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Ivo Petráš

Fractional-Order Nonlinear Systems

Modeling, Analysis and Simulation

分数维非线性系统

建模、分析与仿真

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With 119 figures

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This book is dedicated to my entire family, my parents Štefan and Erika, and my sister Petra for their understanding and support.

Preface

The aim of the book is to present a survey of a new class of chaotic systems, the so-called fractional-order chaotic systems. This book can also be used as a textbook for courses related to nonlinear systems, fractional-order systems, etc. The book is suitable for advanced undergraduate and graduate students. It is a sort of a guide to fractional-order chaotic systems that features material from original research papers, including the author's own studies. The book is organized as follows:

Chapter 1 is a brief introduction to fractional-order chaotic systems.

Chapter 2 provides fundamentals of fractional calculus, its properties and integral transfer methods. Three well-known definitions of fractional derivatives/integrals and methods for their numerical approximation are presented.

Chapter 3 includes a presentation of fractional-order systems, their description and properties. Fractional linear time-invariant (LTI), nonlinear systems, and fractional-order controllers are considered.

Chapter 4 is devoted to stability of the fractional-order (LTI and nonlinear) systems. The stability of interval fractional-order system is also investigated.

Chapter 5 contains a survey of various fractional-order chaotic systems with the total order less than three. The well-known systems such as, for example, Chua's oscillator, Lorenz's system, Rössler's system, Duffing's system, and some other systems, for instance, Volta's system, are analyzed as well.

Chapter 6 begins with the introduction to control strategies of the fractional-order chaotic systems. Three general approaches: feed-back control, sliding mode control, and synchronization, are described. Other strategies are mentioned and discussed as well.

Chapter 7 concludes this book by some additional remarks.

Appendix A lists Matlab functions used for simulation of the fractional-order chaotic systems described in Chapter 5.

Appendix B lists a Laplace transform and inverse Laplace transform table of the functions used in fractional calculus.

Košice, April, 2010 *Ivo Petráš*

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Besides new results, this book also presents results which have been already published by the author in several journal papers and conference articles during last almost 10 years. Because a portion of the material from these publications is reused in this book, the copyright permission was granted from several publishers.

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Acronyms

A list of symbols and abbreviations used in the book:

${}_aD_t^\alpha$	fractional-order derivative/integral
$\mathbf{J}$	Jacobian matrix
T_{sim}	simulation time
L_m	memory length
T	sampling period
h	time step of calculation
$\mathbf{I}$	identity matrix
det	determinant
eig	eigenvalue
arg	argument of complex number
E^*	equilibrium point
Γ	Euler's gamma function
$E_{\alpha,\beta}(z)$	Mittag-Leffler function
[.]	integer part
B_p	breakpoint
sign	signum
$\approx$	approximation

GL	Grünwald-Letnikov definition
RL	Riemann-Liouville definition
PSE	Power Series Expansion
CFE	Continued Fraction Expansion
FIR	Finite Impulse response
IIR	Infinite Impulse Response
ORA	Oustaloup Recursive Approximation
PID	Proportional-Integral-Derivative
FOC	Fractional-Order Controller
CRONE	Commande Robuste Ordre Non Entier
TID	Tilt-Integral-Derivative
CPE	Constant Phase Element
LTI	Linear Time Invariant
FODE	Fractional-Order Differential Equation
FOS	Fractional-Order System
FOLTI	Fractional-Order Linear Time Invariant
BIBO	Bounded-Input Bounded-Output
LMI	Linear Matrix Inequality
FDEG	Fractional Degree
LCM	Least Common Multiple
VPO	Van der Pol
FrVPO	Fractional Van der Pol
CNN	Cellular Neural Network
LE	Lyapunov Exponent
MS	Memristive System
NMR	Nuclear Magnetic Resonace

Contents

Chapter 1
Introduction

Fractional calculus is a topic being more than 300 years old. The idea of fractional calculus has been known since the regular calculus, with the first reference probably being associated with Leibniz and L'Hospital in 1695 where half-order derivative was mentioned. In a correspondence between Johann Bernoulli and Leibniz in 1695, Leibniz mentioned the derivative of general order. In 1730 the subject of fractional calculus did not escape Euler's attention. J. L. Lagrange in 1772 contributed to fractional calculus indirectly, when he developed the law of exponents for differential operators. In 1812, P. S. Laplace defined the fractional derivative by means of integral and in 1819 S. F. Lacroix mentioned a derivative of arbitrary order in his 700-page long text, followed by J. B. J. Fourier in 1822, who mentioned the derivative of arbitrary order. The first use of fractional operations was made by N. H. Abel in 1823 in the solution of tautochrome problem. J. Liouville made the first major study of fractional calculus in 1832, where he applied his definitions to problems in theory. In 1867, A. K. Grünwald worked on the fractional operations. G. F. B. Riemann developed the theory of fractional integration during his school days and published his paper in 1892. A. V. Letnikov wrote several papers on this topic from 1868 to 1872. Oliver Heaviside published a collection of papers in 1892, where he showed the so-called Heaviside operational calculus concerned with linear generalized operators. In the period of 1900 to 1970 the principal contributors to the subject of fractional calculus were, for example, H. H. Hardy, S. Samko, H. Weyl, M. Riesz, S. Blair, etc. From 1970 to the present, they are for instance J. Spanier, K. B. Oldham, B. Ross, K. Nishimoto, O. Marichev, A. Kilbas, H. M. Srivastava, R. Bagley, K. S. Miller, M. Caputo, I. Podlubny, and many others (Cafagna, 2007; Miller and Ross, 1993).

At present, the number of applications of fractional calculus rapidly grows. These mathematical phenomena allow us to describe and model a real object more accurately than the classical "integer" methods. The real objects are generally fractional (Nakagava and Sorimachi, 1992; Oustaloup, 1995; Podlubny, 1999a; Westerlund, 2002), however, for many of them, the fractionality is very low. A typical example of a non-integer (fractional) order system is the voltage-current relation of a semi-infinite lossy transmission line (Wang, 1987) or diffusion of heat through a semi-infinite solid, where the heat flow is equal to the half-derivative of the temperature (Podlubny, 1999a).

The main reason for using integer-order models was the absence of solution methods for fractional differential equations. At present, there are many methods for approximation of the fractional derivative and integral, and fractional calculus can be easily used in wide areas of applications. Fractional-order calculus has played an important role in physics (Parada et al., 2007; Torvik and Bagley, 1984), electrical engineering (electrical circuits theory and fractances) (Arena et al., 2000; Bode, 1949; Carlson and Halijak, 1964; Nakagava and Sorimachi, 1992; Westerlund, 2002), control systems (Axtell and Bise, 1990; Dorčák, 1994; Podlubny, 1999b; Oustaloup, 1995), robotics (Marcos et al., 2008), signal processing (Tseng, 2007; Vinagre et al., 2003), chemical mixing (Oldham and Spanier, 1974), bioengineering (Magin, 2006), and so on. One of the very important areas of application is the chaos theory (West et al., 2002; Zaslavsky, 2005).

At this point we have to note that various mathematical definitions of chaos are known, but all of them express close characteristics of the dynamic systems that are concerned with supersensitivity or sensitive dependence on the initial conditions, which are characterized by Lyapunov instability as a main property of the chaotic oscillation. Roughly speaking, chaotic behaviour arises whenever the system trajectories are globally bounded and locally unstable (Andrievskii and Fradkov, 2003).

The fundamentals of a new mathematical apparatus for studying the chaotic phenomena and the theory of nonlinear oscillations were laid in 1960's and 1970's by A. Poincaré, B. Van der Pol, A. A. Andronov, N. M. Krylov, A. N. Kolmogorov, D. V. Anosov, Ya. G. Sinai, V. K. Mel'nikov, Yu. I. Neimark, L. P. Shil'nikov, G. M. Zaslavsky, and their collaborators. From that time on, the chaotic behavior has been discovered in numerous systems in mechanics, chemistry, physics, biology and medicine, electronic circuits, economics and so on (Andrievskii and Fradkov, 2004).

It is well known that chaos cannot occur in continuous nonlinear systems with the total order less than three (Silva, 1993). This assertion is based on the usual concepts of order, such as the number of states in a system or the total number of separate differentiations or integrations in the system. The model of chaotic system can be rearranged to three single differential equations, where the equations contain the non-integer (fractional) order derivatives. The total order of the system is the sum of each particular order instead of three. To put this fact into context, we can consider the fractional-order dynamical model of the system. Hartley et al. introduced the fractional-order Chua's system (Hartley et al., 1995). In the work (Arena et al., 1998), the fractional-order cellular neural network (CNN) was considered, the fractional Duffing's system was presented in the work (Gao and Yu, 2005), while other fractional-order chaotic systems were described in many other works (e.g., Ahmad, 2005; Deng et al., 2007; Guo, 2005; Li and Chen, 2004; Lu, 2005a,b; Nimmo and Evans, 1999, etc.). In all these cases chaos was exhibited in a system with total order less than three.

The term of "system order" should be mentioned and explained as well. The system order is not equal to the number of differential equations if we consider fractional differential equations. The system order is equal to the highest derivative of the fractional differential equation of the mathematical model. Arena et al. (1998) and Hartley et al. (1995) simply replaced the integer-order derivative by fractional order one. For numerical simulation they used an approximation method proposed by Charef et al. (1992). This approximation of fractional-order operators is in the form of rational polynomial of high order in the frequency domain. Such approximation may produce the so-called fake chaos. As what has been shown in the works (Tavazoei and Haeri, 2007, 2008), it is possible to calculate a minimal order of system, where chaos is still observed. In other cases it is a numerical error, which leads to the fake chaos. It is very important to take this into account.

The above considerations and conclusions in the aforementioned works by Arena et al. and Hartley et al. lead to several notes. We will discuss three of them. The first note is on fractional order of derivatives. We cannot simply replace the integer order with fractional order without any reasons. Some appropriate reasons are described in this book. It could be a fractional order in the capacitor model in the case of electrical chaotic circuits. The second note is on approximation methods used. If we use a high order approximation method then the total order of the system is not equal to the highest derivative of the fractional differential equation but is equal to the highest order of approximation polynomial. Moreover, the system could produce fake chaos through numerical errors in calculations. The third note is on system order. In most of the mentioned papers, the terms of system order, model order, number of initial conditions, number of state space variables and methods for rewriting the state-space representation to fractional differential equations are not clearly defined. In this book, we will discuss how to define them.

This book is also presented a collection of the Matlab functions created for numerical simulation of the described fractional-order chaotic systems. The function codes are downloadable from the website of MathWorks, Inc. and their utilization is described and commented in Appendix A of this book.

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