5 IAM

Interdisciplinary Applied Mathematics

Rüdiger Seydel

# Practical Bifurcation and Stability Analysis

## From Equilibrium to Chaos

Second Edition

# 实用分歧和稳定性分析

第2版

Springer -Verlag
世界图书出版公司

Seydel

# Practical Bifurcation and Stability Analysis

## From Equilibrium to Chaos

Second Edition

With 184 Illustrations

Springer-Verlag

世界图书出版公司

北京 · 广州 · 上海 · 西安

书　　名：Practical Bifurcation and Stability Analysis　2nd ed.
作　　者：R. Seydel
中 译 名：实用分歧和稳定性分析　第 2 版
出 版 者：世界图书出版公司北京公司
印 刷 者：北京世图印刷厂
发　　行：世界图书出版公司北京公司 (北京市朝阳门内大街 137 号　100010
开　　本：1/32　850 × 1168　　印　张：13.5
版　　次：1999 年 6 月第 1 版　1999 年 6 月第 1 次印刷
书　　号：ISBN 7-5062-2683-9/ O•240
版权登记：图字 01-1999-0330
定　　价：58.00 元

Rüdiger Seydel
Abteilung für Numerik
Universität Ulm
Oberer Eselsberg
D-89069 Ulm
Germany



Cover illustration: Poincaré set of a Duffing equation, from Figure 9.7.

Mathematics Subject Classification (1991): 34Axx, 34B, 34C, 34D, 35B32, 64L07, 70Kxx, 70-08, 73Hxx, 76E30, 92-08

Library of Congress Cataloging-in-Publication Data
Seydel, R. (Rüdiger), 1947–
Practical bifurcation and stability analysis: from equilibrium to chaos/Rüdiger Seydel.
p. cm. – (Interdisciplinary applied mathematics)
Includes bibliographical references and index.
ISBN 0-387-94316-1
1. Bifurcation theory. 2. Stability. I. Title. II. Series.
QA380.S49 1994
515′.35–dc20 94-18005

Printed on acid-free paper.



ISBN 0-387-94316-1 Springer-Verlag New York Berlin Heidelberg
ISBN 3-540-94316-1 Springer-Verlag Berlin Heidelberg New York

# Interdisciplinary Applied Mathematics

Volume 5

## Interdisciplinary Applied Mathematics

---

1. *Gutzwiller:* Chaos in Classical and Quantum Mechanics
2. *Wiggins:* Chaotic Transport in Dynamical Systems
3. *Joseph/Renardy:* Fundamentals of Two-Fluid Dynamics:
   Part I: Mathematical Theory and Applications
4. *Joseph/Renardy:* Fundamentals of Two-Fluid Dynamics:
   Part II: Lubricated Transport, Drops and Miscible Liquids
5. *Seydel:* Practical Bifurcation and Stability Analysis:
   From Equilibrium to Chaos

# Preface

Today, the concepts of *stability* and *chaos* are commonplace in the scientific community. Stability is a classical subject, whereas chaos is a recent field. There is one class of mechanisms that control both, namely, *bifurcations*. Solutions of nonlinear equations bifurcate at critical values of parameters. At these bifurcation points, stability may be lost or gained. On the other hand, chaos sets in after a sequence of certain bifurcations. The wide range of states, with equilibrium as the most rigid state and chaos as the most flexible state, is governed by bifurcations. Bifurcations form the machinery of structural changes.

In mathematics, bifurcation theory has been developed over several decades. In this discipline the emphasis traditionally was placed on theoretical investigations. Inspired by the discovery of numerous applications and stimulated by new emerging fields, bifurcation theory had to renew and reshape some of its results. The availability of powerful computers, and simultaneously the appearance of catastrophe theory and singularity theory, were driving forces that gave bifurcation theory a practical significance. And this is what this book is about: The text presents an introduction into a practical bifurcation and stability analysis of states and phenomena between equilibrium and chaos.

This book, *Practical Bifurcation and Stability Analysis*, is my new version of its predecessor *From Equilibrium to Chaos*, which was published in 1988 by Elsevier. The revisions are in part so drastic that a change in the title was justified. The new title of the book reflects the contents and intentions better. The aim of the book is twofold: It gives an introduction to nonlinear phenomena on a practical level and an account of computational methods. Since the quickly developing field has required changes and many additions, about ten new sections have been included with topics such as cellular automata, eigenvalue calculation, bifurcation in the presence of symmetry, homotopy, calculation of heteroclinic orbits, phase locking, inertial manifold methods, and Liapunov exponents of time series. References include 547 items, compared to the previous 394. This increase by more than 35% may indicate the scale of changes and extensions.

The analysis of nonlinear phenomena requires, on the one hand, tools that provide quantitative results and, on the other hand, the theoretical knowledge of nonlinear behavior that allows one to interpret these quantitative results. The tools of which we speak are numerical and analytical. Without questioning the importance of analytical methods, this book places emphasis on numerical methods. It is expected that most future investigations will be based on numerical methods. There is a broad spectrum of such methods, ranging from elementary strategies to sophisticated algorithms. Underlying these numerous methods are but a few basic principles. This book emphasizes basic principles and shows the reader how the methods result from combining and, on occasion, modifying the underlying principles.

This book is written to address the needs of scientists and engineers and to attract mathematicians. Mathematical formalism is kept to a minimum; the style is not technical, and is often motivating rather than proving. Phenomena and methods are illustrated by many examples and numerous figures; exercises and projects complete the text. The book is self-contained, assuming only basic knowledge in calculus. The extensive bibliography includes many references for analytical and numerical methods, applications in science and engineering, and software.

The first part of the book consists of two chapters that introduce stability and bifurcation. Chapter 1 is thought of as a tutorial of a significant part of applied mathematics, and Chapter 2 introduces basic nonlinear phenomena. The second part of the text, consisting of Chapters 3 to 7, concentrates on practical aspects and numerical methods. Chapter 3 shows what computational difficulties can arise and what kind of numerical methods are required to get around these difficulties. Chapter 4 gives an account of the principles of continuation, the procedure by which "parameter studies" for nonlinear problems will be carried out. Chapters 5 and 6 treat basic computational methods for handling bifurcations, for both systems of algebraic equations and ordinary differential equation boundary-value problems. In Chapter 7, branching phenomena of periodic solutions are handled and related numerical methods are outlined. The third and final part of the book, Chapters 8 and 9, focuses on qualitative aspects. Singularity theory and catastrophe theory, which help in the interpretation of numerical results, are introduced in Chapter 8. Chapter 9 is an introduction to chaotic behavior.

In summary, the object of this book is to provide an introduction to the nonlinear phenomena of bifurcation theory. Motivating examples and geometrical interpretations are essential ingredients in the style. The book attempts to provide a practical guide for performing parameter studies. Although not all nonlinear phenomena are treated (delay equations and stochastic bifurcation are not touched upon), the spectrum covered by the book is wide. For readers who have no immediate interest in computational aspects, a path is outlined listing those sections that provide the general introduction. Readers without urgent interest in computational aspects may wish to concentrate on the following:

Sections 1.1 to 1.5;
all of Chapter 2;
Section 3.4;
part of Sections 5.4.2 and 5.4.4, and Sections 5.5.1, 5.5.4, and 5.5.5;
Section 6.1, example in Section 6.4, and Section 6.8;
Sections 7.1 to 7.4, 7.7, and 7.8;
all of Chapter 8; and
all of Chapter 9.

Additional information and less important remarks are set in small print. On first reading, the reader may skip these parts without harm. Readers with little mathematical background are encouraged to read Appendices 1 to 3 first. Solutions to several of the exercises are given later in the text. References are not meant as required reading, but are hints to help those readers interested in further study. The figures framed in boxes are immediate output of numerical software.

This book grew out of teaching materials for courses given by the author at the State University of New York at Buffalo, at the University of Würzburg, and at the University of Ulm, and was enriched by many years of computational experience in bifurcation. The previous version owed much to William N. Gill and John Lavery. I am indebted to several people who helped to transform my old file into PlainTeX. Reinhard Seydel and Stefan Stöhr helped to bring the figures into PostScript format. Felicitas Werner translated the formulas into TeX. Marlene Hartmann completed the text and replaced my macros with those of Springer. Her virtuosity with TeX and her enthusiasm made the completion of the book a pleasure.

I hope that this book inspires readers to perform their own experimental studies. The many examples and figures should provide a basis and motivation to start right away.

Ulm, Germany — Rüdiger Seydel

# Notation

## Problem-Inherent Variables

$\lambda$ : scalar parameter to be varied (branching parameter)
$\mathbf{y}$ : vector of state variables, vector function, solution of an equation
$n$ : number of components of $\mathbf{y}$ and $\mathbf{f}$
$\mathbf{f}$ : vector function, defines the dynamics of the problem that is to be solved; typical equation: $\mathbf{f}(\mathbf{y}, \lambda) = \mathbf{0}$
$t$ : independent variable, often time
$\dot{\mathbf{y}}$ : derivative of $\mathbf{y}$ with respect to time, $\dot{\mathbf{y}} = d\mathbf{y}/dt$
$\mathbf{y}'$ : derivative of $\mathbf{y}$ with respect to a general independent variable
$a, b$ : interval in which $t$ varies, $a \le t \le b$
$x$ : spatial variable, may be scalar or vector with up to three components
$\mathbf{r}$ : vector function, often used to define boundary conditions as in $\mathbf{r}(\mathbf{y}(a), \mathbf{y}(b)) = \mathbf{0}$
$T$ : period in case of a periodic oscillation
$\gamma$ : additional scalar parameter

## Notations for a General Analysis

In particular examples, several of the following meanings are sometimes superseded by a local meaning:

### *Specific Versions of* $\mathbf{y}$ *and* $\lambda$

$\lambda_0$ : specific parameter value of a branch point
$\mathbf{y}_0$ : specific $n$-vector of a branch point
$y_i$ : $i$th component of vector $\mathbf{y}$

$\mathbf{y}^j$ : $j$th continuation step ($j$ is not an exponent here), specific solution
$\lambda_j$ : specific parameter value, corresponds to $\mathbf{y}^j$
$\mathbf{y}^{(\nu)}$ : iterates of a map. For example, Newton iterate;
for $\nu = 1, 2, \ldots$ sequence of vectors converging to a solution $\mathbf{y}$
$\mathbf{y}^s$ : stationary solution ($\mathbf{y}_s$ in Sections 6.6 and 6.7)

## *Integers*

$k$ : frequently, the $k$th component has a special meaning
$N$ : number of nodes of a discretization
$i$, $j$, $l$, $m$, $\nu$ : other integers
(Note that $i$ sometimes denotes the imaginary unit.)

## *Scalars*

$[\mathbf{y}]$ : a scalar measure of $\mathbf{y}$ (cf. Section 2.2)
$\rho$ : radius
$\vartheta$ : angle
$\omega$ : frequency
$\epsilon$ : accuracy, error tolerance
$\delta$ : distance between two solutions, or parameter
$\eta$ : value of a particular boundary condition
$\tau$ : test function indicating bifurcation
$\Delta$ : increment or decrement, sometimes acting as operator on the following variable; for instance, $\Delta\lambda$ means an increment in $\lambda$
$s$ : arclength
$u$, $v$ : functions, often solutions of scalar differential equations
$\sigma$ : step length
$p$ : parameterization, or phase condition or polynomial
$c_i$ : constants
$\mu$ $= \alpha + i\beta$ : complex-conjugate eigenvalue, $i$: imaginary unit
$\zeta$, $\xi$ : further scalars with local meaning

## *Vectors*

$\mathbf{z}$ : $n$-vector in various roles: tangent, or initial values of a trajectory, or emanating solution, or eigenvector
$\mathbf{d}$ : difference between two $n$-vectors
$\mathbf{h}$ : $n$-vector, solution of a linearization; $\mathbf{h}_0$ or $\bar{\mathbf{h}}$ are related $n$-vectors
$\mathbf{e}_i$ : $i$th unit vector (cf. Appendix A.2)
$\varphi$ : $\varphi(t; \mathbf{z})$ is the trajectory starting at $\mathbf{z}$ (Eq. (7.7))
$\mathbf{w}$ : eigenvector, also $\mathbf{w}^k$
$\mu$ $= \alpha + i\beta$ : vector of eigenvalues
$\Lambda$ : vector of parameters

$\mathbf{Y}$ : vector with more than $n$ components, contains $\mathbf{y}$ as subvector
$\mathbf{F}$ : vector with more than $n$ components, contains $\mathbf{f}$ as subvector
$\mathbf{R}$ : vector with more than $n$ components, contains $\mathbf{r}$ as subvector
$\mathbf{P}$ : map, Poincaré map
$\mathbf{q}$ : argument of Poincaré map

## *$n^2$ Matrices ($n$ rows, $n$ columns)*

$\mathbf{I}$ : identity matrix
$\mathbf{J} = \mathbf{f_y}$ : Jacobian matrix of first-order partial derivatives of $\mathbf{f}$ with respect to $\mathbf{y}$
$\mathbf{M}$ : monodromy matrix
$\mathbf{A}$, $\mathbf{B}$ : derivatives of boundary conditions (Eq. (6.12))
$\mathbf{E}$, $\mathbf{G}_j$ : special matrices of multiple shooting (Eq. (6.21), Eq. (6.22))
$\Phi$, $\mathbf{Z}$ : fundamental solution matrices (cf. Section 7.2)
$\mathbf{S}$ : element of a group $\mathcal{G}$

## *Further Notations and Abbreviations*

$\Omega$ : hypersurface
$\mathcal{M}$ : manifold
$\mathcal{G}$ : group, see Appendix A.7
$\in$ : "in", element of a set, $\notin$ for "not in"
$tr$ : as superscript means "transposed"
ODE: ordinary differential equations
PDE: partial differential equations
Re : real part
Im : imaginary part
t.h.o: terms of higher order
$\partial$ : partial derivative
$\bar{\mathbf{y}}$, $\bar{\lambda}$, $\bar{\mathbf{h}}$, $\bar{\mathbf{z}}$ : overbar characterizes approximations
$\nabla u$ : gradient of $u$ ($\nabla$ is the "del" operator)
$\nabla^2 u$: Laplacian operator (summation of second-order derivatives)
$\nabla \cdot u$ : divergence of $u$
$:=$ defining equation; the left side is "new" and is defined by the right side; see, for example, Eq. (4.14)
$O(\sigma)$ : terms of order of $\sigma$
$\| \;\|$ : vector norm, see Appendix A.1

# Contents

# 1
# Introduction and Prerequisites

## 1.1 A Nonmathematical Introduction

Every day of our lives we experience changes that occur either gradually or suddenly. We often characterize these changes as quantitative or qualitative, respectively. For example, consider the following simple experiment (Figure 1.1). Imagine a board supported at both ends, with a load on top. If the load $\lambda$ is small enough, the board will take a bent shape with a deformation depending on the magnitude of $\lambda$ and on the board's material properties (such as stiffness, $K$). This state of the board will remain *stable* in the sense that a small variation in the load $\lambda$ (or in the stiffness $K$) leads to a state that is only slightly perturbed. Such a variation (described by Hooke's law) would be referred to as a quantitative change. The board is deformed within its elastic regime and will return to its original shape when the perturbation in $\lambda$ is removed.

The situation changes abruptly when the load $\lambda$ is increased beyond a certain *critical level* $\lambda_0$ at which the board breaks (Figure 1.2b). This sudden action is an example of a qualitative change; it will also take place when the material properties are changed beyond a certain limit (see Figure 1.2a). Suppose the shape of the board is modeled by some function (solution of an equation). Loosely speaking, we may say that there is a solution for load values $\lambda < \lambda_0$ and that this solution ceases to exist for $\lambda > \lambda_0$. The load $\lambda$ and stiffness $K$ are examples of *parameters*. The outcome of any experiment, any event, and any construction is controlled by parameters. The practical problem is to *control the state* of a system—that is, to find parameters such that the state fulfills our requirements. This role of