MLM2

Morningside Lectures in Mathematics

Lectures on the Analysis of Nonlinear Partial Differential Equations Vol. 2

非线性偏微分方程分析讲义 第二卷

Editors Fanghua Lin
Ping Zhang

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MORNINGSIDE LECTURES IN MATHEMATICS

Preface to the First Volume

In summer of 2001, we initiated a summer school program on the harmonic analysis and its applications in nonlinear partial differential equations, with special emphases on nonlinear Schrödinger equations, kinetic equations of Bolzmann type and classical fluid equations. Over the years, there have been many distinguished mathematicians working in these fields who have come to help our program and to give series of special lectures. The program has been shown to be particularly helpful to young researchers and students. The lectures involved have gradually turned into more formal and regular seminars on *Analysis in Partial Differential Equations* at the Morningside Center of Mathematics Academy of Mathematics and System Science, Chinese Academy of Sciences.

From June 2007 to January 2008, we held a special semester PDE– program. We invited many mathematicians and experts in mathematical theory of fluid mechanics and quantum mechanics. The visitors during that period includes: Bresch Didier, Carles Rémi, Jean-Yves Chemin, Desvillettes Laurent, Lopes Filho Milton C, Nussenzveig Lopes Helena J. Novotny, Antonin, Chao-Jiang Xu, Chongchun Zeng, Ping Zhang and Yuxi Zheng, who gave a series of lectures and provided excellent lecture notes. It is no doubt that these lecture notes would be very useful for many researchers and students. In this volume, we have collected lecture notes by M. C. Lopes concerning the boundary layers of incompressible fluid flow; by C. J. Xu on the micro-local analysis and its applications to the regularities of kinetic equations; by Y. X. Zheng on the weak solutions of variational wave equation from liquid crystals, and by P. Zhang and Z. F. Zhang on the free boundary problem of Euler equations. In addition,we also included the notes by F. Nier on the hypoellipticity of Fokker-Planck operator and Witten-Laplace operator that were given earlier in the summer of 2006.

We have planned to publish in the forthcoming volumes the other lectures notes. Some are from past lectures at our program and some will be collected from the newly scheduled seminars. We hope that the publication of these lecture notes may provide valuable references and up-to-date descriptions of current developments of various related research topics, that will benefit many young researchers or graduate students. We wish to take this opportunity to thank the Morningside Center of Mathematics, the Institute of Mathematics of AMSS that

provides all necessary supports. We are particularly grateful to professor Lo Yang for his constant help, supports and encouragements to our program. We also would like to thank Guilong Gui for his careful preparations of the latex file of the entire book. We finally appreciate for the financial support from the Chinese Academy of Sciences.

Fanghua Lin in New York
Xueping Wang in Nantes
Ping Zhang in Beijing
On November 3, 2008

Preface to the Second Volume

This book is a sequel to the previous volume *Lectures on the Analysis of Nonlinear Partial Differential Equations*, Vol. 1. In this second volume, we have collected lectures by Marco Cannone, Chongsheng Cao and Jiahong Wu, Eduard Feireisl, Thierry Goudon, Jean-Claude Saut, Zhongwei Shen and Vsevolod A. Solonnikov. We greatly appreciate their time and efforts to provide us with their excellent lectures and notes. We believe these lecture notes will serve as valuable references and up-to-date descriptions of current developments in various research topics in nonlinear partial differential equations (PDE).

The notes collected here come from seminars on "Analysis in Partial Differential Equations" that were part of a special year-long program (from April to November 2009), which took place at the Morningside Center of the Academy of Mathematics and Systems Science, Chinese Academy of Sciences (CAS). We thank the Academy of Mathematics and Systems Science as well as the Morningside Center of CAS for financial support. We also wish to thank several other distinguished visitors who delivered excellent special lectures: Hammadi Abidi, Jean-Yves Chemin, Demetrios Christodoulou, Hideo Kozono, Thomas Hou, Marius Paicu, and others.

Finally, we wish to thank Prof. Lo Yang, Prof. Lei Guo, and Prof. Yuefei Wang, for their continued supports over all these years which were essential to the success of this PDE program. We also thank Dr. Guilong Gui for the careful preparation of the Latex files that make up this entire book.

Fanghua Lin in New York
Ping Zhang in Beijing
On June 26, 2010

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Harmonic Analysis and Navier-Stokes Equations with Application to the Boltzmann Equation

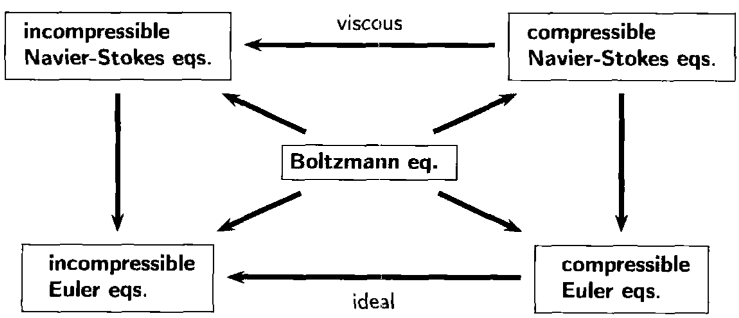
Marco Cannone*

Lecture I: Preliminaries

"It is therefore very desirable that the discussion of the foundations of mechanics be taken up by mathematicians also. Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limit process there merely indicated, which lead from the atomistic view to the laws of motion of continua. Conversely, one may try to derive the laws of the motion of rigid bodies by a limiting process from a system of axioms."

— Lecture delivered by David HILBERT at the International Conference of Mathematics, Paris 1900

Hilbert's Program



* Université Paris-Est Marne-la-Vallée, Laboratoire d'Analyse et de Mathématiques Appliquées, UMR 8050 CNRS, 5 boulevard Descartes, Cité Descartes Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France. E-mail: marco.cannone@univ-mlv.fr

Simple: these problems are locally in time well-posed (Cauchy-Lipschitz).
Difficult: global in time existence for general data ? uniqueness ? regularity ? stability ? blow-up ?

Non-linear ordinary differential equations

We solve $\frac{dy}{dt} = y^\alpha$ with initial condition $y(0) = y_0$.

If $\alpha = 2$ and $y_0 = 1$, then there is only one solution

$$y(t) = \frac{1}{1-t}$$

which **blows up at finite time.**

If $\alpha = \frac{1}{2}$ and $y_0 = 0$, then for $\forall c \geq 0$ there are **infinitely many C^1 solutions**

$$y_c(t) = \begin{cases} (\frac{t-c}{2})^2 & : \quad t \geq c, \\ 0 & : \quad t < c \end{cases}$$

but uniqueness holds in the subclass of C^2 solutions (i.e. when $c = \infty$ and $y_\infty \equiv 0$).

If $\alpha = \frac{1}{2}$ and $y_0 = 1$, then there exists a unique solution

$$y(t) = \begin{cases} (\frac{t+2}{2})^2 & : \quad t \geq -2, \\ 0 & : \quad t < -2. \end{cases}$$

Non-linear partial differential equations

Different types of <u>solutions (i.e. derivatives)</u>

- exact
- classic
- weak
- strong
- mild
- renormalized

Difficulty:

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial t} = |g|^2 - 1,$$
$$g_n(x,t) = \exp(in(x-t)) \quad \text{solution} \quad \forall n$$

but $g_n \rightharpoonup 0$ weakly if $n \to \infty$ and 0 is not a solution.

The Navier-Stokes equations

Problem 15 (on 18) for this century

"*...the solution of this problem might well be a foundamental step toward the very big problem of understanding turbulence ...*"

— Steven SMALE, 1998

Problem 6 (on 7) of the Clay foundation for this millenium

http://www.claymath.org/millennium/

— Charles FEFFERMAN, 2000

Loss of uniqueness ? blow-up in finite time?

We will solve

$$\begin{aligned}\frac{\partial v}{\partial t} - \nu\Delta v &= -(v\cdot\nabla)v - \nabla p + F,\\ \nabla\cdot v &= 0,\\ v(0) &= v_0,\end{aligned}$$

where v velocity, p pressure, ν viscosity, F ext force, v_0 initial data.

J. FOURIER (transform) + K. YOSIDA (semi-groupe) = T. KATO's method

$$\begin{aligned}v(t,x) &= S_t * v_0 - \int_0^t S_{t-\tau} * \mathbb{P}\nabla\cdot(v\otimes v)d\tau + \int_0^t S_{t-\tau} * \mathbb{P}F d\tau\\ p &= -\sum_{j,k=1}^{3} R_j R_k(v_j v_k) + c^t\end{aligned}$$

where $x\in\mathbb{R}^3, t>0$ and c^t is an arbitrary constant.

Riesz's transformation

$$\begin{aligned}R_j &= -i\partial_{x_j}(-\Delta)^{-\frac{1}{2}},\\ \widehat{R_j f}(\xi) &= \frac{\xi_j}{|\xi|}\hat{f}(\xi)\end{aligned} \qquad (j=1,2,3 \text{ and } i^2=-1).$$

Leray's projection

$$\begin{aligned}\mathbb{P} &= \mathbb{I} - \mathbb{R}\otimes\mathbb{R},\\ \widehat{(\mathbb{P}v)}_j(\xi) &= \sum_{k=1}^{3}(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2})\hat{v_k}(\xi) \qquad (j=1,2,3).\end{aligned}$$

Fourier's heat semigroup

$$\begin{aligned}S_t(x) &= (4\pi t)^{-\frac{3}{2}}\exp(-\frac{|x|^2}{4t}),\\ \widehat{S_t}(\xi) &= \exp(-t|\xi|^2).\end{aligned}$$

The fixed point theorem

Solve

$$\boxed{v(t) = S_t * v_0 + B(v,v)(t) + LF(t)}$$

Theorem. (Banach fixed point) *X Banach, $B : X \times X \to X$ bilinear. If moreover B is bi-continuous, say $\|B(x_1, x_2)\| \leq \eta\|x_1\|\|x_2\|$, then $\forall y \in X$ such that $4\eta\|y\| < 1$, the equation $x = y + B(x,x)$ has a solution $x \in X$ such that $\|x\| \leq 2\|y\|$. Uniqueness holds if $\|x\| < \frac{1}{2\eta}$; in particular if $\|x\| \leq 2\|y\|$.*

Thus if

$$|||v(t,\cdot)||| = \sup_{0<t<T} \|v(t,\cdot)\|_X$$

then main issue is to establish

$$|||B(u,v)||| \leq \eta(T)|||u||||||v|||.$$

If for simplicity, we put $F = 0$, then Banach's fixed point applies if

$$4\eta(T) \sup_{0<t<T} \|S_t * v_0\|_X < 1.$$

Now, if the norm in X is invariant by translations, it is easy to show that

$$\sup_{0<t<T} \|S_t * v_0\|_X = \|v_0\|_X,$$

so that

$$\boxed{4\eta(T)\|v_0\|_X < 1}$$

If $\lim_{T\to 0} \eta(T) = 0$, LOCAL EXISTENCE in time for ARBITRARY DATA.
If $\eta(T) = C$, GLOBAL EXISTENCE in time for SMALL DATA.
Uniqueness holds if $\sup_{0<t<T} \|v(t)\|_X \leq 2\|v_0\|_X$.

Uniqueness

In general uniqueness is lost as shown by the simple model equation $x = y + \eta x^2$, that if $4\eta y < 1$ has two distinct real solutions.

But,

$$v = S_t * v_0 + B(v,v),$$
$$\tilde{v} = S_t * v_0 + B(\tilde{v},\tilde{v}),$$

$$\boxed{|||v - \tilde{v}||| \leq \eta(T)(|||v||| + |||\tilde{v}|||)|||v - \tilde{v}|||}$$

Thus uniqueness holds in $\mathcal{C}([0,T);X)$ provided $\lim_{T\to 0} \eta(T) = 0$.

Summarizing, if

$$B(u,v)(t) = -\int_0^t S_{t-\tau} * \mathbb{P}\nabla \cdot (u \otimes v)(\tau)d\tau$$

is such that

$$\sup_{0<t<T} \|B(u,v)\|_X \leq \eta(T)(\sup_{0<t<T} \|u\|_X)(\sup_{0<t<T} \|v\|_X)$$

with $\lim_{T\to 0} \eta(T) = 0$.

> *Then, $\forall v_0 \in X$, $\nabla \cdot v_0 = 0$, there exists a unique solution $v(t) \in \mathcal{C}([0,T);X)$ to the Navier-Stokes equations.*

Navier and Stokes meet Fourier

$$B(u,v)(t) := -\int_0^t S_{t-\tau} * \mathbb{P}\nabla \cdot (u \otimes v)(\tau)d\tau,$$

$$Q_{t-\tau} := S_{t-\tau} * \mathbb{P}\nabla = \{q_j^{kl}(t-\tau)\},$$

$$\widehat{q_j^{kl}(t-\tau)}(\xi) = e^{-(t-\tau)|\xi|^2}(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2})\xi_l,$$

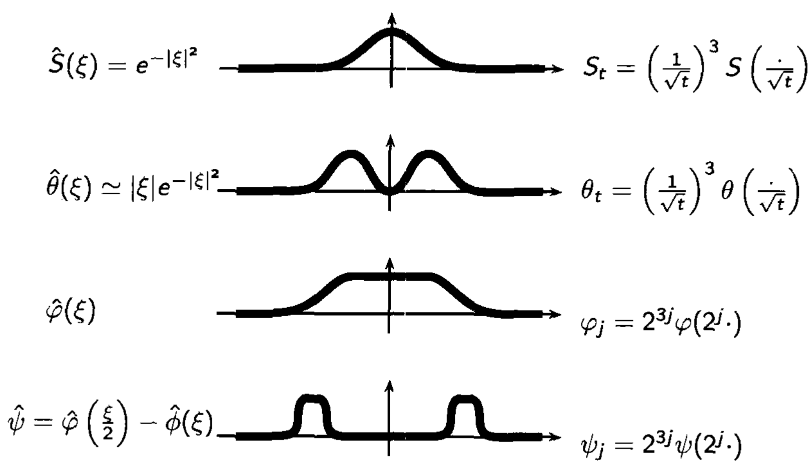
$$\sqrt{t-\tau}Q_{t-\tau} = (\frac{1}{t-\tau})^{\frac{3}{2}}\theta(\frac{\cdot}{\sqrt{t-\tau}})*,$$

$$\theta = \{\theta_j^{kl}\}, \qquad \hat{\theta}(\xi) = e^{-|\xi|^2}(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2})\xi_l,$$

$$\theta \in L^1 \cap L^\infty, \qquad \hat{\theta}(\xi) \simeq |\xi|e^{-|\xi|^2}.$$

Summarizing

$$\boxed{v(t) = S_t * v_0 - \int_0^t (t-s)^{-2}\theta\left(\frac{1}{\sqrt{t-s}}\right) * (v \otimes v)ds}$$



Littlewood-Paley decomposition

Let φ and ψ be defined as before, then the Littlewood-Paley decomposition is

$$\begin{array}{ccccc}
\varphi & \Rightarrow & \varphi_j = 2^{3j}\varphi(2^j\cdot) & \Rightarrow & S_j = \varphi_j * \\
\Downarrow & & \Downarrow & & \Downarrow \\
\psi = 2^3\varphi(2\cdot) - \varphi(\cdot) & & \psi_j = 2^3\varphi_j(2\cdot) - \varphi_j(\cdot) & & \Delta_j = S_{j+1} - S_j \\
\Downarrow & & \Downarrow & & \Downarrow \\
\psi & \Rightarrow & \psi_j = 2^{3j}\psi(2^j\cdot) & \Rightarrow & \Delta_j = \psi_j *
\end{array}$$

In such a way that

$$I = S_0 + \sum_{j\geq 0} \Delta_j$$

and

$$I = \sum_{j\in\mathbb{Z}} \Delta_j \qquad \text{(modulo polynomials).}$$

Besov spaces

Proposition. *Let $1 \leq p, q \leq \infty$ and $\alpha > 0$, then the quantities:*

$$\left[\sum_{j\in\mathbb{Z}} (2^{-\alpha j}\|\Delta_j * f\|_q)^p\right]^{\frac{1}{p}},$$

$$\left[\sum_{j\in\mathbb{Z}} (2^{-\alpha j}\|S_j * f\|_q)^p\right]^{\frac{1}{p}},$$

$$\left[\int_0^\infty (t^{\alpha/2}\|S_t * f\|_q)^p \frac{dt}{t}\right]^{\frac{1}{p}},$$

$$\left[\int_0^\infty (t^{\alpha}\|\theta_t * f\|_q)^p \frac{dt}{t}\right]^{\frac{1}{p}}$$

are equivalent and will be referred by $\|f\|_{\dot{B}_q^{-\alpha,p}}$ where $\dot{B}_q^{-\alpha,p}$ is a homogeneous Besov space of negative order.

In particular if $p = \infty, \alpha > 0$ and $1 \leq q \leq \infty$,

$$\|f\|_{\dot{B}_q^{-\alpha,\infty}} \cong \sup_{t>0} t^{\alpha/2}\|S_t * f\|_q.$$

Moreover, Young's inequality gives

$$\|S_t * f\|_q \leq \|S_t\|_r \|f\|_p = Ct^{-\frac{3}{2}(1-\frac{1}{r})}\|f\|_p$$

with $1 \leq p \leq q \leq \infty$ and $\frac{1}{q} = \frac{1}{r} + \frac{1}{p} - 1$. Thus

$$t^{\frac{1}{2}(1-\frac{3}{q})}\|S_t * f\|_q \leq C\|f\|_3,$$

$$\boxed{\alpha = 1 - \tfrac{3}{q}, \quad 3 \le q \le \infty : \quad \|f\|_{\dot{B}_q^{-\alpha,\infty}} \le C\|f\|_3}$$

i.e.

$$\boxed{L^3 \hookrightarrow \dot{B}_q^{-\alpha,\infty}}\,.$$

More generally, if $3 \le q_1 \le q_2 \le \infty, \alpha_i = 1 - \frac{3}{q_i}$, then the following

EMBEDDINGS hold:

$$L^3 \hookrightarrow \dot{B}_{q_1}^{-\alpha_1,\infty} \hookrightarrow \dot{B}_{q_2}^{-\alpha_2,\infty} \hookrightarrow \dot{B}_{\infty}^{-1,\infty},$$

this follows from Bernstein's inequality,

$$2^{-j\alpha_2}\|\Delta_j * f\|_{q_2} \le C2^{-j\alpha_1}\|\Delta_j * f\|_{q_1} \le C\|\Delta_j * f\|_3 \le C\|f\|_3.$$

HOMOGENEOUS FUNCTIONS:

$$|x|^{-1} \in \dot{B}_q^{-\alpha,\infty}, \qquad \forall\, 3 \le q \le \infty,$$

but $|x|^{-1} \notin L^3$; in fact let $|x|^{-1} = f$, then

$$\sup_{j\in\mathbb{Z}} 2^{-j\alpha}\|S_j * f\|_q = \sup_{j\in\mathbb{Z}} 2^{-j(\alpha+\frac{3}{q}-1)}\|S_0 * f\|_q = \|S_0 * f\|_q.$$

Lecture II: Existence

Mild Navier-Stokes equations

$$v(t) = S_t * v_0 + B(v,v)(t),$$
$$B(u,v)(t) = -\int_0^t (t-s)^{-2}\theta(\frac{\cdot}{\sqrt{t-s}}) * (u \otimes v)(s)ds.$$

Lebesgue spaces

$$v \in \mathcal{C}([0,T); L^p).$$

Young's inequality gives ($p \ge 2$) with $\frac{1}{q} + \frac{2}{p} = \frac{1}{p} + 1$:

$$\|B(u,v)(t)\|_p \le C\|\theta\|_q(\sup_t \|u\|_p)(\sup_t \|v\|_p)\int_0^t (t-s)^{-2+\frac{3}{2q}}ds,$$

and if $p > 3$,

$$(\sup_t \|B(u,v)\|_p) \le C\frac{T^{\frac{1}{2}(1-\frac{3}{p})}}{1-\frac{3}{p}}(\sup_t \|u\|_p \sup_t \|v\|_p),$$

thus $\forall v_0 \in L^p$, $\exists!$ solution $v(t) \in \mathcal{C}([0,T); L^p), p > 3$ and $T = T(\|v_0\|_p)$.

The same idea applies in other super-critical spaces.

Definition. (Well-suited spaces) *A Banach space is well-suited if* $\exists\ \eta_j \in \mathbb{R}_+$ *such that*

$$\|\Delta_j * (fg)\|_X \leq \eta_j \|f\|_X \|g\|_X,$$
$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\eta_j < \infty.$$

Example: $X = L^p(\mathbb{R}^3), \eta_j = \|\psi_j\|_q = C2^{3j/p} \Longrightarrow p > 3.$

Theorem. *Let* X *be a well-suited space,* $\forall v_0 \in X,\ \nabla \cdot v_0 = 0,\ \exists\ T(\|v_0\|)$ *and a unique solution* $v(t) \in \mathcal{C}([0,T); X)$ *to the Navier-Stokes equations.*

Sketch of the proof

$\|t^{-2}\theta(\frac{\cdot}{\sqrt{t}}) * (u \otimes v)\|_X \leq \omega(t)\|u\|_X\|v\|_X,\ \forall t \in (0,1]$ and $\forall f, g \in X$, where X well-suited and with $\int_0^1 \omega(t)dt \leq C\sum_{j\in\mathbb{Z}} 2^{-|j|}\eta_j$.

Evaluating η_j **J.M. Bony**'s paraproduct

$$\Delta_j * (fg) = (\Delta_j * f)(S_{j-2} * g) + (\Delta_j * g)(S_{j-2} * f) + \Delta_j(\sum_{k\geq j} \Delta_k * f \Delta_k * g).$$

Example $X = M_2^p(\mathbb{R}^3)$ Morrey-Campanato: Q_m is a cube with wide 2^{-m}, $f \in M_2^p \Leftrightarrow \|S_0 f\|_\infty \leq C$ and $\int_{Q_m}(\sum_{j\geq m} |\Delta_j * f|^2) \leq C2^{-m(3-\frac{6}{p})}$.

It is easy to get $\eta_j = 2^{\frac{3j}{p}}$ and finally

Theorem. *Let* $p > 3$*, then* $\forall v_0 \in M_2^p, \nabla \cdot v_0 = 0$*, then* $\exists\ T(\|v_0\|)$ *and a unique* $v(t) \in \mathcal{C}([0,T); M_2^p)$ *solution to the mild Navier-Stokes equations.*

Improving smoothness

The solution obtained in the above mentioned theorem can be written in the form

$$v(t,x) = S(t) * v_0 + \omega(t,x),$$

i.e. TENDENCY(heat eq.) + FLUCTUATION (zero integral).

Moreover, $\omega(t,x)$ is more regular than $v(t,x)$, in fact one can prove that $\omega(t,x) \in \mathcal{C}([0,T); \dot{B}_X^{0,1})$.

Examples:
$X = L^p(\mathbb{R}^3)$, $p > 3$, $\omega \in \dot{B}_p^{-1,\infty}$,
$X = \dot{H}^s(\mathbb{R}^3)$, $s > \frac{1}{2}$, $\omega \in \dot{H}^{s-\frac{1}{2}-\epsilon}$ (if $\frac{1}{2} < s \leq \frac{3}{2}$), $\omega \in \dot{H}^{s+1-\epsilon}$ (if $s > \frac{3}{2}$),
$X = C^\alpha(\mathbb{R}^3)$, $\alpha > 0$, $\omega \in C^{\alpha+1}$ (Hölder-Zygmund).
See also **P. Zhang** (2008).