= \{\{x_1(0), x_2(1)\}, \{x_1(1), x_2(0)\}\}.
\end{aligned}$$

Among the 15 directed pregraphs, Par_3, Par_4, Par_5 and Par_6 are 1 pregraph; Par_8 and Par_9 are 1 pregraph; Par_{10} and Par_{11} are 1 pregraph; Par_{14} and Par_{15} are 1 pregraph; and others are 5 pregraphs. Thus, there are 9 pregraphs in all (as shown in Fig. 1.2).

Now, Par= $\{P_1, P_2, \ldots\}$ and $\mathcal{B}$ are, respectively, seen as a mapping $z \mapsto P_i$ ($z \in P_i$, $i \geqslant 1$) and a mapping $z \mapsto \bar{z}(\bar{z} \neq z)$, $\{z, \bar{z}\} \in B(X)$. The *composition* of two mappings α and β on a set $\mathcal{Z}$ is defined to be the mapping

$$(\alpha\beta)z = \bigcup_{y \in \beta z} \alpha y \quad (z \in \mathcal{Z}). \tag{1.3}$$

Let $\Psi_{\{\mathrm{Par},\ \mathcal{B}\}}$ be the semigroup generated by Par=Par(X) and $\mathcal{B} = B(X)$. Since the mappings α =Par and $\mathcal{B}$ have the property that $y \in \alpha z \Leftrightarrow z \in \alpha y$, it can be checked that for any $z, y \in B(X)$, what is determined by

$$\exists \gamma \in \Psi_{\{\mathrm{Par},\ \mathcal{B}\}}, \quad z \in \gamma y$$

is an equivalence. If $B(X)$ itself is an equivalent class, then the semigroup $\Psi_{\{\mathrm{Par},\ \mathcal{B}\}}$ is called *transitive* on $\mathcal{X} = B(X)$. A (directed) pregraph with $\Psi_{\{\mathrm{Par},\ \mathcal{B}\}}$ transitive on $\mathcal{X}$ is called a (*directed*) *graph*.

A (directed) pregraph $G = (V, E)$ that for any two vertices $u, v \in V$, there exists a sequence of edges $e_1, e_2, \ldots, e_s$ for the two ends of e_i ($i = 2, 3, \ldots, s-1$) are in common with those of respective e_{i-1} and e_{i+1} where u and v are, respectively, the other ends of e_1 and e_s, is called *connected* . Such a sequence of edges is called a *trail* between u and v. A trail without edge repetition is a *walk*. A walk without vertex repetition is a *path*. A trail, walk, or path with $u = v$ is, respectively, a *travel*, *tour*, or *circuit*.

Theorem 1.1 A (directed) pregraph is a (directed) graph if, and only if, it is connected.

Proof Necessity. Since $\mathrm{Par}^k = \mathrm{Par}$ ($k \geqslant 1$), and $\mathcal{B}^k = \mathcal{B}$ ($k \geqslant 1$), by the transitivity, for any two elements $y, z \in \mathcal{X}$, there exists γ such that $z \in \gamma y$