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Erhan Çinlar

Probability and Stochastics

概率和随机

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Probability and Stochastics

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PREFACE

This is an introduction to the modern theory of probability and stochastic processes. The aim is to enable the student to have access to the many excellent research monographs in the literature. It might be regarded as an updated version of the textbooks by Breiman, Chung, and Neveu, just to name three.

The book is based on the lecture notes for a two-semester course which I have offered for many years. The course is fairly popular and attracts graduate students in engineering, economics, physics, and mathematics, and a few overachieving undergraduates. Most of the students had familiarity with elementary probability, but it was safer to introduce each concept carefully and in a uniform style.

As Martin Barlow put it once, mathematics attracts us because the need to memorize is minimal. So, only the more fundamental facts are labeled as theorems; they are worth memorizing. Most other results are put as propositions, comments, or exercises. Also put as exercises are results that can be understood only by doing the tedious work necessary. I believe in the Chinese proverb: I hear, I forget; I see, I remember; I do, I know.

I have been considerate: I do not assume that the reader will go through the book line by line from the beginning to the end. Some things are recalled or re-introduced when they are needed. In each chapter or section, the essential material is put first, technical material is put toward the end. Subheadings are used to introduce the subjects and results; the reader should have a quick overview by flipping the pages and reading the headings.

The style and coverage is geared toward the theory of stochastic processes, but with some attention to the applications. The reader will find many instances where the gist of the problem is introduced in practical, everyday language, and then is made precise in mathematical form. Conversely, many a theoretical point is re-stated in heuristic terms in order to develop the intuition and to provide some experience in stochastic modeling.

The first four chapters are on the classical probability theory: random variables, expectations, conditional expectations, independence, and the classical limit theorems. This is more or less the minimum required in a course at graduate level probability. There follow chapters on martingales, Poisson random measures, Lévy processes, Brownian motion, and Markov processes.

The first chapter is a review of measure and integration. The treatment is in tune with the modern literature on probability and stochastic processes. The second chapter introduces probability spaces as special measure spaces, but with an entirely different emotional effect; sigma-algebras are equated to bodies of information, and measurability to determinability by the given information. Chapter III is on convergence; it is routinely classical; it goes through the definitions of different modes of convergence, their connections to each other, and the classical limit theorems. Chapter IV is on conditional expectations as estimates given some information, as projection operators, and as Radon-Nikodym derivatives. Also in this chapter is the construction of probability spaces using conditional probabilities as the initial data.

Martingales are introduced in Chapter V in the form initiated by P.-A. Meyer, except that the treatment of continuous martingales seems to contain an improvement, achieved through the introduction of a "Doob martingale", a stopped martingale that is uniformly integrable. Also in this chapter are two great theorems: martingale characterization of Brownian motion due to Lévy and the martingale characterization of Poisson process due to Watanabe.

Poisson random measures are developed in Chapter VI with some care. The treatment is from the point of view of their uses in the study of point processes, discontinuous martingales, Markov processes with jumps, and, especially, of Lévy processes. As the modern theory pays more attention to processes with jumps, this chapter should fulfill an important need. Various uses of them occur in the remaining three chapters.

Chapter VII is on Lévy processes. They are treated as additive processes just as Lévy and Itô thought of them. Itô-Lévy decomposition is presented fully, by following Itô's method, thus laying bare the roles of Brownian motion and Poisson random measures in the structure of Lévy processes and, with a little extra thought, the structure of most Markov processes. Subordination of processes and the hitting times of subordinators are given extra attention.

Chapter VIII on Brownian motion is mostly on the standard material: hitting times, the maximum process, local times, and excursions. Poisson random measures are used to clarify the structure of local times and Itô's characterization of excursions. Also, Bessel processes and some other Markov processes related to Brownian motion are introduced; they help explain the recurrence properties of Brownian motion, and they become examples for the Markov processes to be introduced in the last chapter.

Chapter IX is the last, on Markov processes. Itô diffusions and jump-diffusions are introduced via stochastic integral equations, thus displaying the process as an integral path in a field of Lévy processes. For such processes, we derive the classical relationships between martingales, generators, resolvents, and transition functions, thus introducing the analytic theory of them. Then we re-introduce Markov processes in the modern setting and explain, for Hunt processes, the meaning and implications of the strong Markov property and quasi-left-continuity.

Over the years, I have acquired indebtedness to many students for their enthusiastic search for errors in the manuscript. In particular, Semih Sezer and Yury Polyanskiy were helpful with corrections and improved proofs. The manuscript was formatted by Emmanuel Sharef in his junior year, and Willie Wong typed the first six chapters during his junior and senior years. Siu-Tang Leung typed the seventh chapter, free of charge, out of sheer kindness. Evan Papageorgiou prepared the figures on Brownian motion and managed the latex files for me. Finally, Springer has shown much patience as I missed deadline after deadline, and the staff there did an excellent job with the production. Many thanks to all.

Frequently Used Notation

$\mathbb{N} = \{0, 1, \ldots\}$, $\overline{\mathbb{N}} = \{0, 1, \ldots, +\infty\}$, $\mathbb{N}^* = \{1, 2, \ldots\}$.

$\mathbb{R} = (-\infty, +\infty)$, $\overline{\mathbb{R}} = [-\infty, +\infty]$, $\mathbb{R}_+ = [0, \infty)$, $\overline{\mathbb{R}}_+ = [0, +\infty]$.

$(a,\ b)$ is the open interval with endpoints a and b; the closed version is $[a,\ b]$; the left-open right-closed version is $(a,\ b]$.

$\exp\ x = e^x$, $\exp_-\ x = e^{-x}$, Leb is the Lebesgue measure.

$\mathbb{R}^d$ is the d-dimensional Euclidean space, for x and y in it,

$$x \cdot y = x_1 y_1 + \cdots + x_d y_d, \quad |x| = \sqrt{x \cdot x} \quad .$$

$(E, \mathcal{E})$ denotes a measurable space, $\mathcal{E}$ is also the set of all $\mathcal{E}$-measurable functions from E into $\overline{\mathbb{R}}$, and $\mathcal{E}_+$ is the set of positive functions in $\mathcal{E}$.

$1_A(x) = \delta_x(A) = I(x, A)$ is equal to 1 if $x \in A$ and to 0 otherwise.

$\mathcal{B}_E$ is the Borel σ-algebra on E when E is topological.

$C(E \mapsto F)$ is the set of all continuous functions from E into F.

$\mathcal{C}_K^2 = C_K^2(\mathbb{R}^d \mapsto \mathbb{R})$ is the set of twice continuously differentiable functions, from $\mathbb{R}^d$ into $\mathbb{R}$, with compact support.

$\mathbb{E}(X|\mathcal{G})$ is the conditional expectation of X given the σ-algebra $\mathcal{G}$.

$\mathbb{E}_t X = \mathbb{E}(X|\mathcal{F}_t)$ when the filtration $(\mathcal{F}_t)$ is held fixed.

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Chapter I

Measure and Integration

This chapter is devoted to the basic notions of measurable spaces, measure, and integration. The coverage is limited to what probability theory requires as the entrance fee from its students. The presentation is in the form and style attuned to the modern treatments of probability theory and stochastic processes.

1 Measurable Spaces

Let E be a set. We use the usual notations for operations on subsets of E:

1.1 $$A \cup B, \quad A \cap B, \quad A \setminus B$$

denote, respectively, the union of A and B, the intersection of A and B, and the complement of B in A. In particular, $E \setminus B$ is called simply the complement of B and is also denoted by B^c. We write $A \subset B$ or $B \supset A$ to mean that A is a subset of B, that is, A is contained in B, or equivalently, B contains A. Note that $A = B$ if and only if $A \subset B$ and $A \supset B$. For an arbitrary collection $\{A_i : i \in I\}$ of subsets of E, we write

1.2 $$\bigcup_{i\in I} A_i, \qquad \bigcap_{i\in I} A_i$$

for the union and intersection, respectively, of all the sets A_i, $i \in I$.

The empty set is denoted by $\emptyset$. Sets A and B are said to be *disjoint* if $A \cap B = \emptyset$. A collection of sets is said to be *disjointed* if its every element is disjoint from every other. A countable disjointed collection of sets whose union is A is called a *partition* of A.

A collection $\mathcal{C}$ of subsets of E is said to be *closed under* intersections if $A \cap B$ belongs to $\mathcal{C}$ whenever A and B belong to $\mathcal{C}$. Of course, then, the

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intersection of every non-empty finite collection of sets in $\mathcal{C}$ is in $\mathcal{C}$. If the intersection of every countable collection of sets in $\mathcal{C}$ is in $\mathcal{C}$, then we say that $\mathcal{C}$ is closed under countable intersections. The notions of being closed under complements, unions, and countable unions, etc. are defined similarly.

Sigma-algebras

A non-empty collection $\mathcal{E}$ of subsets of E is called an *algebra* on E provided that it be closed under finite unions and complements. It is called a *σ-algebra* on E if it is closed under complements and countable unions, that is, if

1.3 a) $A \in \mathcal{E} \quad \Rightarrow \quad E \setminus A \in \mathcal{E}$,
b) $A_1, A_2, \ldots \in \mathcal{E} \quad \Rightarrow \quad \bigcup_n A_n \in \mathcal{E}$.

Since the intersection of a collection of sets is the complement of the union of the complements of those sets, a σ-algebra is also closed under countable intersections.

Every σ-algebra on E includes E and $\emptyset$ at least. Indeed, $\mathcal{E} = \{\emptyset, E\}$ is the simplest σ-algebra on E; it is called the *trivial σ-algebra*. The largest is the collection of all subsets of E, usually denoted by 2^E; it is called the *discrete σ-algebra* on E.

The intersection of an arbitrary (countable or uncountable) family of σ-algebras on E is again a σ-algebra on E. Given an arbitrary collection $\mathcal{C}$ of subsets of E, consider all the σ-algebras that contain $\mathcal{C}$ (there is at least one such σ-algebra, namely 2^E); take the intersection of all those σ-algebras; the result is the smallest σ-algebra that contains $\mathcal{C}$; it is called the σ-algebra *generated* by $\mathcal{C}$ and is denoted by $\sigma\mathcal{C}$.

If E is a topological space, then the σ-algebra generated by the collection of all open subsets of E is called the *Borel σ-algebra* on E; it is denoted by $\mathcal{B}_E$ or $\mathcal{B}(E)$; its elements are called *Borel sets*.

p-systems and d-systems

A collection $\mathcal{C}$ of subsets of E is called a p-system if it is closed under intersections; here, p is for product, the latter being an alternative term for intersection, and next, d is for Dynkin who introduced these systems into probability. A collection $\mathcal{D}$ of subsets of E is called a d-system on E if

1.4 a) $E \in \mathcal{D}$,
b) $A, B \in \mathcal{D}$ and $A \supset B \quad \Rightarrow A \setminus B \in \mathcal{D}$,
c) $(A_n) \subset \mathcal{D}$ and $A_n \nearrow A \quad \Rightarrow A \in \mathcal{D}$.

In the last line, we wrote $(A_n) \subset \mathcal{D}$ to mean that (A_n) is a sequence of elements of $\mathcal{D}$ and we wrote $A_n \nearrow A$ to mean that the sequence is increasing with limit A in the following sense:

1.5 $$A_1 \subset A_2 \subset \ldots, \quad \cup_n A_n = A.$$

It is obvious that a σ-algebra is both a p-system and a d-system, and the converse will be shown next. Thus, p-systems and d-systems are primitive structures whose superpositions yield σ-algebras.

1.6 PROPOSITION. *A collection of subsets of E is a σ-algebra if and only if it is both a p-system and a d-system on E.*

Proof. Necessity is obvious. To show the sufficiency, let $\mathcal{E}$ be a collection of subsets of E that is both a p-system and a d-system. First, $\mathcal{E}$ is closed under complements: $A \in \mathcal{E} \Rightarrow E \setminus A \in \mathcal{E}$, since $E \in \mathcal{E}$ and $A \subset E$ and $\mathcal{E}$ is a d-system. Second, it is closed under unions: $A, B \in \mathcal{E} \Rightarrow A \cup B \in \mathcal{E}$, because $A \cup B = (A^c \cap B^c)^c$ and $\mathcal{E}$ is closed under complements (as shown) and under intersections by the hypothesis that it is a p-system. Finally, this closure extends to countable unions: if $(A_n) \subset \mathcal{E}$, then $B_1 = A_1$ and $B_2 = A_1 \cup A_2$ and so on belong to $\mathcal{E}$ by the preceding step, and $B_n \nearrow \bigcup_n A_n$, which together imply that $\bigcup_n A_n \in \mathcal{E}$ since $\mathcal{E}$ is a d-system by hypothesis. □

The lemma next is in preparation for the main theorem of this section. Its proof is left as an exercise in checking the conditions 1.4 one by one.

1.7 LEMMA. *Let $\mathcal{D}$ be a d-system on E. Fix D in $\mathcal{D}$ and let*

$$\hat{\mathcal{D}} = \{A \in \mathcal{D} : \ A \cap D \in \mathcal{D}\}$$

Then, $\hat{\mathcal{D}}$ is again a d-system.

Monotone class theorem

This is a very useful tool for showing that certain collections are σ-algebras. We give it in the form found most useful in probability theory.

1.8 THEOREM. *If a d-system contains a p-system, then it contains also the σ-algebra generated by that p-system.*

Proof. Let $\mathcal{C}$ be a p-system. Let $\mathcal{D}$ be the smallest d-system on E that contains $\mathcal{C}$, that is, $\mathcal{D}$ is the intersection of all d-systems containing $\mathcal{C}$. The claim is that $\mathcal{D} \supset \sigma\mathcal{C}$. To show it, since $\sigma\mathcal{C}$ is the smallest σ-algebra containing $\mathcal{C}$, it is sufficient to show that $\mathcal{D}$ is a σ-algebra. In view of Proposition 1.6, it is thus enough to show that the d-system $\mathcal{D}$ is also a p-system.

To that end, fix B in $\mathcal{C}$ and let

$$\mathcal{D}_1 = \{A \in \mathcal{D} : \ A \cap B \in \mathcal{D}\}.$$

Since $\mathcal{C}$ is contained in $\mathcal{D}$, the set B is in $\mathcal{D}$; and Lemma 1.7 implies that $\mathcal{D}_1$ is a d-system. It also contains $\mathcal{C}$: if $A \in \mathcal{C}$ then $A \cap B \in \mathcal{C}$ since B is in $\mathcal{C}$ and $\mathcal{C}$ is a p-system. Hence, $\mathcal{D}_1$ must contain the smallest d-system containing $\mathcal{C}$, that is, $\mathcal{D}_1 \supset \mathcal{D}$. In other words, $A \cap B \in \mathcal{D}$ for every A in $\mathcal{D}$ and B in $\mathcal{C}$.

Consequently, for fixed A in $\mathcal{D}$, the collection

$$\mathcal{D}_2 = \{B \in \mathcal{D} : \ A \cap B \in \mathcal{D}\}$$

contains $\mathcal{C}$. By Lemma 1.7, $\mathcal{D}_2$ is a d-system. Thus, $\mathcal{D}_2$ must contain $\mathcal{D}$. In other words, $A \cap B \in \mathcal{D}$ whenever A and B are in $\mathcal{D}$, that is, $\mathcal{D}$ is a p-system. □

Measurable spaces

A *measurable space* is a pair $(E, \mathcal{E})$ where E is a set and $\mathcal{E}$ is a σ-algebra on E. Then, the elements of $\mathcal{E}$ are called *measurable sets*. When E is topological and $\mathcal{E} = \mathcal{B}_E$, the Borel σ-algebra on E, then measurable sets are also called *Borel sets*.

Products of measurable spaces

Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. For $A \subset E$ and $B \subset F$, we write $A \times B$ for the set of all pairs (x, y) with x in A and y in B; it is called the *product* of A and B. If $A \in \mathcal{E}$ and $B \in \mathcal{F}$, then $A \times B$ is said to be a *measurable rectangle*. We let $\mathcal{E} \otimes \mathcal{F}$ denote the σ-algebra on $E \times F$ generated by the collection of all measurable rectangles; it is called the *product σ-algebra*. The measurable space $(E \times F, \mathcal{E} \otimes \mathcal{F})$ is called the product of $(E, \mathcal{E})$ and $(F, \mathcal{F})$, and the notation $(E, \mathcal{E}) \times (F, \mathcal{F})$ is used as well.

Exercises

1.9 *Partition generated σ-algebras.*

a) Let $\mathcal{C} = \{A, B, C\}$ be a partition of E. List the elements of $\sigma\mathcal{C}$.

b) Let $\mathcal{C}$ be a (countable) partition of E. Show that every element of $\sigma\mathcal{C}$ is a countable union of elements taken from $\mathcal{C}$. Hint: Let $\mathcal{E}$ be the collection of all sets that are countable unions of elements taken from $\mathcal{C}$. Show that $\mathcal{E}$ is a σ-algebra, and argue that $\mathcal{E} = \sigma\mathcal{C}$.

c) Let $E = \mathbb{R}$, the set of all real numbers. Let $\mathcal{C}$ be the collection of all singleton subsets of $\mathbb{R}$, that is, each element of $\mathcal{C}$ is a set that consists of exactly one point in $\mathbb{R}$. Show that every element of $\sigma\mathcal{C}$ is either a countable set or the complement of a countable set. Incidentally, $\sigma\mathcal{C}$ is much smaller than $\mathcal{B}(\mathbb{R})$; for instance, the interval $(0, 1)$ belongs to the latter but not to the former.

1.10 *Comparisons.* Let $\mathcal{C}$ and $\mathcal{D}$ be two collections of subsets of E. Show the following:

a) If $\mathcal{C} \subset \mathcal{D}$ then $\sigma\mathcal{C} \subset \sigma\mathcal{D}$
b) If $\mathcal{C} \subset \sigma\mathcal{D}$ then $\sigma\mathcal{C} \subset \sigma\mathcal{D}$
c) If $\mathcal{C} \subset \sigma\mathcal{D}$ and $\mathcal{D} \subset \sigma\mathcal{C}$, then $\sigma\mathcal{C} = \sigma\mathcal{D}$
d) If $\mathcal{C} \subset \mathcal{D} \subset \sigma\mathcal{C}$, then $\sigma\mathcal{C} = \sigma\mathcal{D}$

1.11 *Borel σ-algebra on $\mathbb{R}$.* Every open subset of $\mathbb{R} = (-\infty, +\infty)$, the real line, is a countable union of open intervals. Use this fact to show that $\mathcal{B}_{\mathbb{R}}$ is generated by the collection of all open intervals.

1.12 *Continuation.* Show that every interval of $\mathbb{R}$ is a Borel set. In particular, $(-\infty, x)$, $(-\infty, x]$, $(x, y]$, $[x, y]$ are all Borel sets. For each x, the singleton $\{x\}$ is a Borel set.

1.13 *Continuation.* Show that $\mathcal{B}_{\mathbb{R}}$ is also generated by any one of the following (and many others):

a) The collection of all intervals of the form $(-\infty, x]$.
b) The collection of all intervals of the form $(x, y]$.
c) The collection of all intervals of the form $[x, y]$.
d) The collection of all intervals of the form (x, ∞).

Moreover, in each case, x and y can be limited to be rationals.

1.14 *Lemma 1.7.* Prove.

1.15 *Trace spaces.* Let $(E, \mathcal{E})$ be a measurable space. Fix $D \subset E$ and let

$$\mathcal{D} = \mathcal{E} \cap D = \{A \cap D :\ A \in \mathcal{E}\}.$$

Show that $\mathcal{D}$ is a σ-algebra on D. It is called the *trace* of $\mathcal{E}$ on D, and $(D, \mathcal{D})$ is called the *trace* of $(E, \mathcal{E})$ on D.

1.16 *Single point extensions.* Let $(E, \mathcal{E})$ be a measurable space, and let Δ be an extra point, not in E. Let $\bar{E} = E \cup \{\Delta\}$. Show that

$$\bar{\mathcal{E}} = \mathcal{E} \cup \{A \cup \{\Delta\} :\ A \in \mathcal{E}\}$$

is a σ-algebra on $\bar{E}$; it is the σ-algebra on $\bar{E}$ generated by $\mathcal{E}$.

1.17 *Product spaces.* Let $(E, \mathcal{E})$ and $(F, \mathcal{F})$ be measurable spaces. Show that the product σ-algebra $\mathcal{E} \otimes \mathcal{F}$ is also the σ-algebra generated by $\hat{\mathcal{E}} \cup \hat{\mathcal{F}}$, where

$$\hat{\mathcal{E}} = \{A \times F :\ A \in \mathcal{E}\}, \quad \hat{\mathcal{F}} = \{E \times B :\ B \in \mathcal{F}\}.$$

1.18 *Unions of σ-algebras.* Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be σ-algebras on the same set E. Their union is not a σ-algebra, except in some special cases. The σ-algebra generated by $\mathcal{E}_1 \cup \mathcal{E}_2$ is denoted by $\mathcal{E}_1 \vee \mathcal{E}_2$. More generally, if $\mathcal{E}_i$ is a σ-algebra on E for each i in some (countable or uncountable) index set I, then

$$\mathcal{E}_I = \bigvee_{i \in I} \mathcal{E}_i$$

denotes the σ-algebra generated by $\bigcup_{i \in I} \mathcal{E}_i$ (a similar notation for intersection is superfluous, since $\bigcap_{i \in I} \mathcal{E}_i$ is always a σ-algebra). Let $\mathcal{C}$ be the collection of all sets A having the form

$$A = \bigcap_{i \in J} A_i$$

for some finite subset J of I and sets A_i in $\mathcal{E}_i$, $i \in J$. Show that $\mathcal{C}$ contains all $\mathcal{E}_i$ and therefore $\bigcup_I \mathcal{E}_i$. Thus, $\mathcal{C}$ generates the σ-algebra $\mathcal{E}_I$. Show that $\mathcal{C}$ is a p-system.