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—— 几何及随机方法

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Geometric and Stochastic Methods

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Global Analysis in Mathematical Physics: geometric and stochastic methods
EISBN: 0-387-94867-8

图书在版编目(CIP)数据

数学物理中的全局分析：几何及随机方法＝Global Analysis in Mathematical Physics: geometric and stochastic methods：英文/(俄罗斯)格利克里克(Gliklikh Y.)著．—影印本．—北京：清华大学出版社，2005.1
(天元基金影印系列丛书)
ISBN 7-302-10201-5

Ⅰ．数…　Ⅱ．格…　Ⅲ．数学物理方法－英文　Ⅳ．O411.1

中国版本图书馆 CIP 数据核字(2004)第 139315 号

出 版 者：清华大学出版社　　地　　址：北京清华大学学研大厦
http://www.tup.com.cn　　邮　　编：100084
社 总 机：010-62770175　　客户服务：010-62776969
责任编辑：刘　颖
封面设计：常雪影
印 装 者：北京鑫海金澳胶印有限公司
发 行 者：新华书店总店北京发行所
开　　本：185×230　印张：14.25
版　　次：2005 年 1 月第 1 版　2005 年 11 月第 2 次印刷
书　　号：ISBN 7-302-10201-5/O·432
印　　数：2001～3000
定　　价：29.00 元

Preface to the English Edition

The first edition of this book entitled *Analysis on Riemannian Manifolds and Some Problems of Mathematical Physics* was published by Voronezh University Press in 1989. For its English edition, the book has been substantially revised and expanded. In particular, new material has been added to Sections 19 and 20. I am grateful to Viktor L. Ginzburg for his hard work on the translation and for writing Appendix F, and to Tomasz Zastawniak for his numerous suggestions. My special thanks go to the referee for his valuable remarks on the theory of stochastic processes. Finally, I would like to acknowledge the support of the AMS fSU Aid Fund and the International Science Foundation (Grant NZB000), which made possible my work on some of the new results included in the English edition of the book.

Voronezh, Russia — Yuri Gliklikh
September, 1995

Preface to the Russian Edition

The present book is apparently the first in monographic literature in which a common treatment is given to three areas of global analysis previously considered quite distant from each other, namely, differential geometry and classical mechanics, stochastic differential geometry and statistical and quantum mechanics, and infinite-dimensional differential geometry of groups of diffeomorphisms and hydrodynamics. The unification of these topics under the cover of one book appears, however, quite natural, since the exposition is based on a geometrically invariant form of the Newton equation and its analogs taken as a fundamental law of motion. Our approach, therefore, depends heavily on differential geometry of finite- or infinite-dimensional manifolds.

The monograph is addressed to mathematicians familiar with global analysis and requires certain mathematical culture as well as skills. We assume that the reader is familiar with manifolds, vector and principal bundles, tensors, and differential forms. This material can be found, for example, in the first section of Chap. 6 of [141] and in Chaps. 2 to 4 of [120]. Throughout the book, we also use the notions of connection, covariant derivative, and parallel translation as well as some basic results from differential geometry stated in a coordinate-free form. (See, e.g., [17] and [94].) For the sake of convenience, we recall some results from the theory of connections in Appendix A which, to some extent, can be regarded as a brief introduction to the subject.

Appendixes B to E are devoted to the theory of set-valued maps, theory of stochastic processes, the Itô group and principal bundle, and Sobolev spaces. Here our goal is, on the one hand, to give a necessary background and, on the other hand, to provide the reader with some additional information relevant to these topics but omitted in the main text.

Being limited by the size of a single book, we deliberately do not include the material covered in a variety of other textbooks and monographs. Thus, here we almost ignore Lagrangian and Hamiltonian systems and just briefly touch upon some other important subjects. The leading role in the selection of material was played, of course, by the author's taste and research interest.

The book consists of three parts, which correspond to the aforementioned branches of global analysis. In Chap. 1, we prove some general results concerned with analysis on Riemannian manifolds, which are used in the subsequent chapters and, to the best of our knowledge, have not yet been reviewed in monographs or textbooks. Here we include a necessary and sufficient condition for the completeness of a vector field, the basic construction of the integral operator with Riemannian parallel translation, etc.

In Chap. 2, we describe the differential-geometric approach to classical Newtonian mechanics. The Newton equation is introduced by means of the covariant derivative with respect to the Levi-Civitá connection of the Riemannian metric, giving rise to the kinetic energy on the configuration space. First, we deal with the classical cases of potential mechanical systems, a gyroscopic force, systems on Lie groups, etc. In addition to this, however, we study geometric mechanics of systems with discontinuous forces or forces with delay, geometric mechanics with linear constraints in the sense of Vershik–Faddeev, the integral equations of geometric mechanics (in terms of integral operators with Riemannian parallel translation), the velocity hodograph, etc. In a certain sense, the material of Chap. 2 is a starting point for generalizations studied in the subsequent chapters of the book.

Integral operators with Riemannian parallel translation are defined in Chap. 3 and then used to study the qualitative behavior of geometric mechanical systems. In particular, under a very general hypothesis, we prove that a pair of points (or a point and a submanifold) in a nonflat configuration space can be connected by a trajectory of the mechanical system. (This is true, for example, for a system with a discontinuous force, a system with a constraint, etc.) Note, however, that in contrast to the standard case of a flat configuration space, on a manifold we may have inaccessible points even when the configuration space is compact and the force field is smooth and bounded.

Stochastic differential geometry studied in Chap. 4, which opens the second part of the book, is a currently forming branch of global analysis. It is the discovery of a relationship between differential geometry and the theory of stochastic processes that has led to the invention of this geometrically invariant theory having important applications to mathematical physics. Our exposition is based on the Belopolskaya–Dalecky approach to defining Itô stochastic differential equations on manifolds and on a generalization to the stochastic case of integrals with Riemannian parallel translation introduced by the author. Taken into account that the theory of stochastic processes and differential geometry are customarily regarded as areas quite distant from each other and that this book is mainly addressed to experts in geometry and global analysis, we give a detailed review of the classical theory of stochastic processes in Sect. 12 (Chap. 4). In addition to this, some basic definitions from the theory can be found in Appendix C. Throughout the book all the constructions based on probability theory are accompanied by a number of references, yet major attention is paid to the geometric interpretation of the constructions.

Chapters 5 and 6 of Part II concern applications of stochastic differential geometry. In Chap. 5, we study the Langevin equation, which describes the "physical" Brownian motion on a nonlinear configuration space and then, in Chap. 6, Nelson's stochastic mechanics, known to be equivalent to quantum mechanics. We emphasize that the Langevin equation and the equation of motion of stochastic mechanics are essentially different generalizations of the Newton equation.

Part III is devoted to the weak differential geometry of groups of diffeomorphisms of a compact manifold and the modern geometric Lagrangian formalism of hydrodynamics proposed and developed by Arnold, Ebin, and Marsden. In Sect. 7, we study the properties of Sobolev H^s-diffeomorphisms of an n-dimensional compact manifold with $s > n/2 + 1$. The Lagrangian formalism arises naturally from the Newton equation on the group of such diffeomorphisms taken as the configuration space. The basic example of this construction is the so-called system of diffuse matter. The system of an ideal barotropic fluid can then be obtained from the diffuse matter system by introducing a particular force field. (See Sect. 24.) Similarly, the system of an ideal incompressible fluid, studied in Chap. 8, is described by means of a constraint on the configuration space. Since the kinetic energy is a weak Riemannian metric (i.e., it gives rise to the H^0-topology, which is weaker than H^s), many arguments of finite-dimensional differential geometry fail to apply to systems arising in hydrodynamics, and thus new methods are to be developed to deal with them. Note that the motion of an ideal incompressible fluid is given by a C^∞-smooth vector field on the tangent space to the manifold of diffeomorphisms. The passage to the Euler equation, also studied here, leads to the loss of derivatives. We prove the local (in time) existence and regularity of solutions for manifolds with or without boundary.

Methods of stochastic differential geometry are applied in Chap. 9 to describe the motion of a viscous incompressible fluid. Our model problem is to study the motion on the n-dimensional flat torus. We define a class of processes (analogous, in a certain sense, to geodesics) on the group of volume-preserving diffeomorphisms of the torus whose mathematical expectations are the curves on the group giving rise to the flow of the fluid. This approach is quite different from the ones used before in the Lagrange formalism of hydrodynamics.

A more detailed review of the contents is given in the introductions to the sections and chapters of the book.

Throughout the book, theorems, definitions, and formulas are designated by two numbers, where the first one refers to the section; subsections within a section are labeled by capital Roman letters. For example, Subsection A of Section 7 is referred to as Sect. 7.A.

Remarks play an important role in the book. Sometimes a remark gives some extra information, or it contains material (left without proof) to be referred to later on.

In the book, we use the standard terminology and notation of modern differential geometry. (See, e.g., [99] and [120].) Note, however, the following exception: derivatives of a change of coordinates given in local charts as $\phi_1 \circ \phi$ are denoted by primes, e.g., $(\phi_1 \circ \phi)'$ is the first derivative, $(\phi_1 \circ \phi)''$ the second, etc. Here $\phi_1 \circ \phi$ is understood as a map between domains of a Euclidean space and, for example, the second derivative is thought of as a bilinear map to the same space. (See, e.g., [99].) Tensors obtained by lifting or lowering indexes are said to be physically equivalent under the metric (see, [118]).

The list of references given in this book is by no means complete. The emphasis is put on textbooks or monographs, rather than original papers.

In conclusion, the author would like to express his deep gratitude to his former adviser Yuri G. Borisovich for his constant attention, support, and numerous fruitful discussions. I am in great debt to him for drawing my interest to the subject of this book. I am also grateful to Yuri S. Baranov, Boris D. Gel'dman, and Igor V. Fedorenko, my coauthors in a number of papers cited in this book, for their help and interest in our joint work. My special thanks go to Yana I. Belopolskaya, Yuri L. Dalecky, Vladimir Ya. Gershkovich, Aleksandr I. Shnirel'man, and Anatolii M. Vershik for our useful discussions and to my wife Olga for her infinite patience.

Voronezh, Russia — Yuri Gliklikh
April, 1989

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Part I
Finite-Dimensional Differential Geometry and Mechanics

This part consists of three chapters, the last two of which are devoted to studying classical mechanical systems with finitely many degrees of freedom. Here we use freely the notions of modern differential geometry, with which the reader is assumed to be familiar. (See the Preface.) A brief review of this material can be found in Appendix A. In Chap. 1 we deal with some applications, attributed to the author, of finite-dimensional differential geometry to the analysis on manifolds. These results will be used in the subsequent chapters of the book, in particular, in Chaps. 2 and 3.

Chapter 1. Some Geometric Constructions in Calculus on Manifolds

1. Complete Riemannian Metrics and the Completeness of Vector Fields

In this section, we shall study conditions that guarantee that all integral curves of a vector field on a finite-dimensional manifold exist on the interval $(-\infty, \infty)$. A vector field with this property is called complete. We shall give a criterion for completeness that uses special Riemannian metrics on the phase space.

1.A. A Necessary and Sufficient Condition for the Completeness of a Vector Field

Many criteria for the extendability to $(-\infty, \infty)$ of the solutions of differential equations in vector spaces are known (see, e.g., the Bibliography in [61]). It is easy to see that, under the hypotheses of some of these theorems, one can define a new Riemannian metric on the phase space in such a way that the right-hand side of the equation is bounded by a constant with respect to this metric. Thus, in these cases, the extendability of solutions (the completeness of a vector field) follows from the fact that a solution has bounded length on every finite interval with respect to a complete Riemannian metric and, therefore, is relatively compact.

It turns out that the requirement that the vector field should be bounded with respect to a complete Riemannian metric can be modified in such a way that it becomes necessary and sufficient.

Let M be a finite-dimensional smooth manifold and $X(t, m)$ a vector field which is smooth jointly in t and m. Denote the direct product $M \times \mathbb{R}$ by M^+. Obviously, $T_{(m,t)}M^+ = T_m M \times \mathbb{R}$. Define a vector field X^+ on M^+ setting $X^+_{(m,t)} = \big(X(m, t), 1\big)$.

Theorem 1.1 ([61]). *A field X on M is complete if and only if there exists a complete Riemannian metric on M^+ such that X^+ is uniformly bounded with respect to it.*

Proof. Obviously, the completeness of X is equivalent to the completeness of the vector field X^+.

Assume that there exists a complete Riemannian metric on M^+, with respect to which the field X^+ is bounded. Then every integral curve of X^+ has finite length on every finite interval. Since the metric is complete, the last assertion implies the relative compactness of the integral curve on every finite interval. This yields the completeness of the field.

Let us prove the "only if" assertion. Let X be complete, then so is X^+. Since the field X is smooth by the hypotheses of the theorem, the field X^+ is smooth as well. Consider an arbitrary smooth proper real-valued function g on the manifold M. We will call a map proper if the preimage of a compact set is compact, i.e., in our case, the preimage of a compact set in $\mathbb{R}$ under the map g is compact in M. The function g, satisfying the aforesaid conditions, can be constructed, for example, as follows (see [77]). Choose a countable covering of M by relatively compact open sets; this can be done by virtue of the paracompactness and the local compactness of M. Let us label the elements of the covering with integral numbers, and define a smooth constant function on each element of the covering to be equal to the number of the element. With the help of a partition of unity, glue these functions to obtain a function g on M, which satisfies the above conditions. Pick an inner product depending smoothly on (m,t) on each tangent space $T_{(m,t)}(M \times \{t\})$ to the submanifold $M \times \{t\}$ of the manifold M^+. For example, one can take a Riemannian metric on M and extend it in a natural way. Now we can construct a Riemannian metric $\langle\,,\rangle_1$ on M^+ by regarding the vectors of the field X^+ as being of unit length and orthogonal to the subspaces $T_{(m,t)}(M \times \{t\})$.

Denote by Φ_t the diffeomorphism of the manifold $M \times \{0\}$ to the manifold $M \times \{t\}$ along the trajectories of the field X^+. The function g can be regarded as given on $M \times \{0\}$. Since the integral curves of the field X^+ are globally extendable, the function $f\colon M^+ \to \mathbb{R}$, given by the formula

$$f(m,t) = g\left(\Phi_t^{-1}(m,t)\right) + t\ ,$$

is, obviously, smooth and proper. Clearly, $X^+f = 1$, where X^+f is the derivative of the function f in the direction of the field X^+.

Let us now choose an arbitrary smooth function $\phi\colon M^+ \to \mathbb{R}$ such that

$$\phi(m,t) > \max \exp(Yf)^2\ ,$$

where $Y \in T_{(m,t)}(M \times \{t\})$ and $\|Y\|_1 = 1$. Such a function can be defined as follows. For a relatively compact neighborhood of every point $(m',t') \in M^+$, there exists a constant greater than $\sup \max \exp(Yf)^2$, where, as above, $Y \in T_{(m,t)}(M \times \{t\})$ and $\|Y\|_1 = 1$, and the supremum is taken over all points (m,t) from the neighborhood. Then, using the paracompactness of M^+ and, as a consequence, the existence of a smooth partition of unity, we glue the function ϕ defined on the whole of M^+.

At every point $(m,t) \in M^+$, define the inner product on $T_{(m,t)}M^+$ by the formula

$$\langle Y, Z\rangle_2 = \phi^2(m,t)\,\langle p_m Y, p_m Z\rangle_1 + p_X Y \cdot p_Y X\ ,$$