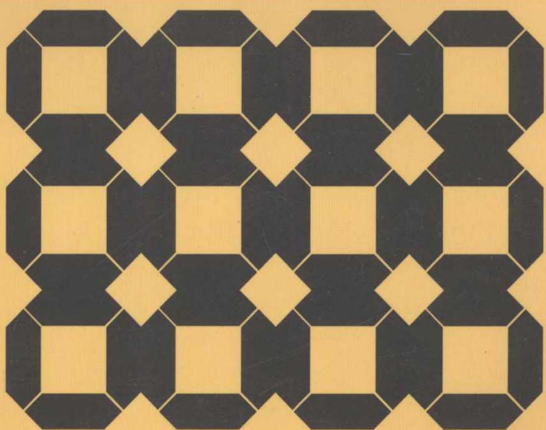
Undergraduate Texts in Mathematics

M. A. Armstrong

Groups and Symmetry

群与对称



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M.A. Armstrong

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(continued after index)

For Jerome and Emily

The beauty of a snow crystal depends on its mathematical regularity and symmetry, but somehow the association of many variants of a single type, all related but no two the same, vastly increases our pleasure and admiration.

D'ARCY THOMPSON
(*On Growth and Form*, Cambridge, 1917.)

En général je crois que les seules structures mathématiques intéressantes, dotées d'une certaine légitimité, sont celles ayant une réalisation naturelle dans le continu.... Du reste, cela se voit très bien dans des théories purement algébriques comme la théorie des groupes abstraits ou on a des groupes plus ou moins étranges apparaissant comme des groupes d'automorphismes de figures continues.

RENÉ THOM
(*Paraboles et Catastrophes*, Flammarion, 1983.)

Preface

Numbers measure size, *groups measure symmetry*. The first statement comes as no surprise; after all, that is what numbers "are for". The second will be exploited here in an attempt to introduce the vocabulary and some of the highlights of elementary group theory.

A word about content and style seems appropriate. In this volume, the emphasis is on *examples* throughout, with a weighting towards the symmetry groups of solids and patterns. Almost all the topics have been chosen so as to show groups in their most natural role, acting on (or permuting) the members of a set, whether it be the diagonals of a cube, the edges of a tree, or even some collection of subgroups of the given group. The material is divided into twenty-eight short chapters, each of which introduces a new result or idea. A glance at the Contents will show that most of the mainstays of a "first course" are here. The theorems of Lagrange, Cauchy, and Sylow all have a chapter to themselves, as do the classification of finitely generated abelian groups, the enumeration of the finite rotation groups and the plane crystallographic groups, and the Nielsen–Schreier theorem.

I have tried to be informal wherever possible, listing only significant results as theorems and avoiding endless lists of definitions. My aim has been to write a book which can be read with or without the support of a course of lectures. It is not designed for use as a dictionary or handbook, though new concepts are shown in bold type and are easily found in the index. Every chapter ends with a collection of exercises designed to consolidate, and in some cases fill out, the main text. It is essential to work through as many of these as possible before moving from one chapter to the next. Mathematics is not for spectators; to gain in understanding, confidence, and enthusiasm one has to participate.

As prerequisites I assume a first course in linear algebra (including matrix multiplication and the representation of linear maps between Euclidean spaces by matrices, though not the abstract theory of vector spaces) plus familiarity with the basic properties of the real and complex numbers. It would seem a pity to teach group theory without matrix groups available as a rich source of examples, especially since matrices are so heavily used in applications.

Elementary material of this type is all common stock, nevertheless it is not static, and improvements are made from time to time. Three such should be mentioned here: H. Wielandt's approach to the Sylow theorems (Chapter 20), James H. McKay's proof of Cauchy's theorem (Chapter 13), and the introduction of groups acting on trees by J.-P. Serre (Chapter 28). Another influence is of a more personal nature. As a student I had the good fortune to study with A.M. Macbeath, whose lectures first introduced me to group theory. The debt of gratitude from pupil to teacher is best paid in kind. If this little book can pass on something of the same appreciation of the beauty of mathematics as was shown to me, then I shall be more than satisfied.

Durham, England M.A.A.
September 1987

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My thanks go to Andrew Jobbings who read and commented on much of the manuscript, to Lyndon Woodward for many stimulating discussions over the years, to Mrs. S. Nesbitt for her good humour and patience whilst typing, and to the following publishers who have kindly permitted the use of previously published material: Cambridge University Press (quotation from *On Growth and Form*), Flammarion (quotation from *Paraboles et Catastrophes*), Dover Publications (Figure 2.1 taken from *Snow Crystals*), Office du Livre, Fribourg (Figure 25.3 and parts of Figure 26.2 taken from *Ornamental Design*), and Plenum Publishing Corporation (parts of Figure 26.2 taken from *Symmetry in Science and Art*).

Contents

CHAPTER 1

Symmetries of the Tetrahedron

How much symmetry has a tetrahedron? Consider a regular tetrahedron T and, for simplicity, think only of rotational symmetry. Figure 1.1 shows two axes. One, labelled L, passes through a vertex of the tetrahedron and through the centroid of the opposite face; the other, labelled M, is determined by the midpoints of a pair of opposite edges. There are four axes like L and two rotations about each of these, through $2\pi/3$ and $4\pi/3$, which send the tetrahedron to itself. The sense of the rotations is as shown: looking along the axis from the vertex in question the opposite face is rotated anticlockwise. Of course, rotating through $2\pi/3$ (or $4\pi/3$) in the opposite sense has the same effect on T as our rotation through $4\pi/3$ (respectively $2\pi/3$). As for axis M, all we can do is rotate through π, and there are three axes of this kind. So far we have $(4 \times 2) + 3 = 11$ symmetries. Throwing in the identity symmetry, which leaves T fixed and is equivalent to a full rotation through 2π about any of our axes, gives a total of twelve rotations.

We seem to have answered our original question. There are precisely twelve rotations, counting the identity, which move the tetrahedron onto itself. But this is not the end of the story. A flat hexagonal plate with equal sides also has twelve rotational symmetries (Fig. 1.2), as does a right regular pyramid on a twelve sided base (Fig. 1.3). For the plate we have five rotations (through $\pi/3$, $2\pi/3$, π, $4\pi/3$, and $5\pi/3$) about the axis perpendicular to it which passes through its centre of gravity. In addition there are three axes of symmetry determined by pairs of opposite corners, three determined by the midpoints of pairs of opposite sides, and we can rotate the plate through π about each of these. Not forgetting the identity, our total is again twelve. The pyramid has only one axis of rotational symmetry. It joins the apex of the pyramid to the centroid of its base, and there are twelve distinct rotations (through $k\pi/6$, $1 \leqslant k \leqslant 12$, in some chosen sense) about this axis. Despite the fact that we

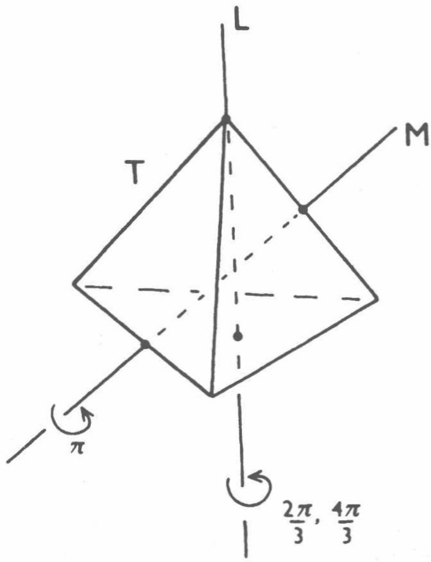

Figure 1.1

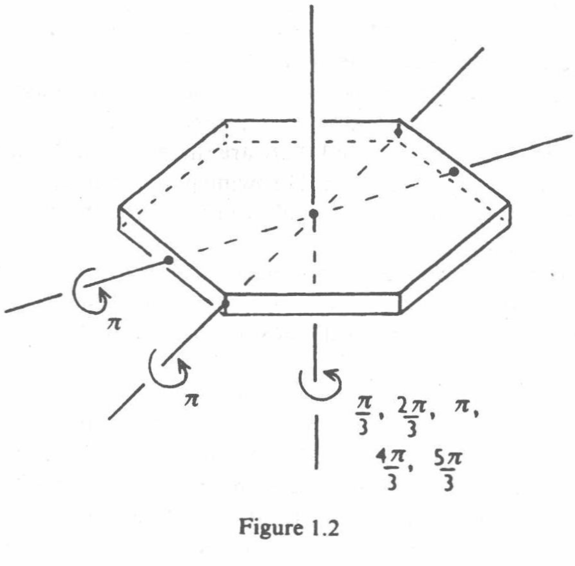

Figure 1.2

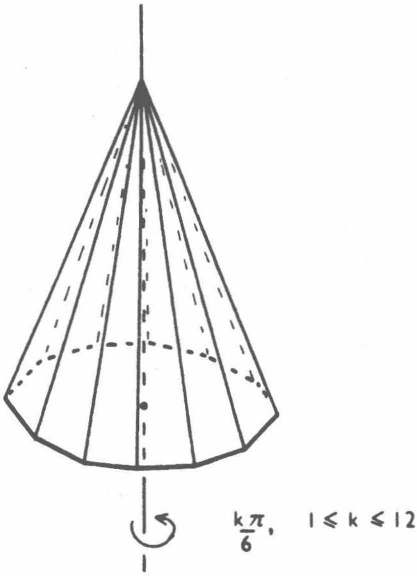

Figure 1.3

have counted twelve rotations in each case, the tetrahedron, the plate, and the pyramid quite clearly do not exhibit the same symmetry.

The most striking difference is that the pyramid possesses just one axis of symmetry. A rotation of $\pi/6$ about this axis has to be repeated (in other words, combined with itself) twelve times before the pyramid returns to its original position. Indeed, by suitable repetition of this basic rotation we can produce all the other eleven symmetries. However, no single rotation of the plate or the tetrahedron when repeated will give us all the other rotations.

If we look more carefully we can spot other differences, all of which have to do, in one way or another, with the way in which our symmetries combine. For example, the symmetries of the pyramid all *commute* with each other. That is to say, if we take any two and perform one rotation after the other, the effect on the pyramid is the same no matter which one we choose to do first. (These rotations all have the same axis, so if, for the sake of argument, we rotate through $\pi/3$ then through $5\pi/6$, we obtain rotation through $7\pi/6$, which is also the result of $5\pi/6$ first followed by $\pi/3$.) This is not the case for the tetrahedron or the plate. We recommend an experiment with the tetrahedron. Labelling the vertices of T as in Figure 1.4 enables us to see clearly the effect of a particular symmetry. Think of the rotations r ($2\pi/3$ about axis L in the sense indicated) and s (π about axis M). Performing first r then s takes vertex 2 back to its initial position and gives a rotation about axis N. But first s then r moves 2 to the place originally occupied by 4, and so cannot be the same rotation. Do

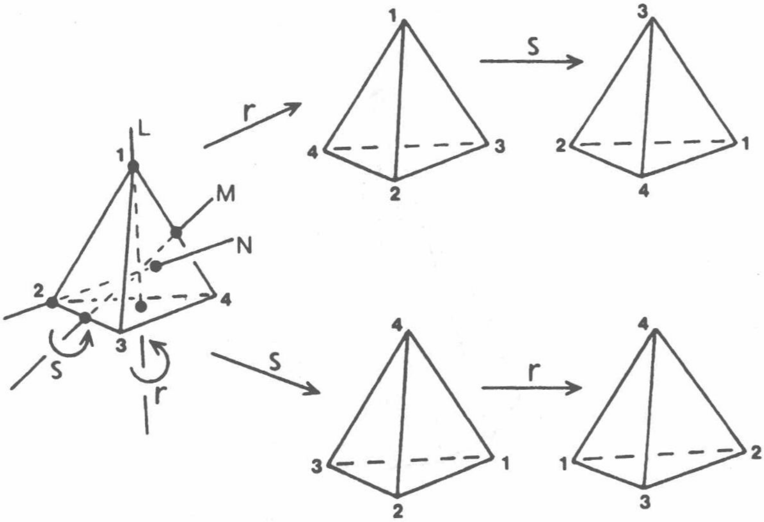

Figure 1.4

not fall into the trap of carrying the axis of s along with you as you do r first. Both r and s should be thought of as rigid motions of space, each of which has an axis that is *fixed* in space, and each of which rotates T onto itself.

Here is a third observation. There is only one rotation of the pyramid which, when combined once with itself, gives the identity; namely, the unique rotation through π. The plate has seven such symmetries and the tetrahedron three. These three rotations through π of the tetrahedron commute with one another, but only one of the seven belonging to the plate commutes with all the other six. Which one? Experiment until you find out.

To obtain a decent measure of symmetry, simply counting symmetries is not enough; we must also take into consideration how they combine with each other. It is the so-called symmetry group which captures this information and which we now attempt to describe.

The set of rotational symmetries of T has a certain amount of "algebraic structure". Given two rotations u and v we can *combine* them, by first doing v, then doing u, to produce a new rotation which also takes T to itself, and which we write uv. (Our choice of uv rather than vu is influenced by the convention for the composition of two functions, where fg usually means first apply g, then apply f.) The *identity* rotation, which we denote by e, behaves in a rather special way. Applying first e then another rotation u, or first u then e, always gives the same result as just applying u. In other words $ue = u$ and $eu = u$ for every symmetry u of T. Each rotation u has a so-called *inverse* u^{-1}, which is also a symmetry of T and which satisfies $u^{-1}u = e$ and $uu^{-1} = e$. To obtain u^{-1}, just rotate about the same axis and through the same angle as for u, but in the opposite sense. (For example, the inverse of the rotation r is rr, because applying r three times gives the identity.) Finally, if we take three of our rotations u,v, and w, it does not matter whether we first do w then the composite rotation uv, or whether we apply vw first and then u. In symbols this reduces to $(uv)w = u(vw)$ for any three (not necessarily distinct) symmetries of T.

The twelve symmetries of the tetrahedron together with this algebraic structure make up its rotational symmetry group.

Exercises

1.1. Glue two copies of a regular tetrahedron together so that they have a triangular face in common, and work out all the rotational symmetries of this new solid.

1.2. Find all the rotational symmetries of a cube.

1.3. Adopt the notation of Figure 1.4. Show that the axis of the composite rotation srs passes through vertex 4, and that the axis of $rsrr$ is determined by the midpoints of edges 12 and 34.

1.4. Having completed the previous exercise, express each of the twelve rotational symmetries of the tetrahedron in terms of r and s.

1.5. Again with the notation of Figure 1.4, check that $r^{-1} = rr$, $s^{-1} = s$, $(rs)^{-1} = srr$, and $(sr)^{-1} = rrs$.

1.6. Show that a regular tetrahedron has a total of twenty-four symmetries if reflections and products of reflections are allowed. Identify a symmetry which is not a rotation and not a reflection. Check that this symmetry is a product of three reflections.

1.7. Let q denote reflection of a regular tetrahedron in the plane determined by its centroid and one of its edges. Show that the rotational symmetries, together with those of the form uq, where u is a rotation, give all twenty-four symmetries of the tetrahedron.

1.8. Find all plane symmetries (rotations and reflections) of a regular pentagon and of a regular hexagon.

1.9. Show that the hexagonal plate of Figure 1.2 has twenty-four symmetries in all. Identify those symmetries which commute with all the others.

1.10. Make models of the octahedron, dodecahedron, and icosahedron (see Fig. 8.1). Try to spot as many symmetries of each of these solids as you can.

CHAPTER 2

Axioms

Without further ado we define the notion of a group, using the symmetries of the tetrahedron as guide. The first ingredient is a set. The second is a rule which allows us to combine any ordered pair x, y of elements from the set and obtain a unique "product" xy *which also lies in the set*. This rule is usually referred to as a "multiplication" on the given set.

*A **group** is a set G together with a multiplication on G which satisfies three axioms*:

(a) *The multiplication is associative, that is to say* $(xy)z = x(yz)$ *for any three (not necessarily distinct) elements from G.*
(b) *There is an element e in G, called an identity element, such that* $xe = x = ex$ *for every x in G.*
(c) *Each element x of G has a (so-called) inverse* x^{-1} *which belongs to the set G and satisfies* $x^{-1}x = e = xx^{-1}$.

How does a formal definition couched in terms of axioms help? So far not at all; indeed, if the only group turned out to be the rotational symmetry group of the tetrahedron, we would be wasting our time. But this is not the case; groups crop up in many different situations.

All of us take the additive group structure of the set of real numbers for granted. Here the rule for combining an ordered pair of numbers x, y is simply to add them to give $x + y$. We accept that $(x + y) + z = x + (y + z)$ for any three real numbers, there is an identity element, namely, zero, and $-x$ is clearly an inverse for the real number x. This example shows why we previously placed the words product and multiplication in quotation marks. The rule which enables us to combine our elements is invariably referred to as a multiplication, but may have nothing to do with multiplication of numbers in the usual sense.

A chemist may be interested in the amount of symmetry possessed by a particular molecule. Methane (CH_4), for example, can be thought of as having a carbon nucleus at the centroid of a regular tetrahedron, with four protons (hydrogen nuclei) arranged at the vertices. The benzene molecule (C_6H_6), on the other hand, is modelled by a hexagonal structure with a carbon and a hydrogen nucleus at each vertex. (Hexagonal symmetry is common in nature, perhaps nowhere more pleasing than in the structure of a snow crystal; see Fig. 2.1.) From our experience with the tetrahedron and the hexagon we know that it matters in which order we combine two symmetries. Hence, the continual reference to *ordered* pairs of elements. It matters whether we take two elements of a group in the order x, y or in the opposite order y, x. In the first case our rule gives the answer xy, in the second yx, and these two need not be equal.

A physicist learning relativity meets the Lorentz group, whose elements are matrices of the form

$$\begin{bmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{bmatrix} \qquad (*)$$

and which are combined via matrix multiplication. Remember that $\cosh u$, $\sinh u$ are the hyperbolic functions, so called because the equations $x = \cosh u$, $y = \sinh u$ determine the hyperbola $x^2 - y^2 = 1$. They satisfy

$$\cosh(u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v,$$

$$\sinh(u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v$$

consequently,

$$\begin{bmatrix} \cosh u & \sinh u \\ \sinh u & \cosh u \end{bmatrix}\begin{bmatrix} \cosh v & \sinh v \\ \sinh v & \cosh v \end{bmatrix} = \begin{bmatrix} \cosh(u+v) & \sinh(u+v) \\ \sinh(u+v) & \cosh(u+v) \end{bmatrix}$$

and this product does give a matrix *of the same form*. The identity matrix fulfils the requirements of an identity, and lies in the given set of matrices because it is equal to

$$\begin{bmatrix} \cosh 0 & \sinh 0 \\ \sinh 0 & \cosh 0 \end{bmatrix}$$

As an inverse for $(*)$ we can use

$$\begin{bmatrix} \cosh(-u) & \sinh(-u) \\ \sinh(-u) & \cosh(-u) \end{bmatrix}$$

which has the required form. Since matrix multiplication is associative, we have a group.

A mathematician thinking about Euclidean geometry finds he is studying those properties of figures which are left unchanged by the elements of a