天元基金 影 印 系 列 丛 书

Olle Stormark 著

偏微分方程组中的李结构法

Lie's Structural Approach to PDE Systems

清华大学出版社

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Originally published by Cambridge University Press in 2000.
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原书由英国剑桥大学出版社于 2000 年出版，本影印版得到剑桥大学出版社授权。

Lie's Structural Approach to PDE Systems
Olle Stormark
EISBN: 0-521-78088-8

北京市版权局著作权合同登记号 图字：01-2004-1912

图书在版编目(CIP)数据

偏微分方程组中的李结构法 = Lie's Structural Approach to PDE Systems：英文 /斯托尔马克（Stormark, O.）著. —影印本. —北京：清华大学出版社，2004.8
（天元基金影印系列丛书）
ISBN 7-302-09087-4

Ⅰ. 偏… Ⅱ. 斯… Ⅲ. ①偏微分方程－研究－英文 ②局部李群－研究－英文 Ⅳ. ①O175.2 ②O152.5

中国版本图书馆 CIP 数据核字（2004）第 072239 号

出 版 者：清华大学出版社　　地　　址：北京清华大学学研大厦
http://www.tup.com.cn　　邮　　编：100084
社总机：(010) 6277 0175　　客户服务：(010) 6277 6969
责任编辑：刘晓艳
印 装 者：北京昌平环球印刷厂
发 行 者：新华书店总店北京发行所
开　　本：185×230　印 张：36.75
版　　次：2004 年 8 月第 1 版　2004 年 8 月第 1 次印刷
书　　号：ISBN 7-302-09087-4/O・379
印　　数：1~1000
定　　价：75.00 元

本书如存在文字不清、漏印以及缺页、倒页、脱页等印装质量问题，请与清华大学出版社出版部联系调换。联系电话：（010）62770175-3103 或（010）62795704

Preface

> Das Ziel der Wissenschaft ist einerseits, neue Tatsachen zu erobern, anderseits, bekannte unter höheren Gesichtspunkten zusammenfassen.
>
> S. Lie

> One of the principal objects of theoretical research is to find the point of view from which the subject appears in its greatest simplicity.
>
> J.W. Gibbs

> Good mathematics = down-to-earth mathematics.
>
> A. Andreotti

Everyone knows that the theory of PDE systems is enormously rich in results—but *what about foundations*? This monograph describes one approach, but there is no claim that it is the only one, or that it is the best possible; just that there is one, and moreover one that has been around for a long time without having been as recognized as it deserves.

The study is restricted to local solvability. If a PDE system $\mathcal{S}$ is defined in a domain $\mathcal{D}$, and if it can be shown that $\mathcal{S}$ possesses local solutions at each point of $\mathcal{D}$, the question of global solvability boils down to whether it is possible to glue together local solutions in order to form global ones—in analogy with the cohomology theory for coherent analytic sheaves. But *first of all one has to know that there are local solutions.*

A major idea is to regard PDE theory from the point of view of *differential geometry*—rather than basing it on analysis, say.

To illustrate this, let us first find a suitable angle from which ODE systems appear in a simple way.

Example. Look for functions $y(t)$, $z(t)$ satisfying the ODE system

$$\begin{cases} y'' = f(t, y, z, y', z'), \\ z'' = g(t, y, z, y', z'). \end{cases}$$

With $x^1 := y$, $x^2 := z$, $x^3 := y'$ and $x^4 := z'$, this goes over into the first order system

$$\begin{cases} dx^1/dt = x^3, \\ dx^2/dt = x^4, \\ dx^3/dt = f(t, x^1, x^2, x^3, x^4), \\ dx^4/dt = g(t, x^1, x^2, x^3, x^4). \end{cases}$$

Therefore the key problem in ODE theory consists in solving first order systems

$$\begin{cases} dx^1/dt = f^1(t, x^1, x^2, \dots, x^n), \\ \dots \\ dx^n/dt = f^n(t, x^1, x^2, \dots, x^n). \end{cases} \tag{*}$$

The classical existence and uniqueness theorem says that for smooth f^k there is precisely one local solution $(x^1(t), \dots, x^n(t))$ satisfying initial data $(x^1(t_0), \dots, x^n(t_0)) = (x_0^1, \dots, x_0^n)$.

In order to make this conceptually clearer, rewrite (*) as a pfaffian system:

$$\begin{cases} dx^1 - f^1(t, x)\, dt = 0, \\ \dots \\ dx^n - f^n(t, x)\, dt = 0, \end{cases}$$

where $x = (x^1, \dots, x^n)$. Setting

$$\theta^k := dx^k - f^k(t, x)\, dt \quad \text{for} \quad k = 1, \dots, n,$$

a solution $(x^1(t), \dots, x^n(t))$ is a function whose graph

$$\mathbb{R}_t \ni t \mapsto (t, x^1(t), \dots, x^n(t)) \in \mathbb{R}_t \times \mathbb{R}_x^n$$

is an integral curve of the pfaffian system $\theta^1 = \cdots = \theta^n = 0$ on $\mathbb{R}_t \times \mathbb{R}_x^n$.

The dual of the latter consists of those vector fields

$$V = a\frac{\partial}{\partial t} + \sum_{k=1}^{n} a^k \frac{\partial}{\partial x^k}$$

which satisfy $0 = \theta^m(V) = a^m - a \cdot f^m(t,x)$ for $m = 1, \dots, n$. Consequently they are all multiples of the vector field

$$X = \frac{\partial}{\partial t} + \sum_{k=1}^{n} f^k(t,x)\frac{\partial}{\partial x^k}.$$

Then $(x^1(t), \dots, x^n(t))$ is a solution of (*) if and only if its graph is an integral curve of X.

Thus solving a first order ODE system is equivalent to finding the integral curves of a vector field. The latter may be done locally by introducing new coordinates making the vector field maximally simple.

The local rectification lemma. *Let*

$$X = \frac{\partial}{\partial t} + \sum_{k=1}^{n} f^k(t,x)\frac{\partial}{\partial x^k}$$

be a vector field defined near the origin of $\mathbb{R}_t \times \mathbb{R}^n_x$. *Then there is a local diffeomorphism*

$$\phi : \mathbb{R}_s \times \mathbb{R}^n_y \xrightarrow{\cong} \mathbb{R}_t \times \mathbb{R}^n_x$$

with $\phi(0) = 0$ *and* $\phi_*(\partial/\partial s) = X$.

This means that the integral curves of X are given by

$$y^k(t,x) = \text{constant} \qquad \text{for } k = 1, \dots, n.$$

From a technical point of view this theorem is equivalent to the classical local existence theorem for first order ODE systems—but the advantage is that it is much more appealing to the intuition (if you agree with this, read on; otherwise stop right here).

Conclusion. *Solving an ODE system is equivalent to rectifying a vector field—which always can be done locally in the smooth category.*

Since this result is most satisfying, it is natural to ask if something similar works for general PDE systems.

Question. *Is it possible to geometrize PDE systems in an analogous way?*

The first step consists in rewriting an arbitrary PDE system as a pfaffian system, or—dually—a vector field system.

To get the idea, consider a first order ODE:

$$F(x, y, y') = 0.$$

The graph of a solution is given by

$$x \mapsto (x, y(x)).$$

Analogously the 1-graph is given by

$$x \mapsto (x, y(x), y'(x)),$$

the 2-graph by

$$x \mapsto (x, y(x), y'(x), y''(x)),$$

and so on. Let us concentrate on the 1-graph, which is a curve in a 3-dimensional space $J^1(\mathbb{R}_x, \mathbb{R}_y)$ with coordinates x, y and p, say:

$$\begin{cases} x = x, \\ y = y(x), \\ p = y'(x). \end{cases}$$

Note that on this 1-graph

$$dy - p\,dx = 0.$$

The one-form $\theta^0 = dy - p\,dx$ appearing here is called the *contact form.*

Conversely, let c be a curve in the (x, y, p)-space on which x can be used as a local coordinate, and suppose that θ^0 vanishes on c. Then c is of the form

$$x \mapsto (x, f(x), g(x))$$

with

$$0 = df - g\,dx = (f'(x) - g(x))\,dx, \qquad \text{i.e.,} \qquad g(x) = f'(x).$$

But this means that c is the 1-graph of the function $f(x)$.

To the ODE $F(x, y, y') = 0$ corresponds the hypersurface $\boldsymbol{M}$ in the (x, y, p)-space defined by

$$F(x, y, p) = 0.$$

Clearly solutions of $F(x, y, y') = 0$ will correspond to 1-graphs contained in $\boldsymbol{M}$, that is, to curves in $\boldsymbol{M}$ satisfying the two requirements

(i) x can be used as a local coordinate on the curve,
(ii) $\theta^0 \equiv dy - p\,dx = 0$ vanishes on the curve.

Let us assume M to be smooth, and set

$$\theta := \theta^0{}_{|M} = (dy - p\,dx)_{|M},$$

so that θ is a one-form on M. Then the solutions of $F(x, y, y') = 0$ correspond precisely to the integral curves of θ on which $dx \neq 0$. If the latter condition is not fulfilled, the integral curve is said to represent a *generalized solution* of the ODE.

Conclusion. *The ODE $F(x, y, y') = 0$ can be regarded as a manifold M equipped with a one-form θ. The solutions of the ODE correspond to integral curves of θ.*

Let us now play the same game for a q^{th} order PDE system $\mathcal{S}$ in n independent variables $x^1, \ldots, x^n$ and m dependent variables $z^1, \ldots, z^m$:

$$\mathcal{S}: \quad F^a\left(x^1, \ldots, x^n; z^1, \ldots, z^m; \ldots, \frac{\partial^k z^j}{\partial x^{i_1} \ldots \partial x^{i_k}}, \ldots\right) = 0,$$

with $a = 1, \ldots, A$ and $k \leq q$. To this set-up corresponds the jet space $J^q(\mathbb{R}^n_x, \mathbb{R}^m_z)$ with coordinates

$$x^1, \ldots, x^n, z^1, \ldots, z^m, \ldots, p^j_{i_1 \ldots i_k}, \ldots,$$

where $p^j_{i_1 \ldots i_k}$ is associated to the derivative

$$\frac{\partial^k z^j}{\partial x^{i_1} \ldots \partial x^{i_k}}.$$

$\mathcal{S}$ corresponds to the subset

$$M: \quad F^a(x^1, \ldots x^n; z^1, \ldots, z^m; \ldots, p^k_{i_1 \ldots i_k}, \ldots) = 0$$

of $J^q(\mathbb{R}^n_x, \mathbb{R}^m_z)$; let us assume M to be smooth.

An n-dimensional submanifold $\mathcal{N}$ of $J^q(\mathbb{R}^n_x, \mathbb{R}^m_z)$ is a q-graph if and only if

(i) $dx^1 \wedge \cdots \wedge dx^n \neq 0$ on $\mathcal{N}$,

(ii) $\mathcal{N}$ is an integral manifold of the contact ideal ${}^qCt^{n,m}$ generated by the one-forms

$$\begin{cases} dz^j - \sum_{i_1=1}^n p^j_{i_1}\, dx^{i_1}, \\ dp^j_{i_1} - \sum_{i_2=1}^n p^j_{i_1 i_2}\, dx^{i_2}, \\ \ldots \\ dp^j_{i_1 \ldots i_{q-1}} - \sum_{i_q=1}^n p^j_{i_1 i_2 \ldots i_q}\, dx^{i_q}. \end{cases}$$

Let $\mathcal{P}$ denote the restriction of ${}^qCt^{n,m}$ to M. Then

> *the set of solutions of $\mathcal{S}$ correspond precisely to the set of n-dimensional integral manifolds of $\mathcal{P}$ on which $dx^1 \wedge \cdots \wedge dx^n \neq 0$.*

Define the dual vector field system $\mathcal{V} = \mathcal{P}^\perp$ on M by

$$X \in \mathcal{V} \Longleftrightarrow \theta(X) = 0 \text{ for all } \theta \in \mathcal{P}.$$

Then

> *solving the PDE system $\mathcal{S}$ is equivalent to finding all n-dimensional integral manifolds of $\mathcal{V}$ on which $dx^1 \wedge \cdots \wedge dx^n \neq 0$.*

Again, suspending the last condition gives *generalized solutions* of $\mathcal{S}$. The classical terminology for finding integral manifolds of a pfaffian or vector field system is to *integrate* the system.

Conclusion. *Solving a PDE system is equivalent to integrating a pfaffian system or its dual vector field system.*

Consider the vector field version: let $\mathcal{V} = (X_1, \ldots X_q)$ be the vector field system generated by the vector fields $X_1, \ldots, X_q$ over the ring of smooth functions on the manifold M. Shrinking M if necessary, the X_i can be assumed to be linearly independent everywhere. The *derived system* $\mathcal{V}'$ is generated by the X_i and their Lie brackets $[X_i, X_j]$; say that $\mathcal{V}' = (X_1, \ldots, X_q; Z_1, \ldots, Z_p)$. Then the *structure equations* of $\mathcal{V}$ are given by

$$[X_i, X_j] \equiv \sum_{k=1}^{p} c_{ij}^k Z_k \pmod{\mathcal{V}} \qquad \text{for } i, j = 1, \ldots r,$$

with certain *structure functions* c_{ij}^k. The latter depend on the basis of $\mathcal{V}$, and it is natural to choose one that kills as many c_{ij}^k as possible. The resulting set is called the *Lie structure* of $\mathcal{V}$, and the general claim is

> *the integrability properties of $\mathcal{V}$ are governed by the Lie structure!*

One reason for this is Cartan's local existence theorem for integral manifolds. The proof consists of two parts: first linear algebra applied to the structure equations gives all *involutions*, and then these are specialized to *integral manifolds* by repeated applications of the Cauchy–Kowalewski theorem. Unfortunately the latter requires power series, and is not valid in the C^∞ category.

Questions:

(i) *What is required in order to obtain integral manifolds without using power series?*

(ii) *Is there a natural intrinsic property from which more information than just existence can be derived?*

In order to tackle these it is convenient to use the classification of Drach: any PDE system is equivalent to either a first or a second order PDE system in one dependent variable.

First order systems in one dependent variable are easily understood by means of Lie's structural methods, so the real difficulties start with second order systems.

The main part of the monograph is devoted to the study of second order PDE systems in one dependent and two or three independent variables. When considering these, one is naturally led to the notion of *Monge characteristic subsystems*, which in their turn provide the following partial answer to the questions above:

> *If a vector field system admits Monge systems with enough first integrals, then it is possible to find integral manifolds without using the Cauchy–Kowalewski theorem. Moreover these Monge systems yield a lot of interesting information beyond that of pure existence—in particular as regards classifications.*

A big surprise is that looking at second order PDE systems in one dependent variable from this angle, the theory will be dominated by local Lie groups and Lie pseudogroups.

Extending these methods to the case of more than three independent variables would be quite complicated, but surely not impossible. And anyway, who would expect the general PDE theory to be simple?

Most topics treated here can be found in the classical works of Lie, Cartan and Vessiot. However, a great effort has been made to present the ideas in a *unified* and very *simple* manner. In order not to obscure the fundamental issues it has been left to the interested reader to fill in the details needed to obtain his or her own desired level of rigour.

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1

Introduction and summary

This is the story of a geometric approach to the theory of PDE systems, initiated by Sophus Lie and developed by two of his disciples, Élie Cartan and Ernest Vessiot.

In chapter **2** it is explained how any decent PDE system $\mathcal{S}$ can be considered as a submanifold of an appropriate jet bundle. The latter is equipped with its canonical contact pfaffian system, the restriction of which to $\mathcal{S}$ makes $\mathcal{S}$ a *manifold with a pfaffian system* $\mathcal{P}$ or, dually, a *manifold with a vector field system* $\mathcal{V}$. The problem of solving the PDE system $\mathcal{S}$ then goes over into that of finding *integral manifolds* of $\mathcal{P}$ (or $\mathcal{V}$) of a prescribed dimension.

Of the three heroes of our tale, Lie and Vessiot favoured the vector field approch, while Cartan is the great champion of differential forms. For our purposes it is important to be able to use both approaches and to have a complete duality, so that each concept for vector field systems has a counterpart for pfaffian systems, and vice versa.

As we are only interested in local properties, $\mathcal{S}$ is assumed to be a small open subset of $\mathbb{R}^r$ (or $\mathbb{C}^r$), and $\mathcal{V}$ is supposed to be generated by vector fields $X_1, \ldots ,X_q$, independent everywhere on $\mathcal{S}$: $\mathcal{V} = (X_1, \ldots, X_q)$.

The simplest case occurs when $\mathcal{V}$ is complete with respect to Lie brackets, that is,

$$[X_i, X_j] \equiv 0 \pmod{\mathcal{V}} \quad \text{for } i, j = 1, \ldots, r.$$

According to the Frobenius theorem it is in this case possible to introduce local coordinates $x^1, \ldots, x^r$ such that $\mathcal{V} = (\partial/\partial x^1, \ldots, \partial/\partial x^q)$, whence the integral manifolds are given by

$$x^{q+1} = \text{constant}, \ldots, x^r = \text{constant}.$$

With the derived system $\mathcal{V}'$ being generated by the X_i and their Lie brackets $[X_i, X_j]$, the Frobenius condition can equivalently be written as $\mathcal{V}' = \mathcal{V}$.

The general case with $\mathcal{V}' \supsetneq \mathcal{V}$ is solved in chapter **3** by means of Cartan's local existence theorem. The key idea is to first look for *maximal involutions*—with an involution being a subsystem $\mathcal{I}$ of the vector field system $\mathcal{V}$ satifying $[\mathcal{I}, \mathcal{I}] \subseteq \mathcal{V}$. Then these involutions are specialized to *complete subsystems*, i.e., subsystems $\mathcal{W}$ of $\mathcal{V}$ with $[\mathcal{W}, \mathcal{W}] \subseteq \mathcal{W}$. Thereupon the Frobenius theorem yields the wanted integral manifolds.

The step from involutions to complete subsystems is based upon a repeated application of the Cauchy–Kowalewski theorem, which unfortunately requires analyticity.

Often one wants n-dimensional integral manifolds on which $\omega^1 \wedge \cdots \wedge \omega^n \neq 0$, where the ω^i are given one-forms. If the general n-dimensional involution $\mathcal{I}_n$ satisfies $\omega^1 \wedge \cdots \wedge \omega^n|_{\mathcal{I}_n} \neq 0$, $\mathcal{V}$ is said to be involutive with respect to the ω^i. In this case Cartan's procedure yields integral manifolds of the kind wanted.

Any vector field system $\mathcal{V}$ can be *prolonged* to a vector field system $\mathcal{V}^{(1)}$ on a higher dimensional manifold, and this in turn can be prolonged to $\mathcal{V}^{(2)}$, and so on. Moreover there is a one-to-one correspondence between the integral manifolds of $\mathcal{V}$ and those of $\mathcal{V}^{(k)}$ for $k = 1, 2, 3, \ldots$.

Chapter **4** sketches the prolongation theorem of Cartan and Janet, which says that by a *finite* number of prolongations it is possible to conclude either that some $\mathcal{V}^{(m)}$ is involutive with respect to $\omega^1 \wedge \cdots \wedge \omega^n$—in which case the wanted integral manifolds are given by Cartan's existence theorem—or that $\mathcal{V}$ does not admit any integral manifold on which $\omega^1 \wedge \cdots \wedge \omega^n \neq 0$.

Drach observed that any PDE system is equivalent to either a first or a second order PDE system in one dependent variable. In chapter **5** we take a preliminary look at a single second order PDE in one dependent variable, and in particular investigate the presence of *singular* vector fields—that is, vector fields in $\mathcal{V}$ commuting modulo $\mathcal{V}$ with a greater number of vector fields than the average one does. There turn out to be either exactly two singular subsystems of $\mathcal{V}$, or none at all. If there are such, the PDE is *hyperbolic* if they are different, and *parabolic* if they coincide.

These observations give rise to the notion of *Monge characteristic subsystems* of the vector field system $\mathcal{V}$: a subsystem $\mathcal{M}$ of $\mathcal{V}$ is Monge if

(i) $\mathcal{M}$ is singular,
(ii) $\mathcal{M} \cap \mathcal{I} \neq 0$ for any maximal involution $\mathcal{I}$ of $\mathcal{V}$, and

(iii) $\mathcal{M} \cap \mathcal{W}$ is complete for any maximal complete subsystem $\mathcal{W}$ of $\mathcal{V}$.

The integral manifolds of $\mathcal{M} \cap \mathcal{W}$ are called *Monge characteristics.*

A special case is the *Cauchy charateristic subsystem* $\mathcal{C}(\mathcal{V})$ *of* $\mathcal{V}$:

$$\mathcal{C}(\mathcal{V}) := \{X \in \mathcal{V} \mid [X, \mathcal{V}] \subseteq \mathcal{V}\}.$$

$\mathcal{C}(\mathcal{V})$ is complete, and is included in any maximal complete subsystem of $\mathcal{V}$.

In chapter **6** we consider the integration of vector field systems satisfying $\dim \mathcal{V}' = \dim \mathcal{V} + 1$, which includes first order PDE systems in one dependent variable as a special case. Such a vector field system is essentially equivalent to a single pfaffian equation $\omega = 0$, which is solved by putting it into a canonical form:

$$\omega = 0 \Longleftrightarrow dz - \sum_{i=1}^{n} p_i \, dx^i = 0.$$

The reduction procedure accomplishing this is a good demonstration of how powerful Lie's ideas are.

By Drach's classification there then remains to consider second order PDE systems in one dependent and n independent variables; the remainder of the monograph is devoted to the cases $n = 2$ and 3. The main method is

look for Monge systems and their first integrals!

At the outset there is no consideration at all of groups—but they will turn up anyway. The first example is the Lie pseudogroup of contact transformations, which consists of all local diffeomorphisms of the jet bundle $J^1(\mathbb{R}^n_x, \mathbb{R}_z)$ preserving the pfaffian equation $dz - \sum_{i=1}^{n} p_i \, dx^i = 0$. A general Lie pseudogroup is a family of local diffeomorphisms constituting the general solution of some PDE system, and being closed under composition whenever this is defined.

Chapter **7** discusses higher order contact transformations and prolongations of local diffeomorphisms to jet bundles.

In chapter **8** the general solution of the defining PDE system is supposed to depend on a *finite* number of parameters only, in which case the Lie pseudogroup is called a *local Lie group.* Acting on $\mathbb{C}^n$ and having r parameters, its elements are given by local diffeomorphisms

$$(x^1, \ldots, x^n) \mapsto (f^1(x^1, \ldots, x^n; a^1, \ldots, a^r), \ldots, f^n(x^1, \ldots, x^n; a^1, \ldots, a^r)),$$