H. Blaine Lawson

Spin Geometry

自旋几何

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For Christie, Didi,
Michelle, and Heather

Preface

In the late 1920's the relentless march of ideas and discoveries had carried physics to a generally accepted relativistic theory of the electron. The physicist P.A.M. Dirac, however, was dissatisfied with the prevailing ideas and, somewhat in isolation, sought for a better formulation. By 1928 he succeeded in finding a theory which accorded with his own ideas and also fit most of the established principles of the time. Ultimately this theory proved to be one of the great intellectual achievements of the period. It was particularly remarkable for the internal beauty of its mathematical structure which not only clarified much previously mysterious phenomena but also predicted in a compelling way the existence of an electron-like particle of negative energy. Indeed such particles were subsequently found to exist and our understanding of nature was transformed.

Because of its compelling beauty and physical significance it is perhaps not surprising that the ideas at the heart of Dirac's theory have also been discovered to play a role of great importance in modern mathematics, particularly in the interrelations between topology, geometry and analysis. A great part of this new understanding comes from the work of M. Atiyah and I. Singer. It is their work and its implications which form the focus of this book.

It seems appropriate to sketch some of the fundamental ideas here. In searching for his theory, Dirac was faced, roughly speaking, with the problem of finding a Lorentz-invariant wave equation $D\psi = \lambda\psi$ compatible with the Klein-Gordon equation $\Box\psi = \lambda\psi$ where $\Box = (\partial/\partial x_0)^2 - (\partial/\partial x_1)^2 - (\partial/\partial x_2)^2 - (\partial/\partial x_3)^2$. Causality required that D be first order in the "time" coordinate x_0. Of course by Lorentz invariance there could be no preferred time coordinate, and so D was required to be first-order in all variables. Thus, in essence Dirac was looking for a first-order differential operator whose square was the laplacian. His solution was to replace the complex-valued wave function ψ with an n-tuple $\Psi = (\psi_1, \ldots, \psi_n)$ of such functions. The operator D then became a first-order system of the form

$$D = \sum_{\mu=0}^{3} \gamma_\mu \frac{\partial}{\partial x_\mu}$$

where $\gamma_0, \ldots, \gamma_3$ were $n \times n$-matrices. The requirement that

$$D^2 = \begin{pmatrix} \Box & & \\ & \ddots & \\ & & \Box \end{pmatrix}$$

led to the equations

$$\gamma_\nu \gamma_\mu + \gamma_\mu \gamma_\nu = \pm 2\delta_{\nu\mu}.$$

These were easily and explicitly solved for small values of n, and the analysis was underway.

This construction of Dirac has a curious and fundamental property. Lorentz transformations of the space-time variables $(x_0, \ldots, x_3)$ induce linear transformations of the n-tuples Ψ which are determined only up to a sign. Making a consistent choice of sign amounts to passing to a non-trivial 2-fold covering $\tilde{L}$ of the Lorentz group L. That is, in transforming the Ψ's one falls upon a representation of $\tilde{L}$ which does not descend to L.

The theory of Dirac had another interesting feature. In the presence of an electromagnetic field the Dirac Hamiltonian contained an additional term added on to what one might expect from the classical case. There were strong formal analogies with the additional term one obtains by introducing internal spin into the mechanical equations of an orbiting particle. This "spin" or internal magnetic moment had observable quantum effects. The n-tuples Ψ were thereby called *spinors* and this family of transformations was called the *spin representation.*

This physical theory touches upon an important and general fact concerning the orthogonal groups. (We shall restrict ourselves for the moment to the positive definite case.) In the theory of Cartan and Weyl the representations of the Lie algebra of SO_n are essentially generated by two basic ones. The first is the standard n-dimensional representation (and its exterior powers). The second is constructed from the representations of the algebra generated by the γ_μ's as above (the *Clifford algebra* associated to the quadratic form defining the orthogonal group). This second representation is called the spin representation. It does not come from a representation of the orthogonal group, but only of its universal covering group, called Spin_n. It plays a key role in an astounding variety of questions in geometry and topology: questions involving vector fields on spheres, immersions of manifolds, the integrality of certain characteristic numbers, triality in dimension eight, the existence of complex structures, the existence of metrics of positive scalar curvature, and perhaps most basically, the index of elliptic operators.

In the early 1960s general developments had led mathematicians to consider the problem of finding a topological formula for the index of any elliptic operator defined on a compact manifold. This formula was to generalize the important Hirzebruch-Riemann-Roch Theorem already established in the complex algebraic case. In considering the problem, Atiyah and Singer noted that among all manifolds, those whose SO_n-structure could be simplified to a $Spin_n$-structure had particularly suggestive properties. Realizing that over such spaces one could carry out the Dirac construction, they produced a globally defined elliptic operator canonically associated to the underlying riemannian metric. The index of this operator was a basic topological invariant called the $\hat{A}$-genus, which was known always to be an integer in this special class of spin manifolds. (It is not an integer in general.) Twisting the Dirac-type operator with arbitrary coefficient bundles led, with some sophistication, to a general formula for the index of any elliptic operator.

Atiyah and Singer went on to understand the index in the more proper setting of K-theory. This led in particular to the formulation of certain KO-invariants which have profound applications in geometry and topology. These invariants touch questions unapproachable by other means. Their study and elucidation was a principal motivation for the writing of this tract.

It is interesting to note in more recent years there has been another profound and beautiful physical theory whose ideas have come to the core of topology, geometry and analysis. This is the non-abelian gauge field theory of C. N. Yang and R. L. Mills which through the work of S. Donaldson and M. Freedman has led to astonishing results in dimension four. Yang-Mills theory can be plausibly considered a highly non-trivial generalization of Dirac's theory which encompasses three fundamental forces: the weak, strong, and electromagnetic interactions. This theory involves modern differential geometry in an essential way. The theory of connections, Dirac-type operators, and index theory all play an important role. We hope this book can serve as a modest introduction to some of these concepts.

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This book owes much to the fundamental work of Michael Atiyah and Iz Singer. Part of our initial motivation in writing the book was to give a leisurely and rounded presentation of their results.

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This author would like to take this opportunity to express her deep appreciation to Mark Mahowald who held out a hand when one was so dearly needed.

M.-L. MICHELSOHN

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Spin Geometry

Introduction

Over the past two decades the geometry of spin manifolds and Dirac operators, and the various associated index theorems have come to play an increasingly important role both in mathematics and in mathematical physics. In the area of differential geometry and topology they have become fundamental. Topics like spin cobordism, previously considered exotic even by topologists, are now known to play an essential role in such classical questions as the existence or non-existence of metrics of positive curvature. Indeed, the profound methods introduced into geometry by Atiyah, Bott, Singer and others are now indispensible to mathematicians working in the field. It is the intent of this book to set out the fundamental concepts and to present these methods and results in a unified way.

A principal theme of the exposition here is the consistent use of Clifford algebras and their representations. This reflects the observed fact that these algebras emerge repeatedly at the very core of an astonishing variety of problems in geometry and topology.

Even in discussing riemannian geometry, the formalism of Clifford multiplication will be used in place of the more conventional exterior tensor calculus. There is a philosophical justification for this bias. Recall that to any vector space V there is naturally associated the exterior algebra $\Lambda^* V$, and this association carries over directly to vector bundles. Applied to the tangent bundle of a smooth manifold, it gives the de Rham bundle of exterior differential forms. In a similar way, to any vector space V equipped with a quadratic form q, there is associated the Clifford algebra $C\ell(V, q)$, and this association carries over directly to vector bundles equipped with fibre metrics. In particular, applied to the tangent bundle of a smooth riemannian manifold, it gives a canonically associated bundle of algebras, called the Clifford bundle. As a vector bundle it is isomorphic to the bundle of exterior forms. However, the Clifford multiplication is strictly richer than exterior multiplication; it reflects the inner symmetries and basic identities of the riemannian structure. In fact fundamental curvature identities will be derived here in the formalism of Clifford multiplication and applied to some basic problems.

Another justification for our approach is that the Clifford formalism gives a transparent unification of all the fundamental elliptic complexes in differential geometry. It also renders many of the technical arguments involved in applying the Index Theorem quite natural and simple.

This point of view concerning Clifford bundles and Clifford multiplication is an implicit, but rarely an explicit theme in the writing of Atiyah and Singer. The authors feel that for anyone working in topology or geometry it is worthwhile to develop a friendly, if not intimate relationship with spin groups and Clifford modules. For this reason we have used them explicitly and systematically in our exposition.

The book is organized into four chapters whose successive themes are algebra, geometry, analysis, and applications. The first chapter offers a detailed introduction to Clifford algebras, spin groups and their representations. The concepts are illuminated by giving some direct applications to the elementary geometry of spheres, projective spaces, and low-dimensional Lie groups. K-theory and KR-theory are then introduced, and the fundamental relationship between Clifford algebras and Bott periodicity is established.

In the second chapter of this book, the algebraic concepts are carried over to define structures on differentiable manifolds. Here one enters properly into the subject of spin geometry. Spin manifolds, spin cobordism, and spinor bundles with their canonical connections are all discussed in detail, and a general formalism of Dirac bundles and Dirac operators is developed. Hodge-de Rham Theory is reviewed in this formalism, and each of the fundamental elliptic operators of riemannian geometry is derived and examined in detail.

Special emphasis is given here to introducing the notion of a $C\ell_k$-linear elliptic operator and discussing its index. This index lives in a certain quotient of the Grothendieck group of Clifford modules. For the fundamental operators (which are discussed in detail here) it is one of the deepest and most subtle invariants of global riemannian geometry. The systematic discussion of $C\ell_k$-linear differential operators is one of the important features of this book.

In the last section of Chapter II a universal identity of Bochner type is established for any Dirac bundle, and the classical vanishing theorems of Bochner and Lichnerowicz are derived from it.

This seems an appropriate time to make some general observations about spin geometry. To begin it should be emphasized that spin geometry is really a special topic in riemannian geometry. The central concept of a spin manifold is often considered to be a topological one. It is just a manifold with a simply-connected structure group. This is understood systematically as follows. On a general differentiable n-manifold ($n \geqq 3$), the tangent bundle has structure group GL_n. The manifold is said to be