Graduate Texts in Mathematics

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Rational Homotopy Theory

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Introduction

Homotopy theory is the study of the invariants and properties of topological spaces X and continuous maps f that depend only on the homotopy type of the space and the homotopy class of the map. (We recall that two continuous maps $f, g : X \longrightarrow Y$ are *homotopic* ($f \sim g$) if there is a continuous map $F : X \times I \longrightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. Two topological spaces X and Y have the *same homotopy type* if there are continuous maps $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ such that $fg \sim id_Y$ and $gf \sim id_X$.) The classical examples of such invariants are the singular homology groups $H_i(X)$ and the homotopy groups $\pi_n(X)$, the latter consisting of the homotopy classes of maps $(S^n, *) \longrightarrow (X, x_0)$. Invariants such as these play an essential role in the geometric and analytic behavior of spaces and maps.

The groups $H_i(X)$ and $\pi_n(X)$, $n \geq 2$, are abelian and hence can be rationalized to the vector spaces $H_i(X; \mathbb{Q})$ and $\pi_n(X) \otimes \mathbb{Q}$. Rational homotopy theory begins with the discovery by Sullivan in the 1960's of an underlying geometric construction: simply connected topological spaces and continuous maps between them can themselves be rationalized to topological spaces $X_{\mathbb{Q}}$ and to maps $f_{\mathbb{Q}} : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$, such that $H_*(X_{\mathbb{Q}}) = H_*(X; \mathbb{Q})$ and $\pi_*(X_{\mathbb{Q}}) = \pi_*(X) \otimes \mathbb{Q}$. The rational homotopy type of a CW complex X is the homotopy type of $X_{\mathbb{Q}}$ and the rational homotopy class of $f : X \longrightarrow Y$ is the homotopy class of $f_{\mathbb{Q}} : X_{\mathbb{Q}} \longrightarrow Y_{\mathbb{Q}}$, and rational homotopy theory is then the study of properties that depend only on the rational homotopy type of a space or the rational homotopy class of a map.

Rational homotopy theory has the *disadvantage* of discarding a considerable amount of information. For example, the homotopy groups of the sphere S^2 are non-zero in infinitely many degrees whereas its rational homotopy groups vanish in all degrees above 3. By contrast, rational homotopy theory has the *advantage* of being remarkably computational. For example, there is not even a conjectural description of all the homotopy groups of any simply connected finite CW complex, whereas for many of these the rational groups can be explicitly determined. And while rational homotopy theory is indeed simpler than ordinary homotopy theory, it is exactly this simplicity that makes it possible to address (if not always to solve) a number of fundamental questions.

This is illustrated by two early successes:

- (Vigué-Sullivan [152]) *If M is a simply connected compact riemannian manifold whose rational cohomology algebra requires at least two generators then its free loop space has unbounded homology and hence (Gromoll-Meyer [73]) M has infinitely many geometrically distinct closed geodesics.*

- (Allday-Halperin [3]) *If an r torus acts freely on a homogeneous space G/H (G and H compact Lie groups) then*

$$r \leq \operatorname{rank} G - \operatorname{rank} H ,$$

as well as by the list of open problems in the final section of this monograph.

The computational power of rational homotopy theory is due to the discovery by Quillen [135] and by Sullivan [144] of an explicit algebraic formulation. In each case the rational homotopy type of a topological space is the same as the *isomorphism class* of its algebraic model and the rational homotopy type of a continuous map is the same as the algebraic homotopy class of the corresponding morphism between models. These models make the rational homology and homotopy of a space transparent. They also (in principle, always, and in practice, sometimes) enable the calculation of other homotopy invariants such as the cup product in cohomology, the Whitehead product in homotopy and rational Lusternik-Schnirelmann category.

In its initial phase research in rational homotopy theory focused on the identification of rational homotopy invariants in terms of these models. These included the homotopy Lie algebra (the translation of the Whitehead product to the homotopy groups of the loop space ΩX under the isomorphism $\pi_{*+1}(X) \cong \pi_*(\Omega X)$), LS category and cone length.

Since then, however, work has concentrated on the properties of these invariants, and has uncovered some truly remarkable, and previously unsuspected phenomena. For example

- *If X is an n-dimensional simply connected finite CW complex, then either its rational homotopy groups vanish in degrees $\geq 2n$, or else they grow exponentially.*

- *Moreover, in the second case any interval $(k, k+n)$ contains an integer i such that $\pi_i(X) \otimes \mathbb{Q} \neq 0$.*

- *Again in the second case the sum of all the solvable ideals in the homotopy Lie algebra is a finite dimensional ideal R, and*

$$\dim R_{\text{even}} \leq \operatorname{cat} X_{\mathbb{Q}} \,.$$

- *Again in the second case for all elements $\alpha \in \pi_{\text{even}}(\Omega X) \otimes \mathbb{Q}$ of sufficiently high degree there is some $\beta \in \pi_*(\Omega X) \otimes \mathbb{Q}$ such that the iterated Lie brackets $[\alpha, [\alpha, \dots, [\alpha, \beta] \cdots]]$ are all non-zero.*

- *Finally, rational LS category satisfies the product formula*

$$\operatorname{cat}(X_{\mathbb{Q}} \times Y_{\mathbb{Q}}) = \operatorname{cat} X_{\mathbb{Q}} + \operatorname{cat} Y_{\mathbb{Q}} \,,$$

  in sharp contrast with what happens in the 'non-rational' case.

The first bullet divides all simply connected finite CW complexes X into two groups: the *rationally elliptic spaces* whose rational homotopy is finite dimensional, and the *rationally hyperbolic spaces* whose rational homotopy grows exponentially. Moreover, because $H_*(\Omega X; \mathbb{Q})$ is the universal enveloping algebra on the graded Lie algebra $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$, it follows from the first two bullets that whether X is rationally elliptic or rationally hyperbolic can be determined from the numbers $b_i = \dim H_i(\Omega X; \mathbb{Q})$, $1 \leq i \leq 3n - 3$, where $n = \dim X$. Rationally elliptic spaces include Lie groups, homogeneous spaces, manifolds supporting a codimension one action and Dupin hypersurfaces (for the last two see [77]). However, the 'generic' finite CW complex is rationally hyperbolic.

The theory of Sullivan replaces spaces with algebraic models, and it is extensive calculations and experimentation with these models that has led to much of the progress summarized in these results. More recently the fundamental article of Anick [11] has made it possible to extend these techniques for finite CW complexes to coefficients $\mathbb{Z}\left(\frac{1}{p_1}, \ldots, \frac{1}{p_r}\right)$ with only finitely many primes invested, and thereby to obtain analogous results for $H_*(\Omega X; \mathbb{F}_p)$ for large primes p. Moreover, the rational results originally obtained via Sullivan models often suggest possible extensions beyond the rational realm. An example is the 'depth theorem' originally proved in [54] via Sullivan models and established in this monograph (§35) topologically for any coefficients. This extension makes it possible to generalize many of the results on loop space homology to completely arbitrary coefficients.

However, for reasons of space and simplicity, in this monograph we have restricted ourselves to rational homotopy theory itself. Thus our monograph has three main objectives:

- *To provide a coherent, self-contained, reasonably complete and usable description of the tools and techniques of rational homotopy theory.*

- *To provide an account of many of the main structural theorems with proofs that are often new and/or considerably simplified from the original versions in the literature.*

- *To illustrate both the use of the technology, and the consequences of the theorems in a rich variety of examples.*

We have written this monograph for graduate students who have already encountered the fundamental group and singular homology, although our hope is that the results described will be accessible to interested mathematicians in other parts of the subject and that our rational homotopy colleagues may also find it useful. To help keep the text more accessible we have adopted a number of simplifying strategies:

- coefficients are usually restricted to fields $\Bbbk$ of characteristic zero.

- topological spaces are usually restricted to be simply connected.

- Sullivan models for spaces (and their properties) are derived first and only then extended to the more general case of fibrations, rather than being deduced from the latter as a special case.

- complex diagrams and proofs by diagram chase are almost always avoided.

Of course this has meant, in particular, that theorems and technology are not always established in the greatest possible generality, but the resulting saving in technical complexity is considerable.

It should also be emphasized that this is a monograph about topological spaces. This is important, because the models themselves at the core of the subject are strictly algebraic and indeed we have been careful to define them and establish their properties in purely algebraic terms. The reader who needs the machinery for application in other contexts (for instance local commutative algebra) will find it presented here. However the examples and applications throughout are drawn largely from topology, and we have not hesitated to use geometric constructions and techniques when this seemed a simpler and more intuitive approach.

The algebraic models are, however, at the heart of the material we are presenting. They are all graded objects with a *differential* as well as an algebraic structure (algebra, Lie algebra, module, ...), and this reflects an understanding that emerged during the 1960's. Previously objects with a differential had often been thought of as merely a mechanism to compute homology; we now know that they carry a homotopy theory which is much richer than the homology. For example, if X is a simply connected CW complex of finite type then the work of Adams [1] shows that the homotopy type of the cochain algebra $C^*(X)$ is sufficient to calculate the loop space homology $H_*(\Omega X)$ which, on the other hand, cannot be computed from the cohomology algebra $H^*(X)$. This algebraic homotopy theory is introduced in [134] and studied extensively in [20].

In this monograph there are three differential graded categories that are important:

(i) modules over a differential graded algebra (dga), (R, d).

(ii) commutative cochain algebras.

(iii) differential graded Lie algebras (dgl's).

In each case both the algebraic structure and the differential carry information, and in each case there is a fundamental modelling construction which associates to an object A in the category a morphism

$$\varphi : M \longrightarrow A$$

such that $H(\varphi)$ is an isomorphism (φ is called a *quasi-isomorphism*) and such that the algebraic structure in M is, in some sense "free".

These models (the cofibrant objects of [134]) are the exact analogue of a free resolution of an arbitrary module over a ring. In our three cases above we find, respectively:

(i) A *semi-free resolution* of a module over (R, d) which is, in particular a complex of free R-modules.

(ii) A *Sullivan model* of a commutative cochain algebra which is a quasi-isomorphism from a commutative cochain algebra that, in particular, is free as a commutative graded algebra. (These cochain algebras are called *Sullivan algebras.*)

(iii) A *free Lie model* of a dgl, which is a quasi-isomorphism from a dgl that is free as a graded Lie algebra.

These models are the main algebraic tools of the subject.

The combination of this technology with its application to topological spaces constitutes a formidable body of material. To assist the reader in dealing with this we have divided the monograph into forty sections grouped into six Parts. Each section presents a single aspect of the subject organized into a number of distinct topics, and described in an introduction at the start of the section. The table of contents lists both the titles of the sections and of the individual topics within them. Reading through the table of contents and scanning the introductions to the sections should give the reader an excellent idea of the contents.

Here we present an overview of the six Parts, indicating some of the highlights and the role of each Part within the book.

Part I: Homotopy Theory, Resolutions for Fibrations and P–local Spaces.

This Part is a self-contained short course in homotopy theory. In particular, §0 is merely a summary of definitions and notation from general topology, while §3 is the analogue for (graded) algebra. The text proper begins with the basic geometric objects, CW complexes and fibrations in §1 and §2, and culminates with the rationalization in §9 of a topological space. Since CW complexes and fibrations are often absent from an introductory course in algebraic topology we present their basic properties for the convenience of the reader. In particular, we construct a CW model for any topological space and establish Whitehead's homotopy lifting theorem, since this is the exact geometric analogue, and the motivating example, for the algebraic models referred to above.

Then, in §6, we introduce the first of these algebraic models: the semifree resolution of a module over a differential graded algebra. These resolutions are of key importance throughout the text. Now modules over a dga arise naturally in topology in at least two contexts:

- *If $f : X \longrightarrow Y$ is a continuous map then the singular cochain algebra $C^*(X)$ is a module over $C^*(Y)$ via $C^*(f)$.*
- *If $X \times G \longrightarrow X$ is the action of a topological monoid then the singular chains $C_*(X)$ are a module over the chain algebra $C_*(G)$.*

In §7 we consider the first case when f is a fibration, and use a semifree resolution to compute the cohomology of the fibre (when Y is simply connected with homology of finite type). In §8 we consider the second case when the action is that of a principal G–fibration $X \longrightarrow Y$ and use a semifree resolution to compute $H_*(Y)$. Both these results are due essentially to J.C. Moore.

The second result turns out to give an easy, fast and spectral-sequence-free proof of the Whitehead-Serre theorem that for a continuous map $f : X \longrightarrow Y$ between simply connected spaces and for $\Bbbk \subset \mathbb{Q}$, $H_*(f;\Bbbk)$ is an isomorphism if and only if $\pi_*(f) \otimes \Bbbk$ is an isomorphism. We have therefore included this as an interesting application, especially as the theorem itself is fundamental to the rationalization of spaces constructed in §9.

Aside from these results it is in Part I that we establish the notation and conventions that will be used throughout (particularly in §0–§5) and state the theorems in homotopy theory we will need to quote. Since it turned out that with the definitions and statements in place the proofs could also be included at very little additional cost in space, we indulged ourselves (and perhaps the reader) and included these as well.

Part II: Sullivan Models

This Part is the core of the monograph, in which we identify the rational homotopy theory of simply connected spaces with the homotopy theory of commutative cochain algebras. This occurs in three steps:

- *The construction in §10 of Sullivan's functor from topological spaces X to commutative cochain algebras $A_{PL}(X)$, which satisfies $C^*(X) \simeq A_{PL}(X)$.*
- *The construction in §12 of the Sullivan model*

  $$(\Lambda V, d) \xrightarrow{\simeq} (A, d)$$

  for any commutative cochain algebra satisfying $H^0(A,d) = \Bbbk$. (Here, following Sullivan ([144]), and the rest of the rational homotopy literature, ΛV denotes the free commutative graded algebra on V.)
- *The construction in §17 of Sullivan's realization functor which converts a Sullivan algebra, $(\Lambda V, d)$, (simply connected and of finite type) into a rational topological space $|\Lambda V, d|$ such that $(\Lambda V, d)$ is a Sullivan model for $A_{PL}(|\Lambda V, d|)$.*

Along the way we show that these functors define bijections:

$$\left\{\begin{array}{c}\text{rational homotopy types}\\ \text{of spaces}\end{array}\right\} \xleftrightarrow{\cong} \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{minimal Sullivan algebras}\end{array}\right\}$$

and

$$\left\{\begin{array}{c}\text{homotopy classes of}\\ \text{maps between rational spaces}\end{array}\right\} \xleftrightarrow{\cong} \left\{\begin{array}{c}\text{homotopy classes of}\\ \text{maps between minimal}\\ \text{Sullivan algebras}\end{array}\right\}$$

where we restrict to spaces and cochain algebras that are simply connected with cohomology of finite type.

Sullivan's functor A_{PL} was motivated by the classical commutative cochain algebra $A_{DR}(M)$ of smooth differential forms on a manifold. In §11 we review the construction of $A_{DR}(M)$ and prove Sullivan's result that $A_{DR}(M)$ is quasi-isomorphic to $A_{PL}(M;\mathbb{R})$. This implies (§12) that they have the same Sullivan model.

The rest of Part II is devoted to the technology of Sullivan algebras, and to geometric applications. We construct models of adjunction spaces, identity the generating space V of a Sullivan model with the dual of the rational homotopy groups and identity the quadratic part of the differential with the dual of the Whitehead product. Here the constructions are in §13 but some of the proofs are deferred to §15.

In §14 we construct relative Sullivan algebras and decompose any Sullivan algebra as the tensor product of a minimal and a contractible Sullivan algebra. In §15 we use relative Sullivan algebras to model fibrations and show (applying the result from §7) that the Sullivan fibre of the model is a Sullivan model for the fibre. Finally, in §16 this material is applied to the structure of the homology algebra $H_*(\Omega X;\Bbbk)$ of the loop space of X.

Part III: Graded Differential Algebra (Continued).

In §3 we were careful to limit ourselves to those algebraic constructions needed in Parts I and II. Now we need more: the bar construction of a cochain algebra, spectral sequences (finally, we held off as long as possible!) and some elementary homological algebra.

Part IV: Lie Models

In Part I we introduced the first of our algebraic categories (modules over a dga), in Part II we focused on commutative cochain algebras and now we introduce and study the third category: differential graded Lie algebras.

In §21 we introduce graded Lie algebras and their universal enveloping algebras and exhibit the two fundamental examples in this monograph: the homotopy Lie algebra $L_X = \pi_*(\Omega X)\otimes \Bbbk$ of a simply connected topological space, and the homotopy Lie algebra L of a minimal Sullivan algebra $(\Lambda V, d)$. The latter vector space is defined by $L_k = \operatorname{Hom}\left(V^{k+1}, \Bbbk\right)$ with Lie bracket given by the quadratic part of d. Moreover, if $(\Lambda V, d)$ is the Sullivan model for X then $L_X \cong L$.

In §22 we construct the free Lie models for a dgl, (L,d). We also construct (in §22 and §23) the classical homotopy equivalences

$$(L,d) \rightsquigarrow C^*(L,d) \quad \text{and} \quad (A,d) \rightsquigarrow \mathcal{L}_{(A,d)}$$

between the categories of dgl's (with $L = L_{\geq 1}$ of finite type) and commutative cochain algebras (with simply connected cohomology of finite type). In particular a Lie model for a *free topological space* X is a free Lie model of $\mathcal{L}_{(\Lambda V,d)}$, where $(\Lambda V, d)$ is a Sullivan model for X.

Given a dgl (L, d) that is free as a Lie algebra on generators v_i of degree n_i we show in §24 how to construct a CW complex X with a single $(n_i + 1)$-cell for each v_i, and whose free Lie model is exactly (L, d). This provides a much more geometric approach to the passage algebra $\longrightarrow$ topology then the realization functor in §17.

Finally, §24 and §25 are devoted to Majewski's theorem [119] that if (L, d) is a free Lie model for X then there is a chain algebra quasi-isomorphism $U(L, d) \xrightarrow{\simeq} C_*(\Omega X; \Bbbk)$ which preserves the diagonals up to dga homotopy.

Part V: Rational Lusternik-Schnirelmann Category

The *LS category*, $\operatorname{cat} X$, of a topological space X is the smallest number m (or infinity) such that X can be covered by $m + 1$ open sets each of which is contractible in X. In particular:

- $\operatorname{cat} X$ *is an invariant of the homotopy type of* X.
- *If* $\operatorname{cat} X = m$ *then the product of any* $m + 1$ *cohomology classes of* X *is zero.*
- *If* X *is a CW complex then* $\operatorname{cat} X \leq \dim X$ *but the inequality may be strict: indeed for the wedge of spheres* $X = \bigvee_{i=1}^{\infty} S^i$ *we have* $\dim X = \infty$ *and* $\operatorname{cat} X = 1$.

The *rational LS category*, $\operatorname{cat}_0 X$, of X is the LS category of a rational CW complex in the rational homotopy type of X.

Part V begins with the presentation in §27 of the main properties of LS category for 'ordinary' topological spaces. We have included this material here for the convenience of the reader and because, to our knowledge, much of it is not available outside the original articles scattered through the research literature.

We then turn to rational LS category (§28) and its calculation in terms of Sullivan models (§29). A key point is the Mapping Theorem: *Given a continuous map* $f : X \longrightarrow Y$ *between simply connected spaces, then*

$$\pi_*(f) \otimes \mathbb{Q} \quad \textit{injective} \quad \Rightarrow \operatorname{cat}_0 X \leq \operatorname{cat}_0 Y \ .$$

In particular, the Postnikov fibres in a Postnikov decomposition of a simply connected finite CW complex all have finite rational LS category. (The integral analogue is totally false!).

A second key result is Hess' theorem (Mcat = cat), which is the main step in the proof of the product formula $\operatorname{cat} X_{\mathbb{Q}} \times Y_{\mathbb{Q}} = \operatorname{cat} X_{\mathbb{Q}} + \operatorname{cat} Y_{\mathbb{Q}}$ in §30. Finally, in §31 we prove a beautiful theorem of Jessup which gives circumstances under which the rational LS category of a fibre must be strictly less than that of the total space of a fibration. The "α, β" theorem described at the start of this introduction is an immediate corollary.

Part VI: The Rational Dichotomy: Elliptic and Hyperbolic Spaces AND Other Applications

In this Part we use rational homotopy theory to derive the results referred to at the start of this introduction (and others) on the structure of $H_*(\Omega X; \mathbb{k})$, when X is a simply connected finite CW complex. These are outlined in the introductions to the sections, and we leave it to the reader to check there, rather than repeating them here.

As the overview above makes evident, this monograph makes no pretense of being a complete account of rational homotopy theory, and indeed important aspects have been omitted. For example we do not treat the iterated integrals approach of Chen ([37], [79], [145]) and therefore have not been able to include the deep applications to algebraic geometry of Hain and others (e.g. [80], [81], [101]). Equivariant rational homotopy theory as developed by Triantafillou and others ([151]) is another omission, as is any serious effort to treat the non-simply connected case, even though at least nilpotent spaces are covered by Sullivan's original theory. We have not described the Sullivan-Haefliger model ([144], [78]) for the section space of a fibration even in the simpler case of mapping space, except for the simple example of the free loop space X^{S^1}, nor have we included the Sullivan-Barge classification ([144], [18]) of closed manifolds up to rational homotopy type. And we have not given Lemaire's construction [108] of a finite CW complex whose homotopy Lie algebra is not finitely generated as a Lie algebra.

Moreover, this monograph does not pursue the connections outside or beyond rational homotopy theory. Such connections include the algebraic homotopy theory developed by Baues [20] following Quillen's homotopical algebra [134]. There is no mention in the text (except in the problems at the end) of Anick's extension of the theory to coefficients with only finitely many primes inverted ([11]) and its application to loop space homology,and there is equally no mention of how the results in Part VI generalize to arbitrary coefficients [56]. And finally, we have not dealt with the interaction with the homological study of local commutative rings [14] that has been so significantly exploited by Avramov and others.

We regret that limitations of time and energy (as well as our publisher's insistence on limiting the number of pages!) have made it necessary simply to refer the reader to the literature for these important aspects of the subject, in the hope that what is presented here will make that task an easier one.

In the last twenty five years a number of monographs have appeared that presented various parts of rational homotopy theory. These include Algèbres Connexes et Homologie des Espaces de Lacets by Lemaire [109], On PL de Rham Theory and Rational Homotopy Type by Bousfield and Gugenheim ([30]), Théorie Homotopique des Formes Différentielles by Lehmann ([107]), Rational Homotopy Theory and Differential Forms by Griffiths and Morgan ([72]), Homotopie Rationnelle: Modèles de Chen, Quillen, Sullivan by Tanré ([145]), Lectures on Minimal Models by Halperin [82], La Dichotomie Elliptique - Hyperbolique en Homotopic Rationnelle by Félix ([50]), and Homotopy Theory and Models by Aubry ([12]). Our hope is that the present work will complement the real contribution these make to the subject.

This monograph brings together the work of many researchers, accomplished as a co-operative effort for the most part over the last thirty years. A clear account of this history is provided in [91], and we would like merely to indicate here a few of the high points. First and foremost we want to stress our individual and collective appreciation to Daniel Lehmann, who led the development of the rational homotopy group at Lille that provided the milieu in which all three of us became involved in the subject. Secondly, we want to emphasize the importance of the memoir [109] by Lemaire and of the article [21] by Baues and Lemaire which have formed the foundation for the use of Lie models.

In this context the mini-conference held at Louvain in 1979 played a key role. It brought together the two approaches (Lie and Sullivan) and crystalized the questions around LS category that proved essential in subsequent work. Another mini-conference in Bonn in 1981 (organized jointly by Baues and the second author) led to a trip to Sofia to meet Avramov and the intensification of the infusion into rational homotopy of the intuition from local algebra begun by Anick, Avramov, Löfwall and Roos.

This monograph was conceived of in 1992 and in the intervening eight years we have benefited from the advice and suggestions of countless colleagues and students. It is a particular pleasure to acknowledge the contributions of Cornea, Dupont, Hess, Jessup, Lambrechts and Murillo, who have all worked with us as students or postdocs and have all beaten problems we could not solve. We also wish to thank Peter Bubenik whose careful reading uncovered an unbelievable number of mistakes, both typographical and mathematical.

The actual writing and rewriting have been a team effort accomplished by the three of us working together in intensive sessions at sites that rotated through the campuses at which we have held faculty positions: the Université Catholique de Louvain (Felix), the University of Toronto and the University of Maryland (Halperin) and the Université de Lille 1 and the Université d'Angers (Thomas). The Fields Institute for Research in Mathematical Sciences in Toronto provided us with a common home during the spring of 1996, and the Centre International pour Mathématiques Pures et Appliquées, hosted and organized a two week summer school at Breil where we presented a first version of the text. Our granting agencies (Centre National de Recherche Scientifique, FNRS, National Sciences and Engineering Research Council of Canada, North Atlantic Treaty Organization) all provided essential financial support. To all of these organizations we express our appreciation.

Above all, however, we wish to express our deep gratitude and appreciation to Lucile Lo, who converted thousands of pages of handwritten manuscript to beautifully formatted final product with a speed, accuracy, intelligence and good humor that are unparalleled in our collective experience.

And finally, we wish to express our appreciation to our families who have lived through this experience, and most especially to Agnes, Danielle and Janet for their constant nurturing and support, and to whom we gratefully dedicate this book.

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