Graduate Texts in Mathematics

Günter Ewald

Combinatorial Convexity and Algebraic Geometry

组合凸性和代数几何

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Combinatorial Convexity and Algebraic Geometry

With 130 Illustrations

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To Hanna
and our children
Daniel, Sarah, Anna, Esther, David

Preface

The aim of this book is to provide an introduction for students and nonspecialists to a fascinating relation between combinatorial geometry and algebraic geometry, as it has developed during the last two decades. This relation is known as the theory of toric varieties or sometimes as torus embeddings.

Chapters I–IV provide a self-contained introduction to the theory of convex polytopes and polyhedral sets and can be used independently of any applications to algebraic geometry. Chapter V forms a link between the first and second part of the book. Though its material belongs to combinatorial convexity, its definitions and theorems are motivated by toric varieties. Often they simply translate algebraic geometric facts into combinatorial language. Chapters VI–VIII introduce toric varieties in an elementary way, but one which may not, for specialists, be the most elegant.

In considering toric varieties, many of the general notions of algebraic geometry occur and they can be dealt with in a concrete way. Therefore, Part 2 of the book may also serve as an introduction to algebraic geometry and preparation for farther reaching texts about this field.

The prerequisites for both parts of the book are standard facts in linear algebra (including some facts on rings and fields) and calculus. Assuming those, all proofs in Chapters I–VII are complete with one exception (IV, Theorem 5.1). In Chapter VIII we use a few additional prerequisites with references from appropriate texts.

The book covers material for a one year graduate course. For shorter courses with emphasis on algebraic geometry, it is possible to start with Part 2 and use Part 1 as references for combinatorial geometry.

For each section of Chapters I–VIII, there is an addendum in the appendix of the book. In order to avoid interruptions and to minimize frustration for the beginner, comments, historical notes, suggestions for further reading, additional exercises, and, in some cases, research problems are collected in the Appendix.

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This text is based on lectures I gave several times at Bochum University. Many collegues and students have contributed to it in one way or another.

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Also Markus Eikelberg, Rolf Gärtner, Ralph Lehmann, and Uwe Wessels made important contributions. Michel Brion, Dimitrios Dais, Bernard Teissier, Günter Ziegler added remarks, and Hassan Azad, Katalin Bencsath, Peter Braß, Sharon Castillo, Reinhold Matmann, David Morgan, and Heinke Wagner made corrections to the text. Elke Lau and Elfriede Rahn did the word processing of the computer text.

I thank all who helped me, in particular, those who are not mentioned by name.

Günter Ewald

Introduction

Studying the complex zeros of a polynomial in several variables reveals that there are properties which depend not on the specific values of the coefficients but only on their being nonzero. They depend on the exponent vectors showing up in the polynomial or, more precisely, on the lattice polytope which is the convex hull of such vectors. This had already been discovered by Newton and was taken into consideration by Minding and some other mathematicians in the nineteenth century. However, it had practically been forgotten until its rediscovery around 1970, when Demazure, Oda, Mumford, and others developed the theory of toric varieties.

The starting point lay in algebraic groups. Properties of zeros of polynomials that depend only on the exponent vectors do not change if each coordinate of any solution is multiplied by a nonvanishing constant. Such transformations are effected by diagonal matrices with nonzero determinants. They form a group which can be represented by $\mathbb{C}^{*n}$ where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ is the multiplicative group of complex numbers. $\mathbb{C}^{*n}$ (for $n = 2$ having, topologically, an ordinary torus as a retract) is called an algebraic torus. Demazure succeeded in combinatorially characterizing those regular algebraic varieties on which a torus operates with an open orbit. Oda, Mumford, and others extended this to the nonregular case and termed the introduced varieties torus embeddings or toric varieties.

Once the combinatorial characterization had been achieved, it gave way to defining toric varieties without starting from algebraic groups by use of combinatorial concepts like lattice cones and the algebras defined by monoids of all lattice points in cones. This is the path we follow in the present book.

Toric varieties—being a class of relatively concrete algebraic varieties—may appear to relate combinatorics to old-fashioned, say, up to 1950, algebraic geometry. This is not the case. Actually, the more recent way of thought provides the tools for building a wide bridge between combinatorial and algebraic geometry. Notions like sheaves, blowups, or the use of homology in algebraic geometry are such tools.

In the first part of the book, we have naturally limited the topics to those which are needed in the second part. However, there was not much to be omitted. Coming from combinatorial convexity, it is quite a surprise how many of the traditional notions like support function or mixed volume now appear in a new light.

In our attempt to present a compact introduction to the theory of convex polytopes, we have sought short proofs. Also, a coordinate-free approach to Gale transforms seemed to fit particularly well into the needs of later applications. Similarly, in Part 2 we spent much energy on simplifications. Our definition of intersection numbers and a discussion of the Hodge inequality working without the tools of algebraic topology are some of the consequences.

A natural question concerning the relationship between combinatorial and algebraic geometry is "Does the algebraic geometric side benefit more from the combinatorial side than the combinatorial side does from the algebraic geometric one?" In this text the former is true. We prove algebraic geometric theorems from combinatorial geometric facts, "turning around" the methods often applied in the literature. There is only one exception in the very last section of the book. We quote a toric version of the Riemann–Roch–Hirzebruch theorem without proof and draw combinatorial conclusions from it. A purely combinatorial version of the theorem due to Morelli [1993a] would require more work on so-called polytope algebra.

Many related topics have been omitted, for example, matroid theory or the theory of Stanley–Reisner rings and their powerful combinatorial implications. The reader familiar with such topics may recognize their links to those covered here and detect the common spirit of mathematical development in all of them.

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Part 1

Combinatorial Convexity

I

Convex Bodies

1. Convex sets

Most of the sets considered in the first part of the book are subsets of Euclidean n-space. Many definitions and theorems could be stated in an affinely invariant manner. We do not, however, stress this point. If we use the symbol $\mathbb{R}^n$, it should be clear from the context whether we mean real vector space, real affine space, or Euclidean space. In the latter case, we assume the ordinary scalar product

$$\langle x, y\rangle = \xi_1\eta_1 + \cdots + \xi_n\eta_n \qquad \text{for } x = (\xi_1, \ldots, \xi_n), \quad y = (\eta_1, \ldots, \eta_n)$$

so that the square of Euclidean distance between points x and y equals

$$\|x - y\|^2 = \langle x - y, x - y\rangle.$$

Recall that an open ball with center x and radius r is the set $\{y \mid \|x - y\| < r\}$. By $\langle K, y\rangle \geq 0$, we mean $\langle x, y\rangle \geq 0$ for every $x \in K$. We assume the reader to be somewhat familiar with n-dimensional affine and Euclidean geometry.

1.1 Definition. A set $C \subset \mathbb{R}^n$ is called *convex* if, for all $x, y \in C, x \neq y$, the line segment

$$[x, y] := \{\lambda x + (1 - \lambda)y \mid 0 \leq \lambda \leq 1\}$$

is contained in C (Figure 1).

Examples of convex sets are a point, a line, a circular disc in $\mathbb{R}^2$, the platonic solids (see Figure 10 in section 6) in $\mathbb{R}^3$. Also $\emptyset$ and $\mathbb{R}^n$ are convex.

If B is an open circular disc in $\mathbb{R}^2$ and M is any subset of the boundary circle ∂B of B, then $B \cup M$ is also convex. So, a convex set need be neither open nor closed. In general we shall restrict ourselves to closed convex sets.

There is a simple way to construct new convex sets from given ones:

1.2 Lemma. *The intersection of an arbitrary collection of convex sets is convex.*

PROOF. If a line segment is contained in every set of the collection, it is also contained in their intersection. □