弹性力学学习指导及习题集

GUIDE AND PRACTICE FOR THEORY OF ELASTICITY

主　编　于桂兰
副主编　余自若　徐艳秋

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内 容 简 介

本书是《弹性力学》课程双语教学的辅导教材。由绪论、平面问题基本理论、直角坐标系下平面问题的求解、极坐标系下平面问题的求解、空间问题基本理论等五部分组成。每部分包含内容总结及典型问题的求解与分析。本书有利于读者加深对弹性力学基本概念和基本理论与方法的掌握，提高解决问题的能力，并使读者掌握一定量的专业词汇和专业术语的英文表达。本书可供高校工科类本科生学习弹性力学课程参考，也可作为高校教师弹性力学课程双语教学的参考书。

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Preface

This book is intended as the supporting materials for the course of "Theory of Elasticity" in bilingual languages offered in universities and colleges of engineering. It consists of five parts including introduction, theory of plane problems, solutions to plane problems in rectangular coordinates, solutions to plane problems in polar coordinates and a brief introduction to the theory of spatial problems. Each part contains a summary of the basic contents and key points as well as a large number of typical problems. Solutions to these problems are provided in detail which will be helpful for undergraduate students understanding. This book can also be used as a reference for teachers.

The authors will be pleased to make acknowledgement of the helpful suggestions contributed by the readers of the book.

Guilan Yu
Ziruo Yu
Yanqiu Xu

Nomenclature

x, y, z—rectangular coordinates

ρ, φ—polar coordinates

f_x, f_y, f_z— body force components in rectangular coordinates

f_ρ, f_φ— body force components in polar coordinates

$\bar{f}_x$, $\bar{f}_y$, $\bar{f}_z$— surface force components in rectangular coordinates

$\bar{f}_\rho$, $\bar{f}_\varphi$— surface force components in polar coordinates

u, v, w— displacement components in rectangular coordinates

u_ρ, u_φ— displacement components in polar coordinates

σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}— stress components in rectangular coordinates

σ_ρ, σ_φ, $\tau_{\rho\varphi}$— stress components in polar coordinates

ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{zx}— strain components in rectangular coordinates

ε_ρ, ε_φ, $\gamma_{\rho\varphi}$— strain components in polar coordinates

Θ— bulk stress

θ— bulk strain

Φ— Airy stress function

E—Young's modulus (modulus of elasticity)

G— shear modulus (modulus of rigidity)

μ— Poison's ratio

l, m, n— direction cosines

M— bending moment

F_s— shearing force

I— moment of inertia

S— first moment of area

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Chapter 1

Introduction

Basic Contents and Key Points

1.1 Basic Contents

1.1.1 Contents of theory of elasticity

Theory of elasticity, often called elasticity for short, is the branch of solid mechanics which deals with the stresses and deformations in elastic solids produced by external forces or changes in temperature.

1.1.2 Some important concepts in theory of elasticity

Body forces: External forces, or the loads, distributed over the volume of the body, are called body forces. Such as gravitational forces, magnetic forces, or in the case of a body in motion, inertia forces. Its dimension is $L^{-2}MT^{-2}$.

Surface forces: External forces, or the loads, distributed over the surface of the body, are called surface forces. Such as the pressure of one body on another or hydrostatic pressure. Its dimension is $L^{-1}MT^{-2}$.

Stress: Under the action of external forces, internal forces will be produced between the parts of a body, and are usually measured by the intensity, i.e., by the amount of force per unit area of the surface on which they act. This intensity is called traction or stress vector of which the dimension is $L^{-1}MT^{-2}$. Notice that the traction vector depends on both the spatial location and the unit normal vector to the surface under study. Thus, even though we may be investigating the same point, the traction vector still varies as a function of the orientation of the surface normal.

The nine quantities σ_x, σ_y, σ_z, τ_{yz}, τ_{zy}, τ_{zx}, τ_{xz}, τ_{xy}, τ_{yx} are the components of the traction vector on each of three coordinate planes. These nine components are called the stress components at a point with σ_x, σ_y, σ_z referred to as normal stresses and τ_{yz}, τ_{zy}, τ_{zx}, τ_{xz}, τ_{xy}, τ_{yx} called the shearing stresses. The six shearing stresses are mutually equal in pairs, i. e., $\tau_{xy}=\tau_{yx}$, $\tau_{yz}=\tau_{zy}$, $\tau_{zx}=\tau_{xz}$.

If the outward normal to a side of the element is in the positive direction of a coordinate axis, the stress component on this side will be considered positive as it acts in the positive direction of corresponding axis; if the outward normal to a side of the element is in the negative direction of a coordinate axis, the stress component on this side will be considered positive as it acts in the negative direction of corresponding axis.

Strain: Under the application of external loading, elastic solids deform. The deformation of a body may be expressed by changes in lengths and angles of the parts. The change per unit length of fibers oriented in the n-direction, is defined as the normal strain, denoted by the letter ε_n; the change in angle between two originally orthogonal directions i and j is defined as the shear strain by the letter γ_{ij}, which is often referred to as the engineering shear strain. Normal strain is positive if fibers increase in length and shear strain is positive if the right angle between the positive directions of the two axes decreases. The nine quantities ε_x, ε_y, ε_z, γ_{yz}, γ_{zy}, γ_{xz}, γ_{zx}, γ_{yx} and γ_{xy} are called the components of strain at a point. The six shearing strains are mutually equal in pairs, i. e., $\gamma_{xy}=\gamma_{yx}$, $\gamma_{yz}=\gamma_{zy}$, $\gamma_{zx}=\gamma_{xz}$.

Displacement: The change of position at any point of a body. The displacement vector is often expressed by its projections on the x, y and z axes in rectangular coordinates, denoted by u, v and w, respectively. These three projections are called the displacement components at a point.

1.1.3 Basic assumptions in theory of elasticity

(1) The body is continuous, i. e., the whole volume of the body is filled with continuous matter, without any void. Only under this assumption, can the physical quantities in the body, such as stresses, strains and displacements, be continuously distributed and thereby expressed by continuous functions.

(2) The body is perfectly elastic, i. e., the original configuration of the body is recovered after unload, and furthermore, the body obeys Hooke's Law which shows the linear relations between stress and strain. Under this assumption, the elastic constants will be independent of the magnitudes of stresses or strains.

(3) The body is homogeneous, i. e., the elastic properties are the same throughout the body. Thus, the elastic constants will be independent of the location in the body. Under this assumption, the differential equations with constant coefficients are obtained, and we may analyze an elementary volume isolated from the body and then apply the results of analysis to the entire body.

(4) The body is isotropic, i. e. , the elastic properties are the same in all directions. Thus, the elastic constants will be independent of the orientation of coordinate axes. Under this assumption, the material property can be completely determined by two independent constants.

(5) The body undergoes small deformation, i. e. , the displacements of the body are very small in comparison with its original dimensions, and the strain components are much smaller than unity. Thus, when we formulate the equilibrium equations relevant to the deformed state, we may use the lengths and angles of the body before deformation. In addition, we may neglect the squares and products of the strains to linearize the strain-displacement relations.

These assumptions are necessary to simplify the governing equations in elasticity for their easier solutions.

1.2 Key Points

Definition and sign convention for body force, surface force, stress, strain and displacement.

Problems

1-1. Fill in the blanks.

(1) The theory of elasticity deals with the ________ and ________ in elastic solids produced by external forces or changes in temperature.

(2) The dimension of the body force component is ________; The dimension of the surface force component is ________; The dimension of the stress component is ________ .

(3) That the body is homogeneous means ________ are the same throughout the body.

1-2. Choose the right one from the four choices.

(1) The shearing strain γ in Fig. 1-1 can be denoted as ________ .

A. γ_{xy} B. γ_{yz}

C. γ_{zx} D. γ_{yx}

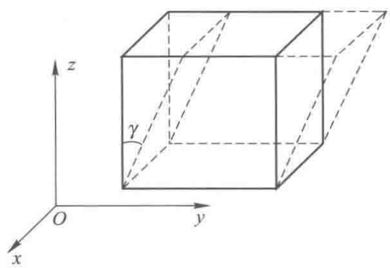


Fig. 1-1

(2) ________ can be considered as isotropic material.

A. Wood B. Bamboo

C. Concrete D. Sandwich plate

1 – 3. Give some examples of homogeneous anisotropic body, nonhomogeneous isotropic body, and nonhomogeneous anisotropic body.

1 – 4. Show the positive directions of body forces and surface forces for the body shown in Fig. 1 – 2.

1 – 5. Show the positive directions of normal and shear stresses on the element in Fig. 1 – 3.

1 – 6. Determine the sign of the shear stress shown in Fig. 1 – 4 according to the sign conventions in Mechanics of Materials and in Elasticity respectively.

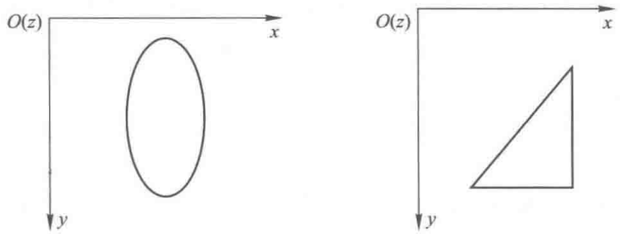


Fig. 1 – 2

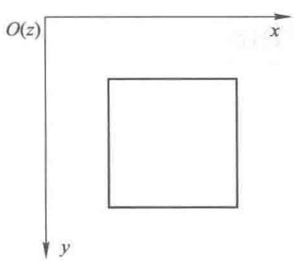


Fig. 1 – 3

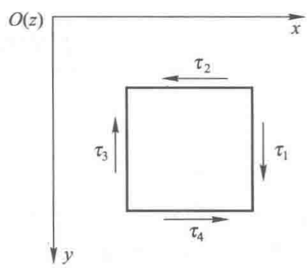


Fig. 1 – 4

Chapter 2
Theory of Plane Problems

Basic Contents and Key Points

2.1 Basic Contents

2.1.1 Definition of two kinds of plane problems

(1) Plane stress

If a thin plate is loaded by forces applied at the boundary, parallel to the plane of the plate and distributed uniformly over the thickness, as shown in Fig. 2 - 1, the stress components σ_z, τ_{zx} and τ_{zy} are zero on both faces of the plate. It may be assumed, tentatively, that they are zero within the plate. The state of stress is then specified by σ_x, σ_y and τ_{xy} only which are independent of the z coordinate. This is called a state of plane stress in the xy plane.

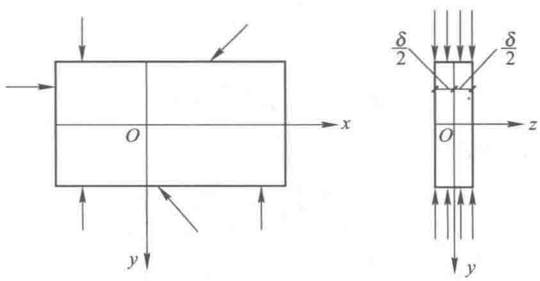


Fig. 2 - 1

(2) Plane strain

If an infinitely long prismatic body is loaded by body forces and surface forces on the lateral surface, and these forces are parallel to the cross sections of the body and do not vary along the axial direction, then the displacement component in the axial direction vanishes and the displacement components in the cross sections are independent of the axial z coordinate. In this case, the strain components ε_z, γ_{zx} and γ_{zy} are zero and the state of strain is specified by ε_x, ε_y and γ_{xy} only. This deformation is referred to as a state of plane strain in the xy plane.

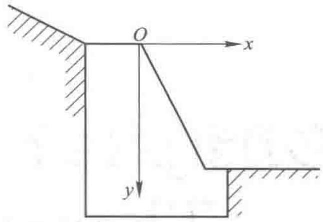


Fig. 2 - 2

2.1.2 Governing equations

Differential equations of equilibrium, geometrical equations and physical equations are the governing elasticity equations.

(1) Differential equations of equilibrium

$$\left.\begin{aligned}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+f_x=0\\ \frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+f_y=0\end{aligned}\right\}.$$

(2) Geometrical equations (also called strain-displacement relations)

$$\varepsilon_x=\frac{\partial u}{\partial x},\quad \varepsilon_y=\frac{\partial v}{\partial y},\quad \gamma_{xy}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}.$$

Compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y}.$$

(3) Physical equations (also called stress-strain relations or Hooke's Law or constitutive equations)

$$\left.\begin{aligned}\varepsilon_x&=\frac{1}{E}(\sigma_x-\mu\sigma_y)\\ \varepsilon_y&=\frac{1}{E}(\sigma_y-\mu\sigma_x)\\ \gamma_{xy}&=\frac{2(1+\mu)}{E}\tau_{xy}\end{aligned}\right\},$$

for plane stress problems, and

$$\left.\begin{aligned}\varepsilon_x&=\frac{1-\mu^2}{E}\left(\sigma_x-\frac{\mu}{1-\mu}\sigma_y\right)\\ \varepsilon_y&=\frac{1-\mu^2}{E}\left(\sigma_y-\frac{\mu}{1-\mu}\sigma_x\right)\\ \gamma_{xy}&=\frac{2(1+\mu)}{E}\tau_{xy}\end{aligned}\right\},$$

for plane strain problems.

(If we replace E by $\frac{E}{1-\mu^2}$ and μ by $\frac{\mu}{1-\mu}$ in the physical equations for plane stress problems, the physical equations for plane strain problems are obtained.)

Notation:

The compatibility relations ensure that the displacements are continuous and single-valued and are necessary when the strains are arbitrarily specified.

If, however, the displacements are included in the problem formulation, the solution normally generates single - valued displacements and strain compatibility is automatically satisfied. Thus, in discussing the governing equations of elasticity, the compatibility relations are normally set aside, to be used only with the stress formulation.

Therefore, the field equations for plane problems refers to the eight relations including two equilibrium relations, three strain-displacement relations and three stress-strain relations. These field equations involve eight unknowns including two displacements, three strains and three stresses. Therefore, the number of equations matches the number of unknowns to be determined.

2.1.3 Stress at a point

If the stress components at a point P are known as σ_x, σ_y and τ_{xy}, the direction cosines of the outward normal to the plane passing through this point is denoted by (l, m), then we can obtain

(1) The traction (p_x, p_y) at point P on this plane

$$p_x = l\sigma_x + m\tau_{yx}, \quad p_y = m\sigma_y + l\tau_{xy}.$$

(2) The normal stress and shearing stress (σ_n, τ_n) at point P on this plane

$$\sigma_n = l^2\sigma_x + m^2\sigma_y + 2lm\tau_{xy}, \quad \tau_n = lm(\sigma_y - \sigma_x) + (l^2 - m^2)\tau_{xy}.$$

(3) The principal stress (σ_1, σ_2) at point P

$$\left.\begin{matrix}\sigma_1\\ \sigma_2\end{matrix}\right\} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}.$$

The principal stress directions: $\tan\alpha_1 = \frac{\sigma_1 - \sigma_x}{\tau_{xy}}$ or $\tan\alpha_2 = -\frac{\tau_{xy}}{\sigma_1 - \sigma_x}$, where α_i $(i=1, 2)$ denotes the angle between σ_i and x axis.

(4) The maximum and minimum values of normal stress and shearing stress (if $\sigma_1 \geqslant \sigma_2$)

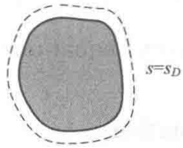
$$\sigma_{\max} = \sigma_1, \quad \sigma_{\min} = \sigma_2, \quad \tau_{\max} = \frac{\sigma_1 - \sigma_2}{2}, \quad \tau_{\min} = -\frac{\sigma_1 - \sigma_2}{2},$$

the maximum or minimum shearing stress acts on the plane bisecting the angle between the two principal stresses.

2.1.4 Boundary conditions

The solutions of the above field equations require appropriate boundary conditions on the body under study. The three typical boundary conditions are

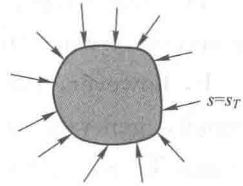
(1) displacement boundary conditions



$$(u)_s=\bar{u}(s),\quad (v)_s=\bar{v}(s),$$

where $\bar{u}(s)$ and $\bar{v}(s)$ are prescribed displacement.

(2) traction boundary conditions



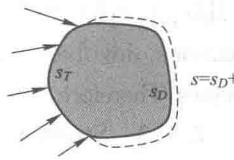
$$\left.\begin{array}{l}(l\sigma_x+m\tau_{yx})_s=\bar{f}_x(s)\\(m\sigma_y+l\tau_{xy})_s=\bar{f}_y(s)\end{array}\right\},$$

where $\bar{f}_x(s)$ and $\bar{f}_y(s)$ are prescribed tractions.

(3) mixed boundary conditions

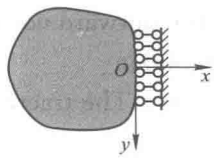
Two cases of mixed boundary conditions are often dealt with.

(a) Tractions are specified on some portion of the boundary and displacements are given on the remaining portion of the boundary



$$(u)_{s_D}=\bar{u}(s),\quad (v)_{s_D}=\bar{v}(s),\quad \left.\begin{array}{l}(l\sigma_x+m\tau_{yx})_{s_T}=\bar{f}_x(s)\\(m\sigma_y+l\tau_{xy})_{s_T}=\bar{f}_y(s)\end{array}\right\}.$$

(b) A displacement and a traction are specified in two different orthogonal directions at the same boundary point



$$(u)_{x=0}=0,\quad (\tau_{xy})_{x=0}=0.$$

Thereby the three fundamental boundary-value problems are classified in the theory of elasticity, including traction boundary-value problem, displacement boundary-value problem and mixed boundary-value problem.

2.1.5 Principle of superposition

The principle of superposition applies to any problem that is governed by linear equations.

Under the assumption of small deformations and linear elastic constitutive behavior, all the elasticity field equations are linear. Furthermore, the usual boundary conditions are also linear. Thus the principle of superposition can be applied.

The general statement of the principle can be expressed as follows:

For a given problem, if the state $\{\sigma_{ij}^{(1)},\ \varepsilon_{ij}^{(1)},\ u_i^{(1)}\}$ is a solution to the fundamental elasticity equations with prescribed body forces $f_i^{(1)}$ and surface tractions $\bar{f}_i^{(1)}$, and the state $\{\sigma_{ij}^{(2)},\ \varepsilon_{ij}^{(2)},\ u_i^{(2)}\}$ is a solution to the fundamental equations with prescribed body forces $f_i^{(2)}$ and surface tractions $\bar{f}_i^{(2)}$, then the state $\{\sigma_{ij}^{(1)}+\sigma_{ij}^{(2)},\ \varepsilon_{ij}^{(1)}+\varepsilon_{ij}^{(2)},\ u_i^{(1)}+u_i^{(2)}\}$ will be a solution to the problem with body forces $f_i^{(1)}+f_i^{(2)}$ and surface tractions $\bar{f}_i^{(1)}+\bar{f}_i^{(2)}$.

2.1.6 Saint Venant's Principle

Saint Venant's Principle can be stated as: the stress, strain, and displacement fields caused by two different statically equivalent forces distributed on parts of the body far away from the loading points are approximately the same.

We often encounter difficulties in having all the boundary conditions completely satisfied. When the pointwise boundary conditions can't be satisfied exactly or only the resultants instead of the distribution of surface forces are known on a small part of the boundary, Saint Venant's Principle is often used which may be of much help to solve the problem.

2.1.7 Solutions to plane problems in terms of displacements

In the solution of a plane problem in terms of displacements, the displacement components $u(x, y)$ and $v(x, y)$ are the basic unknowns, and they must satisfy the governing equations throughout the body, and also satisfy the boundary conditions on the surface of the body.

(1) Equilibrium equations

$$\left.\begin{aligned}\frac{E}{1-\mu^2}\left(\frac{\partial^2 u}{\partial x^2}+\frac{1-\mu}{2}\frac{\partial^2 u}{\partial y^2}+\frac{1+\mu}{2}\frac{\partial^2 v}{\partial x\,\partial y}\right)+f_x=0\\ \frac{E}{1-\mu^2}\left(\frac{\partial^2 v}{\partial y^2}+\frac{1-\mu}{2}\frac{\partial^2 v}{\partial x^2}+\frac{1+\mu}{2}\frac{\partial^2 u}{\partial x\,\partial y}\right)+f_y=0\end{aligned}\right\}\text{(for a plane stress problem).}$$

(2) Boundary conditions

Displacement boundary conditions

$$(u)_s=\bar{u}(s),\quad (v)_s=\bar{v}(s).$$

Traction boundary conditions

$$\left.\begin{aligned}\frac{E}{1-\mu^2}\left[l\left(\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}\right)+m\frac{1-\mu}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right]_s=\bar{f}_x(s)\\ \frac{E}{1-\mu^2}\left[m\left(\frac{\partial v}{\partial y}+\mu\frac{\partial u}{\partial x}\right)+l\frac{1-\mu}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]_s=\bar{f}_y(s)\end{aligned}\right\}\text{(for a plane stress problem).}$$

For a plane strain problem, it is necessary to replace E by $\frac{E}{1-\mu^2}$ and μ by $\frac{\mu}{1-\mu}$ in equilibrium equations and traction boundary conditions.

2.1.8 Solutions to plane problems in terms of stresses

In the solution of a plane problem in terms of stresses, the stress components must satisfy the differential equations of equilibrium

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+f_x=0\\ \frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+f_y=0\end{aligned}\right\},$$

and the compatibility equation

$$\nabla^2(\sigma_x+\sigma_y)=-(1+\mu)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right) \text{for plane stress problems, or}$$

$$\nabla^2(\sigma_x+\sigma_y)=-\frac{1}{1-\mu}\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right) \text{ for plane strain problems.}$$

Besides, they must satisfy the traction boundary conditions.

It is found that the compatibility equation is a necessary and sufficient condition for continuous, single-valued displacements only for **simply connected** regions. That is to say, the stress components in a simply connected body are determined if they satisfy the differential equations of equilibrium, the compatibility equation and the traction boundary conditions. However, for **multiply connected** domains, compatibility relations provide only necessary but not sufficient conditions, the single-valued displacement condition should be considered.

The term **simply connected** refers to regions of space for which all simple closed curves drawn in the region can be continuously shrunk to a point without going outside the region. Domains not having this property are called **multiply connected.**

2.1.9 Stress function

In the case of constant body forces, the stress components can be obtained from the differential equations of equilibrium

$$\sigma_x=\frac{\partial^2\Phi}{\partial y^2}-f_x x,\quad \sigma_y=\frac{\partial^2\Phi}{\partial x^2}-f_y y,\quad \tau_{xy}=\tau_{yx}=-\frac{\partial^2\Phi}{\partial x\,\partial y},$$

where the function $\Phi(x, y)$ is known as the Airy stress function for two-dimensional problems.

The compatibility equation in terms of the stress function becomes

$$\nabla^4\Phi=0.$$

It can be seen that the entire set of field equations reduces to a biharmonic equation in terms of this stress function.

2.2 Key Points

1. The similarities and differences between the two kinds of plane problems
2. Governing equations of plane problems in rectangular coordinates
3. Boundary conditions
4. Saint Venant's Principle and its applications