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Editors

Advances in Fractional Calculus

Theoretical Developments and Applications in Physics and Engineering

分数阶微积分

Springer
世界图书出版公司
www.wpcbj.com.cn

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Theoretical Developments and Applications in Physics and Engineering

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图书在版编目（CIP）数据

分数阶微积分：英文/（法）萨巴蒂尔著．—北京：世界图书出版公司北京公司，2014．7

ISBN 978－7－5100－7784－5

Ⅰ．①分…　Ⅱ．①萨…　Ⅲ．①微积分—英文　Ⅳ．①O172

中国版本图书馆 CIP 数据核字（2014）第 057892 号

书　　名：	Advances in Fractional Calculus：Theoretical Developments and Applications in Physics and Engineering
作　　者：	J. Sabatier，O. P. Agrawal，J. A. Tenreiro Machado（Editors）
中 译 名：	分数阶微积分
责任编辑：	高蓉　刘慧
出 版 者：	世界图书出版公司北京公司
印 刷 者：	三河市国英印务有限公司
发　　行：	世界图书出版公司北京公司（北京朝内大街 137 号 100010）
联系电话：	010－64021602，010－64015659
电子信箱：	kjb@ wpcbj. com. cn
开　　本：	24 开
印　　张：	23. 5
版　　次：	2014 年 7 月
版权登记：	图字：01－2014－3455
书　　号：	978－7－5100－7784－5　　定　　价：99. 00 元

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-1-4020-6041-0 (HB)
ISBN 978-1-4020-6042-7 (eBook)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

www.springer.com

Reprint from English language edition:
Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering
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We dedicate this book to the honorable memory of our colleague and friend Professor Peter W. Krempl

Preface

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders (including complex orders), and their applications in science, engineering, mathematics, economics, and other fields. It is also known by several other names such as Generalized Integral and Differential Calculus and Calculus of Arbitrary Order. The name "Fractional Calculus" is holdover from the period when it meant calculus of ration order. The seeds of fractional derivatives were planted over 300 years ago. Since then many great mathematicians (pure and applied) of their times, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grünwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann M. Riesz, and H. Weyl, have contributed to this field. However, most scientists and engineers remain unaware of Fractional Calculus; it is not being taught in schools and colleges; and others remain skeptical of this field. There are several reasons for that: several of the definitions proposed for fractional derivatives were inconsistent, meaning they worked in some cases but not in others. The mathematics involved appeared very different from that of integer order calculus. There were almost no practical applications of this field, and it was considered by many as an abstract area containing only mathematical manipulations of little or no use.

Nearly 30 years ago, the paradigm began to shift from pure mathematical formulations to applications in various fields. During the last decade Fractional Calculus has been applied to almost every field of science, engineering, and mathematics. Some of the areas where Fractional Calculus has made a profound impact include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory. Although some of the mathematical issues remain unsolved, most of the difficulties have been overcome, and most of the documented key mathematical issues in the field have been resolved to a point where many of the mathematical tools for both the integer- and fractional-order calculus are the same. The books and monographs of Oldham and Spanier (1974), Oustaloup (1991, 1994, 1995), Miller and Ross (1993), Samko, Kilbas, and Marichev (1993), Kiryakova (1994), Carpinteri and Mainardi (1997), Podlubny (1999), and Hilfer (2000) have been helpful in introducing the field to engineering, science, economics and finance, pure and applied mathematics communities. The progress in this field continues. Three recent books in this field are by West, Grigolini, and Bologna (2003), Kilbas, Srivastava, and Trujillo (2005), and Magin (2006).

One of the major advantages of fractional calculus is that it can be considered as a super set of integer-order calculus. Thus, fractional calculus has the potential to accomplish what integer-order calculus cannot. We believe that many of the great future developments will come from the applications of fractional calculus to different fields. For this reason, we are promoting this field. We recently organized five symposia (the first symposium on Fractional Derivatives and Their Applications (FDTAs), ASME-DETC 2003, Chicago, Illinois, USA, September 2003; IFAC first workshop on Fractional Differentiations and its Applications (FDAs), Bordeaux, France, July 2004; Mini symposium on FDTAs, ENOC-2005, Eindhoven, the Netherlands, August 2005; the second symposium on FDTAs, ASME-DETC 2005, Long Beach, California, USA, September 2005; and IFAC second workshop on FDAs, Porto, Portugal, July 2006) and published several special issues which include Signal Processing, Vol. 83, No. 11, 2003 and Vol. 86, No. 10, 2006; Nonlinear dynamics, Vol. 29, No. 1–4, 2002 and Vol. 38, No. 1–4, 2004; and Fractional Differentiations and its Applications, Books on Demand, Germany, 2005. This book is an attempt to further advance the field of fractional derivatives and their applications.

In spite of the progress made in this field, many researchers continue to ask: "What are the applications of this field?" The answer can be found right here in this book. This book contains 37 papers on the applications of Fractional Calculus. These papers have been divided into seven categories based on their themes and applications, namely, analytical and numerical techniques, classical mechanics and particle physics, diffusive systems, viscoelastic and disordered media, electrical systems, modeling, and control. Applications, theories, and algorithms presented in these papers are contemporary, and they advance the state of knowledge in the field. We believe that researchers, new and old, would realize that we cannot remain within the boundaries of integral order calculus, that fractional calculus is indeed a viable mathematical tool that will accomplish far more than what integer calculus promises, and that fractional calculus is the calculus for the future.

Most of the papers in this book are expanded and improved versions of the papers presented at the Mini symposium on FDTAs, ENOC-2005, Eindhoven, The Netherlands, August 2005, and the second symposium on FDTAs, ASME-DETC 2005, Long Beach, California, USA, September 2005. We sincerely thank the ASME for allowing the authors to submit modified versions of their papers for this book. We also thank the authors for submitting their papers for this book and to Springer-Verlag for its

publication. We hope that readers will find this book useful and valuable in the advancement of their knowledge and their field.

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Part 1

Analytical and Numerical Techniques

THREE CLASSES OF FDEs AMENABLE TO APPROXIMATION USING A GALERKIN TECHNIQUE

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Abstract

We have recently presented elsewhere a Galerkin approximation scheme for fractional order derivatives, and used it to obtain accurate numerical solutions of second-order (mechanical) systems with fractional-order damping terms. Here, we demonstrate how that approximation can be used to find accurate numerical solutions of three different classes of fractional differential equations (FDEs), where for simplicity we assume that there is a single fractional-order derivative, with order between 0 and 1. In the first class of FDEs, the highest derivative has integer order greater than one. An example of a traveling point load on an infinite beam resting on an elastic, fractionally damped, foundation is studied. The second class contains FDEs where the highest derivative has order 1. Examples of the so-called generalized Basset's equation are studied. The third class contains FDEs where the highest derivative is the fractional-order derivative itself. Two specific examples are considered. In each example studied in the paper, the Galerkin-based numerical approximation is compared with analytical or semi-analytical solutions obtained by other means. In each case, the Galerkin approximation is found to be very good. We conclude that the Galerkin approximation can be used with confidence for a variety of FDEs, including possibly nonlinear ones for which analytical solutions may be difficult or impossible to obtain.

Keywords

Fractional derivative, Galerkin, finite element, Basset's problem, relaxation, creep.

1 Introduction

A fractional derivative of order α is given using the Riemann–Louville definition [1, 2], as

$$\mathrm{D}^{\alpha}[x(t)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau \right] ,$$

J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, 3–14.

where $0 < \alpha < 1$. Two equivalent forms of the above with zero initial conditions (as in, e.g., [3]) are given as

$$\mathrm{D}^{\alpha}[x(t)] = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}} d\tau = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(t-\tau)}{\tau^{\alpha}} d\tau \,. \tag{1}$$

Differential equations with a single-independent variable (usually "time"), which involve fractional-order derivatives of the dependent variable(s) are called fractional differential equations or FDEs. In this work, we consider FDEs where the fractional derivative has order between 0 and 1 only. Such FDEs, for our purposes, are divided into three categories, depending on whether the highest-order derivative in the FDE is an integer greater than 1, is exactly equal to 1, or is a fraction between 0 and 1.

In this article, we will demonstrate three strategies for these three classes of FDEs, whereby a new Galerkin technique [4] for fractional derivatives can be used to obtain simple, quick, and accurate numerical solutions. The Galerkin approximation scheme of [4] involves two calculations:

$$\mathbf{A}\dot{\mathbf{a}} + \mathbf{B}\,\mathbf{a} = \mathbf{c}\,\dot{x}(t) \tag{2}$$

and

$$\mathrm{D}^{\alpha}[x(t)] \approx \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\,\mathbf{c}^{T}\mathbf{a}, \tag{3}$$

where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices (specified by the scheme; see [4]), $\mathbf{c}$ is an $n \times 1$ vector also specified by the scheme[1], and $\mathbf{a}$ is an $n \times 1$ vector n internal variables that approximate the infinite-dimensional dynamics of the actual fractional order derivative. The T superscript in Eq. (3) denotes matrix transpose.

As will be seen below, the first category of FDEs (section 2) poses no real problem over and above the examples already considered in [4]. That is, in [4], the highest derivatives in the examples considered had order 2; while in the example considered in section 2 below, the highest derivative will be or order 4. However, the example of section 2 is a boundary-value problem on an infinite domain. Our approximation scheme provides significant advantages for this problem. The second category of FDEs (section 3) also leads to numerical solution of ODEs (not FDEs). The specific example considered here is relevant to the physical problem of a sphere falling slowly under gravity through a viscous liquid, but not yet at steady state. Again, the approximation scheme leads to an algorithmically simple, quick and accurate solution. However, the equations are stiff and suitable for a routine that can handle stiff systems, such as Matlab's "ode23t". Finally, the third category of FDEs (section 4) leads to a system of differential algebraic equations (DAEs), which can be solved simply and accurately using an index one DAE solver such as Matlab's "ode23t".

[1] A Maple-8 worksheet to compute the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{c}$ is available on [5].

We emphasize that we have deliberately chosen linear examples below so that analytical or semi-analytical alternative solutions are available for comparing with our results using the Galerkin approximation. However, it will be clear that the Galerkin approximation will continue to be useful for a variety of nonlinear problems where alternative solution techniques might run into serious difficulties.

2 Traveling Load on an Infinite Beam

The governing equation for an infinite beam on a fractionally damped elastic foundation, and with a moving point load (see Fig. 1), is

$$u_{xxxx} + \frac{\bar{m}}{EI} u_{tt} + \frac{c}{EI} \mathrm{D}_t^{1/2} u + \frac{k}{EI} u = -\frac{1}{EI} \delta(x - vt)\,, \tag{4}$$

where $\mathrm{D}^{1/2}$ has a t-subscript to indicate that x is held constant. The boundary conditions of interest are

$$u(\pm\infty, t) \equiv 0.$$

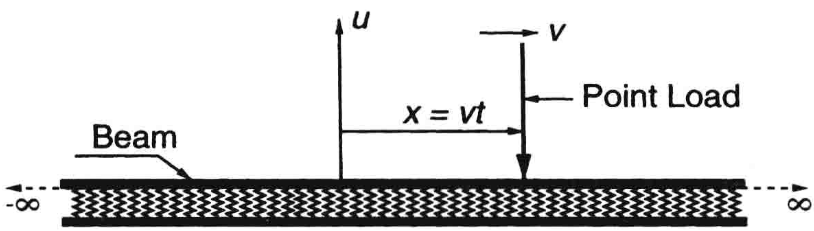

Fig. 1. Traveling point load on an infinite beam with a fractionally damped elastic foundation.

We seek steady-state solutions to this problem.

2.1 With Galerkin

With the Galerkin approximation of the fractional derivative, we get the new PDEs

$$u_{xxxx} + \frac{\bar{m}}{EI} u_{tt} + \frac{c}{EI\,\Gamma(1/2)\Gamma(3/2)} \mathbf{c}^T \mathbf{a} + \frac{k}{EI} u = -\frac{1}{EI} \delta(x - vt)$$

and

$$\mathbf{A}\dot{\mathbf{a}} + \mathbf{B}\mathbf{a} = \mathbf{c} u_t\,,$$