一元函数微积分与线性算子

(Calculus for Functions with Single Variable and Linear Operators)

(双语教材)

王栓宏　编著

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内 容 简 介

本书介绍一元函数的极限(Limits)、导数(Derivatives)、积分 (Integrals)、微分方程(Differential Equations)和线性算子(Linear Operators)的基本概念和理论, 并给出与这些概念相关的大学自主招生考试试题的解析与提高练习. 本书由浅入深, 结合双语课程的特点, 兼顾高中生参加大学自主招生考试的内容.

本书可供普通高等院校数学和相关专业的大学生以及从事双语课程教学的高校教师作为教材使用, 也可供准备参加自主招生考试的高中生及高中教师阅读和参考.

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前　言

目前, 国内许多大学已经意识到本科教学国际化的重要性, 教学计划中也规定了双语课程. 于是, 双语教学如何开展与评估, 尤其是双语教材如何编写, 学生应该使用什么样的教材就成为一个很大的问题, 有待探索与研究.

本科教学的国际化最终是本科生的国际化, 这包括了出国交换学习、参加国际会议、在国内参加国外专家的各种讲座与学术报告会. 即使是研究生甚至是教师, 与外国同行交流专业知识时, 概念和性质的叙述也是非常重要的, 必须用英语完成. 至于性质的证明过程则是一种深入的研究过程. 根据作者多年来与国外同行专家合作研究以及参加各种国际会议的经历, 本科生的双语教材的主要目的是学会应用英语语言来描述和掌握概念、性质、例子和应用, 淡化应用英语来证明性质、定理等的研究过程. 也就是说, 我们用中文给出所有证明过程, 目的是让学生真正领会并会应用严密的数学推导来理解抽象的概念, 避免因为对英语语言的误解而没有理解对数学专业知识的探究.

正是基于这样的考虑, 本书在内容与写作上有如下三个特点：第一, 内容处理上兼顾与所讲概念相关的国际前沿研究性问题; 第二, 增加理论的应用内容; 第三, 用英文描述概念、性质、定理、例子、应用与习题等, 所有性质证明用中文描述. 我们相信, 通过这样的双语教材的学习, 本科生与国际同行专家交流一元函数微积分方面的数学知识将会得心应手.

同时, 作者曾经对自主招生内容进行过研究与培训, 感觉到本书也可以兼顾自主招生考生与高中教师的需求, 因此, 本书中也收集了一些近几年来的自主招生考试试题并进行解证.

在完成本书时, 作者要感谢博士生张晓辉、赵晓凡、鹿道伟和周楠的认真阅读和打印工作. 本书得到了东南大学双语教材项目的资助, 在此一并致谢.

限于作者水平, 不足之处在所难免, 恳请读者批评指正.

作　者

2014 年 11 月于南京

目　录

Chapter 1　极限 (Limits)

详细叙述极限(limits)思想的第一人是牛顿(Newton), 随后数学家柯西(Cauchy)澄清了牛顿的这个极限思想, 再后来, 由魏尔斯特拉斯 (Weierstrass) 给出了极限的严格定义. 在现代的数学分析教科书中, 几乎所有基本概念 (连续、微分、积分等)都是建立在极限概念的基础之上的.

1.1　极限概念 (The Notion of Limits)

本节介绍函数极限的定义且给出具体例子.

Definition 1.1.1　Let $f(x)$ be a function defined on an open interval about x_0, except possibly at x_0 itself. We say that **the limit of $f(x)$ is a number a as x approaches x_0 (在 x 趋向于 x_0 时, $f(x)$ 的极限为 a)** if, for every number $\varepsilon > 0$, there exists a corresponding number δ such that for all x

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - a| < \varepsilon,$$

and we write

$$\lim_{x \to x_0} f(x) = a.$$

Example 1.1.1　Use the definition to prove that $\lim\limits_{x \to 0} a^x = 1$, where $a > 1$.

证明　对任给的 $\varepsilon > 0$, 下证存在 $\delta > 0$ 使得

$$0 < |x| < \delta \Rightarrow |a^x - 1| < \varepsilon.$$

不失一般性, 不妨设 $0 < \varepsilon < 1$. 由于

$$|a^x - 1| < \varepsilon \Leftrightarrow 1 - \varepsilon < a^x < 1 + \varepsilon \Leftrightarrow \log_a(1 - \varepsilon) < x < \log_a(1 + \varepsilon),$$

取 $\delta = \min\{-\log_a(1-\varepsilon), \log_a(1+\varepsilon)\}$, 则当 $0 < |x| < \delta$ 时, 即可证明.　□

Example 1.1.2　Prove that if $x_0 > 0$, then $\lim\limits_{x \to x_0} \sqrt{x} = \sqrt{x_0}$.

证明　对任意的 $\varepsilon > 0$, 下证存在 δ 满足

$$0 < |x - x_0| < \delta \Rightarrow |\sqrt{x} - \sqrt{x_0}| < \varepsilon.$$

由于 $\sqrt{x}$ 的定义域为 $[0,+\infty)$, 所以, 选取 $\delta>0$ 使得 $(x_0-\delta,x_0+\delta)\subseteq[0,+\infty)$, 为此取 $\delta<x_0$, 进而 $x>0$, $\sqrt{x}$ 也有意义.

又

$$|\sqrt{x}-\sqrt{x_0}|=\left|\frac{x-x_0}{\sqrt{x}+\sqrt{x_0}}\right|<\frac{|x-x_0|}{\sqrt{x_0}}<\varepsilon,$$

即 $|x-x_0|<\varepsilon\sqrt{x_0}$. 只需令 $\delta=\min\{\sqrt{x_0}\varepsilon,x_0\}$, 则当 $0<|x-x_0|<\delta$ 时, 证明完毕. □

Example 1.1.3 Use the definition to prove that $\lim\limits_{x\to2}\dfrac{x-3}{x+1}=-\dfrac{1}{3}$.

证明 任取 $\varepsilon>0$, 下证存在 δ 使得

$$0<|x-2|<\delta\Rightarrow\left|\frac{x-3}{x+1}+\frac{1}{3}\right|<\varepsilon.$$

由

$$\left|\frac{x-3}{x+1}+\frac{1}{3}\right|=\left|\frac{3x-9+x+1}{3(x+1)}\right|=\left|\frac{4x-8}{3(x+1)}\right|=\frac{4}{3}\frac{|x-2|}{|x+1|},$$

易知 $\dfrac{1}{x+1}$ 在 x 取 -1 时无意义. 注意到

$$|x+1|=|x-2+3|\geqslant|3-|x-2||,$$

因而我们令 $\delta<1$, 此时 $|x+1|\geqslant2$, 进而 $\dfrac{1}{x+1}$ 有意义. 令 $\delta<\varepsilon$, 则有

$$\left|\frac{x-3}{x+1}+\frac{1}{3}\right|<\frac{4}{3}\frac{|x-2|}{|x+1|}<\frac{4}{3}\cdot\frac{1}{2}|x-2|<|x-2|<\varepsilon.$$

只需令 $\delta=\min\{1,\varepsilon\}$, 则 $0<|x-2|<\delta$ 成立. □

It is possible for a function to approach a limiting value as x approaches from only one side, either from the right or from the left. In this case we say that f has a **one-sided (either right-hand or left-hand) limit(单侧极限或右、左极限)** at x_0.

Definition 1.1.2 A function $f(x)$ converges to a finite limit a on the right side of $x=x_0$ if $f(x)$ approaches a as $x>x_0$ and approaches x_0. In this case, we say that f has **right-hand limit(右极限)** a at x_0, and denote

$$\lim_{x\to x_0^+}f(x)=a\quad\text{or}\quad f(x_0+0)=a.$$

The **left-hand limit(左极限)** at x_0 is defined similarly. Denoted as

$$\lim_{x\to x_0^-}f(x)=a\quad\text{or}\quad f(x_0-0)=a.$$

By the definition of the right-hand and left-hand limits, it is easily to see the following theorem.

Theorem 1.1.1 A function $f(x)$ has a limit as x approaches x_0 if and only if it has left-hand and right-hand limits there, and these one-sided limits are equal:

$$\lim_{x\to x_0} f(x)=a \quad \Leftrightarrow \quad \lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)=a.$$

For example, the sign function, which is also called a step function,

$$\text{sgn}x=\begin{cases}1, & x>0,\\ 0, & x=0,\\ -1, & x<0.\end{cases}$$

We have

$$\lim_{x\to 0^-} f(x)=-1, \quad \lim_{x\to 0^+} f(x)=1.$$

Since the two sided limits are not equal, $\lim\limits_{x\to 0} f(x)$ does not exist.

Now we will give a new kind of limits. In analogy with our "ε-δ" definition for ordinary limits, we make the following definitions for limits as x approaches ∞. The definition of $x\to+\infty$ and $x\to-\infty$ are similar, we omit here.

Definition 1.1.3 Let the function $f(x)$ be defined on $(-\infty,+\infty)$. We say that $\lim\limits_{x\to\infty} f(x)=a$ if for $\varepsilon>0$ there is a corresponding number M such that

$$|x|>M \Rightarrow |f(x)-a|<\varepsilon.$$

Example 1.1.4 Show that $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0$.

证明 对任意的 $\varepsilon>0$, 下证存在 M 使得下式成立:

$$\left|\frac{\sin x}{x}-0\right|=\left|\frac{\sin x}{x}\right|\leqslant\frac{1}{|x|}<\varepsilon.$$

为此, 令 $M=\dfrac{1}{\varepsilon}$, 则由 $|x|>M$ 可知, $\left|\dfrac{\sin x}{x}\right|\leqslant\dfrac{1}{|x|}<\varepsilon.$ □

下面, 为了参加自主招生考试的考生及其辅导教师阅读方便, 我们将采用全中文叙述, 用 $\mathbb{N}^*$ 表示自然数的集合.

Example 1.1.5 数列极限的"ε-N"定义法.

设 $\{a_n\}$ 为一个数列, 对某个常数 a, 如果 $\forall\varepsilon>0$, $\exists N\in\mathbb{N}^*$, 使得当 $n>N$ 时, 有 $|a_n-a|<\varepsilon$ 成立, 则称数列 $\{a_n\}$ 的极限是 a, 或称数列 $\{a_n\}$ 是收敛数列, 且收敛于 a, 记作

$$\lim_{n\to+\infty} a_n=a.$$

Remark (1) 把定义中 $|a_n - a| < \varepsilon$, 改为 $|a_n - a| \leqslant \varepsilon$ 或 $|a_n - a| < c\varepsilon$ (c 是给定的正的常数), 其定义都等价.

(2) 数列 $\{a_n\}$ 不以 a 为极限 $\Leftrightarrow \exists \varepsilon_0 > 0$, 对 $\forall N \in \mathbb{N}^*$, $\exists n_0 > N$, 有 $|a_{n_0} - a| \geqslant \varepsilon_0$.

(3) 在定义中, 若 $a = 0$, 即 $\lim\limits_{n\to+\infty} a_n = 0$, 则称 $\{a_n\}$ 为无穷小量.

(4) 用 "ε-N" 定义法证明数列 $\{a_n\}$ 以 a 为极限, 关键是对任意给的小数 $\varepsilon > 0$, 找出合适的 N, 使 $n > N$ 时, 有 $|a_n - a| < \varepsilon$. 为此, 通常要根据已知条件对 $|a_n - a|$ 适当放大, 使 $|a_n - a| < \alpha_n$, α_n 为包含 n 的表达式, 然后由 $\alpha_n < \varepsilon$ 来确定出所需要的 N. 更一般的情形是, 这种放大过程是分段逐步实现的.

Example 1.1.6 数列无穷大量的定义.

设 $\{a_n\}$ 为一个数列, 如果 $\forall M > 0$, $\exists N \in \mathbb{N}^*$, 使得当 $n > N$ 时, 有 $a_n > M$ 成立, 则称数列 $\{a_n\}$ 是正无穷大量, 或称 $\{a_n\}$ 的极限是 $+\infty$. 记作

$$\lim_{n\to+\infty} a_n = +\infty.$$

类似地, 如果 $\forall M > 0$, $\exists N \in \mathbb{N}^*$, 使得当 $n > N$ 时, 有 $a_n < -M$ 成立, 则称数列 $\{a_n\}$ 是负无穷大量, 或称 $\{a_n\}$ 的极限是 $-\infty$. 记作

$$\lim_{n\to+\infty} a_n = -\infty.$$

Remark 一个常用的极限

$$\lim_{n\to\infty} a^n = \begin{cases} 0, & |a| < 1, \\ 1, & a = 1, \\ \infty, & |a| > 1, \\ \text{不存在}, & a = -1. \end{cases}$$

☞ 自主招生真题讲解

Example 1.1.7(2013 年 "华约"①) 设数列 $\{a_n\}$ 各项均为正数, 且对任意的 $n \in \mathbb{N}^*$ 满足 $a_{n+1} = a_n + ca_n^2$ ($c > 0$为常数).

(1) 求证: 对于任意正数 M, 存在 $N \in \mathbb{N}^*$, 当 $n > N$ 时有 $a_n > M$.

(2) 设 $b_n = \dfrac{1}{1 + ca_n}$, S_n 是 $\{b_n\}$ 前 n 项和, 求证: 对任意 $d > 0$, 存在 $N \in \mathbb{N}^*$, 当 $n > N$ 时, 有 $0 < \left|S_n - \dfrac{1}{ca_1}\right| < d$.

①"华约" 是指清华大学、上海交通大学、浙江大学、中国科学技术大学、西安交通大学、南京大学和中国人民大学等大学的招生实行自主招生联合测试.

证明 (1) 分析: 这等价于 $\lim\limits_{n\to+\infty} a_n = +\infty$.

因为 $\{a_n\}$ 各项均为正数, 所以 $a_{n+1} = a_n + ca_n^2 > a_n$, 特别地, $a_2 - a_1 = ca_1^2$. 又因为 $c > 0$, 所以

$$a_{n+1} - a_n = ca_n^2 - ca_{n-1}^2 + a_n - a_{n-1} > a_n - a_{n-1} > \cdots > a_2 - a_1,$$

且用"累加法"得

$$a_n = a_n - a_{n-1} + a_{n-1} - a_{n-2} + \cdots + a_2 - a_1 + a_1 > (n-1)(a_2 - a_1) = (n-1)ca_1^2.$$

所以对于任意正数 M, 存在 $N = \max\left\{1, \left[\dfrac{M}{ca_1^2} + 1\right] + 1\right\} \in \mathbb{N}^*$, 当 $n > N$ 时有 $a_n > M$.

(2) 由 $a_{n+1} = a_n + ca_n^2$ 得 $a_{n+1} = a_n + ca_n^2 = a_n(ca_n + 1)$, 所以

$$b_n = \frac{1}{1 + ca_n} = \frac{a_n}{a_{n+1}} = \frac{ca_n^2}{ca_n a_{n+1}} = \frac{a_{n+1} - a_n}{ca_n a_{n+1}} = \frac{1}{ca_n} - \frac{1}{ca_{n+1}},$$

所以

$$S_n = \sum_{i=1}^{n} b_i = \frac{1}{ca_1} - \frac{1}{ca_{n+1}}.$$

于是, 有 $\left|S_n - \dfrac{1}{ca_1}\right| = \dfrac{1}{ca_{n+1}} > 0$. 且由 (1) 可知 $a_{n+1} > nca_1^2$, 所以 $\dfrac{1}{ca_{n+1}} < \dfrac{1}{nc^2a_1^2}$. 于是, 对于任意的 $d > 0$, 存在 $N = \max\left\{1, \left[\dfrac{1}{dc^2a_1^2}\right]\right\}$, 当 $n > N$ 时, 有 $0 < \left|S_n - \dfrac{1}{ca_1}\right| < d$. □

Example 1.1.8(2011 年 "卓越联盟"①) 设数列 $\{a_n\}$ 满足 $a_1 = a, a_2 = b$, $2a_{n+2} = a_{n+1} + a_n$.

(1) 设 $b_n = a_{n+1} - a_n$, 证明: 若 $a \neq b$, 则 $\{b_n\}$ 是等比数列.

(2) 若 $\lim\limits_{n\to\infty} (a_1 + a_2 + \cdots + a_n) = 4$, 求 a, b 的值.

证明 (1) 由已知条件可得 $a_{n+2} - a_{n+1} = -\dfrac{1}{2}(a_{n+1} - a_n)$, 即有 $b_{n+1} = -\dfrac{1}{2}b_n$. 于是, 若 $a \neq b$, 则 $\{b_n\}$ 是等比数列.

(2) 因为

①"卓越联盟" 是指北京理工大学、重庆大学、大连理工大学、东南大学、哈尔滨工业大学、华南理工大学、天津大学、同济大学和西北工业大学等大学的招生实行自主招生联合测试.

$$\begin{aligned} a_n &= a_n - a_{n-1} + a_{n-1} - \cdots + a_2 - a_1 + a_1 \\ &= a_1 + b_1 + b_2 + \cdots + b_{n-1} \\ &= a + (b-a)\frac{1-\left(-\frac{1}{2}\right)^{n-1}}{1-\left(-\frac{1}{2}\right)}, \end{aligned}$$

所以

$$a_1 + a_2 + \cdots + a_n = na + \frac{2}{3}(b-a)\left[n - \frac{1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)}\right]$$

$$= \left[a + \frac{2}{3}(b-a)\right]n - \frac{4}{9}(b-a) + \frac{4}{9}(b-a)\left(-\frac{1}{2}\right)^n.$$

由题设条件: $\lim\limits_{n\to\infty}(a_1+a_2+\cdots+a_n) = 4$, 可知 $a + \frac{2}{3}(b-a) = 0$, $-\frac{4}{9}(b-a) = 4$, 所以 $a = 6$, $b = -3$. □

In the end of this section, we will discuss infinitesimal and infinite of functions. For conveniens, we only discuss infinitesimal and infinite using the limit as $x \to x_0$.

$$\lim_{x\to x_0} f(x).$$

The result is same as the other limits, such as,

$$\lim_{x\to x_0^+} f(x),\ \lim_{x\to x_0^-} f(x),\ \lim_{x\to +\infty} f(x),\ \lim_{x\to -\infty} f(x),\ \lim_{x\to \infty} f(x),\ \text{and}\ \lim_{n\to \infty} f(n).$$

Definition 1.1.4 If $\lim\limits_{x\to x_0} f(x) = 0$, then we say $f(x)$ is an **infinitesimal (无穷小量)** as $x \to x_0$.

It is easy to prove the following proposition.

Proposition 1.1.1 The following properties hold for infinitesimal.

(i) Suppose that $f(x), g(x)$ are infinitesimals as $x \to x_0$, then for any constants α, β, $\alpha f(x) + \beta g(x)$ is also an infinitesimal as $x \to x_0$, and so does $f(x)g(x)$.

(ii) Suppose $f(x)$ is an infinitesimal as $x \to x_0$, and $g(x)$ is bounded on some deleted interval about x_0, then $f(x)g(x)$ is an infinitesimal as $x \to x_0$.

Since $\lim\limits_{x\to 0} x = 0$, and $\left|\sin\frac{1}{x}\right| \leqslant 1$, by the Proposition 1.1.1 (ii), we have

$$\lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

We know that $\lim_{x\to x_0} f(x) = a \Leftrightarrow \lim_{x\to x_0}[f(x)-a] = 0$, thus one has the following theorem.

Theorem 1.1.2 $\lim_{x\to x_0} f(x) = a \Leftrightarrow f(x)-a$ is an infinitesimal as $x \to x_0$.

Consider the following infinitesimal $x, x^2, 2x^2, x^3$ as $x \to 0$, their convergence speeds are quite different (see Table 1.1).

Table 1.1

x	10^{-1}	10^{-2}	10^{-3}	10^{-4}	$\cdots$	$\to 0$
x^2	10^{-2}	10^{-4}	10^{-6}	10^{-8}	$\cdots$	$\to 0$
$2x^2$	$2\cdot 10^{-2}$	$2\cdot 10^{-4}$	$2\cdot 10^{-6}$	$2\cdot 10^{-8}$	$\cdots$	$\to 0$
x^3	10^{-3}	10^{-6}	10^{-9}	10^{-12}	$\cdots$	$\to 0$

So we classify the infinitesimal as the following.

Definition 1.1.5 Suppose that $u(x), v(x)$ are infinitesimals as $x \to x_0$.

If $\lim_{x\to x_0} \dfrac{u(x)}{v(x)} = 0$, then we say $u(x)$ is **higher order infinitesimal(高阶无穷小量)** of $v(x)$ as $x \to x_0$. Denoted by $u(x) = o(v(x))$.

If $\lim_{x\to x_0} \dfrac{u(x)}{v(x)} = c \neq 0$, then we say that $u(x)$ is **the same order infinitesimal(同阶无穷小量)** as $v(x)$ as $x \to x_0$. Denoted by $u(x) = O(v(x))$.

If $\lim_{x\to x_0} \dfrac{u(x)}{v(x)} = 1$, then we say that $u(x)$ is **equivalent infinitesimal(等价无穷小量)** with $v(x)$ as $x \to x_0$. Denoted by $u(x) \sim v(x)$.

If $u(x)$ and $(x-x_0)^k (k>0)$ are the same order infinitesimal, then we say that $u(x)$ is k-th **order infinitesimal(k-阶无穷小量)** as $x \to x_0$.

Theorem 1.1.3 The following statements hold for equivalent infinitesimal.

(i) If $u(x) \sim v(x)$, then

$$u(x) - v(x) = o(u(x)) = o(v(x)).$$

(ii) If $u(x) \sim v(x)$, and $\lim_{x\to x_0}(v(x)w(x))$ exists, then

$$\lim_{x\to x_0}(u(x)w(x)) = \lim_{x\to x_0}(v(x)w(x)).$$

证明 (i)

$$\lim_{x\to x_0} \frac{u(x)-v(x)}{v(x)} = \lim_{x\to x_0}\left(\frac{u(x)}{v(x)} - 1\right) = 1 - 1 = 0.$$

于是得 $u(x) - v(x) = o(v(x))$. 同理可得 $u(x) - v(x) = o(u(x))$.

(ii)
$$\begin{aligned}\lim_{x\to x_0}(u(x)w(x)) &= \lim_{x\to x_0}\left(\frac{u(x)}{v(x)}\cdot v(x)w(x)\right)\\ &= \lim_{x\to x_0}\frac{u(x)}{v(x)}\cdot\lim_{x\to x_0}(v(x)w(x))\\ &= \lim_{x\to x_0}(v(x)w(x)).\end{aligned}$$
□

This result implies that equivalent infinitesimal can be changed in limit product.

Example 1.1.9 Find $\lim\limits_{x\to 0}\dfrac{\sin x-\tan x}{x^2\arctan x}$.

解 当 $x\to 0$ 时, 易得 $\sin x\sim x$, $\arctan x\sim x$ 和 $1-\cos x\sim\dfrac{1}{2}x^2$, 此时有

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x-\tan x}{x^2\arctan x} &= \lim_{x\to 0}\frac{\sin x\left(1-\dfrac{1}{\cos x}\right)}{x^3}\\ &= \lim_{x\to 0}\frac{\dfrac{1}{\cos x}(\cos x-1)}{x^2}=-\frac{1}{2}.\end{aligned}$$
□

Example 1.1.10 Find $\lim\limits_{x\to 0^+}\dfrac{1-\sqrt{\cos x}}{x(1-\cos\sqrt{x})}$.

解 当 $x\to 0$ 时, 易得 $1-\cos x\sim\dfrac{1}{2}x^2$. 此时

$$\begin{aligned}\lim_{x\to 0^+}\frac{1-\sqrt{\cos x}}{x(1-\cos\sqrt{x})} &= \lim_{x\to 0^+}\frac{1-\cos x}{\dfrac{1}{2}x(\sqrt{x})^2(1+\sqrt{\cos x})}\\ &= \lim_{x\to 0^+}\frac{\dfrac{1}{2}x^2}{2\cdot\dfrac{1}{2}x^2}=\frac{1}{2}.\end{aligned}$$
□

Definition 1.1.6 Let f be defined on an open interval about x_0, except possibly at x_0 itself. If for every $G>0$, there exists $\delta>0$, such that

$$0<|x-x_0|<\delta\Rightarrow|f(x)|>G,$$

then we called $f(x)$ is an **infinite(无穷大量)** as $x\to x_0$, written by

$$\lim_{x\to x_0}f(x)=\infty\quad\text{or}\quad f(x)\to\infty\ (x\to x_0).$$

If $|f(x)|>G$ is replaced by $f(x)>G$, we get the definition

$$\lim_{x\to x_0}f(x)=+\infty.$$

If replaced by $f(x) < -G$, we get

$$\lim_{x\to x_0} f(x) = -\infty.$$

Moreover, the limit such as

$$\lim_{x\to -\infty} f(x) = +\infty$$

can also be similarly defined.

The infinite limit has many properties similar to the usual finite limit.

Proposition 1.1.2 The following statements hold for the infinite.

(i) If $\lim\limits_{x\to x_0} f(x) = A > 0$, $\lim\limits_{x\to x_0} g(x) = \pm\infty$, then $\lim\limits_{x\to x_0} f(x)g(x) = \pm\infty$.

(ii) If $\lim\limits_{x\to x_0} f(x) = A$, $\lim\limits_{x\to x_0} g(x) = \pm\infty$, then $\lim\limits_{x\to x_0} (f(x) + g(x)) = \pm\infty$.

(iii) If $\lim\limits_{x\to x_0} f(x) = +\infty$, and $\lim\limits_{x\to x_0} g(x) = +\infty$, then $\lim\limits_{x\to x_0} (f(x) + g(x)) = +\infty$.

(iv) If $f(x) \leqslant g(x)$, $\lim\limits_{x\to x_0} f(x) = +\infty$, then $\lim\limits_{x\to x_0} g(x) = +\infty$.

(v) If $\lim\limits_{x\to x_0} f(x) = +\infty$, then $\lim\limits_{x\to x_0} (-f(x)) = -\infty$.

(vi) If $\lim\limits_{x\to x_0} f(x) = \infty$, then $\lim\limits_{x\to x_0} \dfrac{1}{f(x)} = 0$; Conversely, if $\lim\limits_{x\to x_0} f(x) = 0$, and $f(x) \neq 0$, then $\lim\limits_{x\to x_0} \dfrac{1}{f(x)} = \infty$.

1.2 极限的性质与运算 (Properties and Rules of Limits)

本节讨论函数极限的性质与运算.

Proposition 1.2.1[唯一性 (Uniqueness)] If $\lim\limits_{x\to x_0} f(x)$ exists, then it must be unique.

证明 用反证法. 不妨设 $\lim\limits_{x\to x_0} f(x) = a$, $\lim\limits_{x\to x_0} f(x) = b$, $a \neq b$. 令 $\varepsilon = \dfrac{|b-a|}{2}$. 由 $\lim\limits_{x\to x_0} f(x) = a$, 知存在 $\delta_1 > 0$, 使得

$$0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - a| < \varepsilon.$$

又由 $\lim\limits_{x\to x_0} f(x) = b$, 知存在 $\delta_2 > 0$, 使得

$$0 < |x - x_0| < \delta_2 \Rightarrow |f(x) - b| < \varepsilon.$$

进而只需令 $\delta = \min\{\delta_1, \delta_2\}$, 可得

$$0 < |x - x_0| < \delta \Rightarrow |b - a| \leqslant |f(x) - a| + |f(x) - b| < 2\varepsilon = |b - a|.$$

于是得到矛盾. 所以 $a = b$. □

Proposition 1.2.2[局部有界性 (Boundedness)] If $\lim\limits_{x\to x_0} f(x)$ exists, then $f(x)$ is bounded on some deleted interval about x_0.

证明 设 $\varepsilon = 1$. 由 $\lim\limits_{x\to x_0} f(x) = a$, 知存在 $\delta > 0$, 使得

$$0 < |x - x_0| < \delta \Rightarrow |f(x) - a| < 1.$$

进而对 $0 < |x - x_0| < \delta$, 有

$$|f(x)| \leqslant |f(x) - a| + |a| < 1 + |a|.$$ □

Proposition 1.2.3 [局部有序性 (Order Rule)] Suppose $\lim\limits_{x\to x_0} f(x) = a$, $\lim\limits_{x\to x_0} g(x) = b$ and $a < b$, then $f(x) < g(x)$ for all x in some deleted interval about x_0.

证明 设 $\varepsilon = \dfrac{b-a}{2}$. 由 $\lim\limits_{x\to x_0} f(x) = a$, 知存在 $\delta_1 > 0$, 使得

$$0 < |x - x_0| < \delta_1 \Rightarrow |f(x) - a| < \frac{b-a}{2},$$

进而

$$f(x) < a + \frac{b-a}{2} = \frac{a+b}{2}.$$

又由 $\lim\limits_{x\to x_0} g(x) = b$, 知存在 $\delta_2 > 0$, 使得

$$0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - b| < \frac{b-a}{2}.$$

于是又

$$g(x) > b - \frac{b-a}{2} = \frac{a+b}{2}.$$

只需令 $\delta = \min\{\delta_1, \delta_2\}$, 则对于 $0 < |x - x_0| < \delta$,

$$f(x) < \frac{b-a}{2} < g(x).$$ □

Corollary 1.2.1 (1) If $\lim\limits_{x\to x_0} f(x) < 0$, then $f(x) < 0$ for all x in some deleted interval about x_0.

(2) If $\lim\limits_{x\to x_0} f(x) = a$, $\lim\limits_{x\to x_0} g(x) = b$ and $f(x) \leqslant g(x)$ for all x in some deleted interval about x_0, then $a \leqslant b$.

Theorem 1.2.1 The following rules hold if $\lim\limits_{x\to x_0} f(x) = a$ and $\lim\limits_{x\to x_0} g(x) = b$:

(1) Sum Rule: $\lim\limits_{x\to x_0} [f(x) + g(x)] = a + b$;

(2) Difference Rule: $\lim\limits_{x\to x_0} [f(x) - g(x)] = a - b$;

(3) Product Rule: $\lim\limits_{x \to x_0} [f(x) \cdot g(x)] = a \cdot b$;
(4) Constant Multiple Rule: $\lim\limits_{x \to x_0} kf(x) = ka$ (any number k);

(5) Quotient Rule: $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{a}{b}, b \neq 0$.

证明 我们只证明性质 (5). 其余的可类似得到.

设 $b \neq 0$, $\dfrac{|b|}{2} > 0$. 由 $\lim\limits_{x \to x_0} g(x) = b$, 知存在 $\delta_1 > 0$ 使得 $0 < |x - x_0| < \delta_1 \Rightarrow |g(x) - b| < \dfrac{|b|}{2}$, 进而

$$|g(x)| = |g(x) - b + b| \geqslant |b| - |g(x) - b| > \frac{|b|}{2}.$$

对任意的 $\varepsilon > 0$, 存在 $\delta_2 > 0$, 使得

$$0 < |x - x_0| < \delta_2 \Rightarrow |g(x) - b| < \varepsilon.$$

又由 $\lim\limits_{x \to x_0} f(x) = a$, 知存在 $\delta_3 > 0$, 使得

$$0 < |x - x_0| < \delta_3 \Rightarrow |f(x) - a| < \varepsilon.$$

取定 $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, 则 $0 < |x - x_0| < \delta$, 进而

$$\left|\frac{f(x)}{g(x)} - \frac{a}{b}\right| = \left|\frac{bf(x) - ag(x)}{bg(x)}\right| \leqslant \frac{2}{|b|^2}[|b||f(x) - a| + |a||g(x) - b|] < \frac{2}{|b|^2}[|b| + |a|]\varepsilon.$$

对任一正数 $\dfrac{2}{|b|^2}[|b| + |a|]\varepsilon$, 结论成立. □

多项式极限 (Limits of Polynomials) If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to x_0} P(x) = P(x_0) = a_n x_0^n + a_{n-1}x_0^{n-1} + \cdots + a_0.$$

有理函数极限 (Limits of Rational Functions) If $P(x)$ and $Q(x)$ are polynomials and $Q(x_0) \neq 0$, then

$$\lim_{x \to x_0} \frac{P(x)}{Q(x)} = \frac{P(x_0)}{Q(x_0)}.$$

消零分母因子法 (Eliminating Zero Denominators Algebraically) The above theorem applies only when the denominator of the rational function is not zero at the limit point x_0. If the denominator is zero, canceling common factors in the numerator and denominator will sometimes reduce the fraction to one whose