国外数学名著系列

（影印版）73

Pierre Collet Jean-Pierre Eckmann

Concepts and Results in Chaotic Dynamics: A Short Course

混沌动力系统的概念和结果

科学出版社

国外数学名著系列(影印版) 73

Concepts and Results in Chaotic Dynamics: A Short Course

混沌动力系统的概念和结果

Pierre Collet　Jean-Pierre Eckmann

科 学 出 版 社

北 京

图字：01-2011-3330

Pierre Collet, Jean-Pierre Eckmann: Concepts and Results in chaotic Dynamics: A Short Course

图书在版编目 (CIP) 数据

混沌动力系统的概念和结果= Concepts and Results in Chaotic Dynamics: A Short Course/（法）科莱(Collet, P.),（瑞士）埃克曼(Eckmann, J-P.)编著. —影印版. —北京：科学出版社, 2011

(国外数学名著系列; 73)

ISBN 978-7-03-031389-8

I. ①混… II. ①科… ②埃… III. ①混沌理论-动力学-英文 IV. O415.5

中国版本图书馆 CIP 数据核字(2011) 第 104996 号

责任编辑：赵彦超　汪　操/责任印刷：钱玉芬/封面设计：陈　敬

科学出版社 出版

北京东黄城根北街16号

邮政编码：100717

http://www.sciencep.com

双青印刷厂 印刷

科学出版社发行　各地新华书店经销

*

2011年6月第一版　开本：B5(720 × 1000)

2011年6月第一次印刷　印张：15 1/2

印数：1—2 000　字数：294 000

定价：68.00 元

(如有印装质量问题，我社负责调换)

《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来，需要数学家淡泊名利并付出更艰苦地努力。另一方面，我们也要从客观上为数学家创造更有利的发展数学事业的外部环境，这主要是加强对数学事业的支持与投资力度，使数学家有较好的工作与生活条件，其中也包括改善与加强数学的出版工作。

从出版方面来讲，除了较好较快地出版我们自己的成果外，引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说，施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书，使我国广大数学家能以较低的价格购买，特别是在边远地区工作的数学家能普遍见到这些书，无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权，一次影印了 23 本施普林格出版社出版的数学书，就是一件好事，也是值得继续做下去的事情。大体上分一下，这 23 本书中，包括基础数学书 5 本，应用数学书 6 本与计算数学书 12 本，其中有些书也具有交叉性质。 这些书都是很新的，2000 年以后出版的占绝大部分，共计 16 本，其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿，例如基础数学中的数论、代数与拓扑三本，都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点，基础数学类的书以“经典”为主，应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家，例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士，曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然，23 本书只能涵盖数学的一部分，所以，这项工作还应该继续做下去。更进一步，有些读者面较广的好书还应该翻译成中文出版，使之有更大的读者群。

总之，我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持，并盼望这一工作取得更大的成绩。

王　元

2005 年 12 月 3 日

Preface

This book is devoted to the subject commonly called Chaotic Dynamics, namely the study of complicated behavior in time of maps and flows, called dynamical systems.

The theory of chaotic dynamics has a deep impact on our understanding of Nature, and we sketch here our view on this question. The strength of this theory comes from its generality, in that it is not limited to a particular equation or scientific domain. It should be viewed as a conceptual framework with which one can capture properties of systems with complicated behavior. Obviously, such a general framework cannot describe a system down to its most intricate details, but it is a useful and important guideline on how a certain kind of complex systems may be understood and analyzed.

The theory is based on a description of idealized systems, such as "hyperbolic" systems. The systems to which the theory applies should be similar to these idealized systems. They should correspond to a *fixed* evolution equation, which, however, need to be neither modeled nor explicitly known in detail. Experimentally, this means that the conditions under which the experiment is performed should be as constant as possible. The same condition applies to analysis of data, which, say, come from the evolution of glaciations: One cannot apply "chaos theory" to systems under varying external conditions, but only to systems which have some self-generated chaos under fixed external conditions.

So, what *does* the theory allow us to do? We can measure indicators of chaos, and study their dependence on those fixed external conditions. Is the system's behavior regular or chaotic? This can be, for example, inferred by measuring Lyapunov exponents. In general, the theory tells us that complex systems should be analyzed statistically, and not, as was mostly done before the 1960s, by all sorts of Fourier-mode- and linearized, analysis. We hope that the present book and in particular Sect. 9 shows what the useful and robust indicators are.

The material of this book is based on courses we have given. Our aim is to give the reader an overview of results which seem important to us, and which are here to stay. This book is not a mathematical treatise, but a course, which tries to combine two slightly contradicting aims: On one hand to present the main ideas in a simple way and to support them with many examples; on the other to be mathematically sufficiently precise, without undue detail. Thus, we do not aim to present the most general results on a given subject, but rather explain its ideas with a simple statement and many examples. A typical instance of this restriction is that we tacitly assume enough regularity to allow for a simpler exposition.

The proofs of the main results are often only sketched, because we believe that it is more important to understand how the concepts fit together in leading to the results than to present the full details. Thus, we usually spend more space on explaining the ideas than for the proofs themselves. This point of view should enable the reader to grasp the essence of a large body of ideas, without getting lost in technicalities. For the same reason, the examples are carefully chosen so that the general ideas can be understood in a nutshell.

The level of the book is aimed at graduate students in theoretical physics and in mathematics. Our presentation requires a certain familiarity with the language of mathematics but should be otherwise mostly self-contained.

The reader who looks for a mathematical treatise which is both detailed and quite complete, may look at (de Melo and van Strien 1993; Katok and Hasselblatt 1995). For the reader who looks for more details on the physics aspects of the subject a large body of literature is available, with different degrees of mathematical rigor: (Eckmann 1981) and (Eckmann and Ruelle 1985a) deal with experiments of the early 1980s; (Manneville 1990; 2004) deals with many experimental setups; (Abarbanel 1996) is a short course for physicists; (Peinke, Parisi, Rössler, and Stoop 1992) concentrates on semiconductor experiments; (Golubitsky and Stewart 2002) has a good mix of mathematical and experimental examples. Finally, (Kantz and Schreiber 2004) deal with nonlinear time series analysis.

The references in the text cover many (but obviously not all) original papers, as well as work which goes much beyond what we explain. In this way, the reader may use the references as a guide for further study. The reader interested in more of an overview will also find references to textbooks and monographs which shed light on our subject either from different angle, or in the way of more complete treatises.

Like any such project, to remain of reasonable size, we have omitted several subjects which might have been of interest; in particular, bifurcation theory (Arnold 1978; Ruelle 1989b; Guckenheimer and Holmes 1990), topological dynamics, complex dynamics, the Kolmogorov–Arnold–Moser (KAM) theorem and many others. In particular, we mostly avoid repeating material from our earlier book (Collet and Eckmann 1980).

After a few introductory chapters on dynamics, we concentrate on two main subjects, namely hyperbolicity and its consequences, and statistical properties of interest to measurements in real systems.

To make the book easier to read, several definitions and some examples are repeated, so that the reader is not obliged to go back and forth too often.

We hope that the many illustrations, simple examples, and exercises help the reader to penetrate to the core of a beautiful and varied subject, which brings together ideas and results developed by mathematicians and physicists.

This book has profited from the questions, suggestions and reactions of the numerous students in our courses: We warmly thank them all. Furthermore, our work was financially supported by the Fonds National Suisse, the European Science Foundation, and of course our host institutions. We are grateful for this support.

Paris and Geneva,
May 2006

Pierre Collet
Jean-Pierre Eckmann

Contents

1

A Basic Problem

Before we start with the subject proper, it is perhaps useful to look at a concrete physical example, which can be easily built in the laboratory. It is a pendulum with a magnet at the end, which oscillates above three symmetrically arranged fixed magnets, which attract the oscillating magnet, as shown in Fig. 1.1. When one holds the magnet slightly eccentrically and let it go, it will dance around the three magnets, and finally settle at one of the three, when friction has slowed it down enough.

The interesting question is whether one can predict where it will land. That this is a difficult issue is visible to anyone who does the experiment, because the pendulum will hover above one of the magnets, "hesitate" and cross over to another one, and this will happen many times until the movement changes to a small oscillation around one of the magnets and ends the uncertainty of where it will go. Let us call the three magnets "red," "yellow," "blue"; one can ask for every initial position from which the magnet is started (with 0 speed) *where* it will eventually land. The result of the numerical simulation, to some resolution, is shown in Fig. 1.2. The incredible richness of this figure gives an inkling of the complexity of this problem, although we only deal with a simple classical pendulum.

Exercise 1.1. *Program the pendulum equation and check the figure. The equations for the potential U are, for $q \in \mathbb{R}^2$,*

$$U(q) = \frac{3}{8}|q|^2 - \sum_{j=0}^{2} V(q - q_j) , \tag{1.1}$$

where $q_j = \big(\cos(2\pi j/3), \sin(2\pi j/3)\big)$ and $V(q) = 1/|q|$. The equations of motion are

$$\dot{q} = p , \quad \dot{p} = -\gamma p - \nabla_q U(q) ,$$

where $\dot{q}$ is a shorthand for $\mathrm{d}q(t)/\mathrm{d}t$. The friction coefficient is $\gamma = 0.13$.

The domains of same color are very complicated, and the surprising thing about them is that their boundaries actually coincide: If x is in the boundary ∂R of the red region, it is also in the boundary of yellow and blue: $\partial R = \partial Y = \partial B$. (This fact has

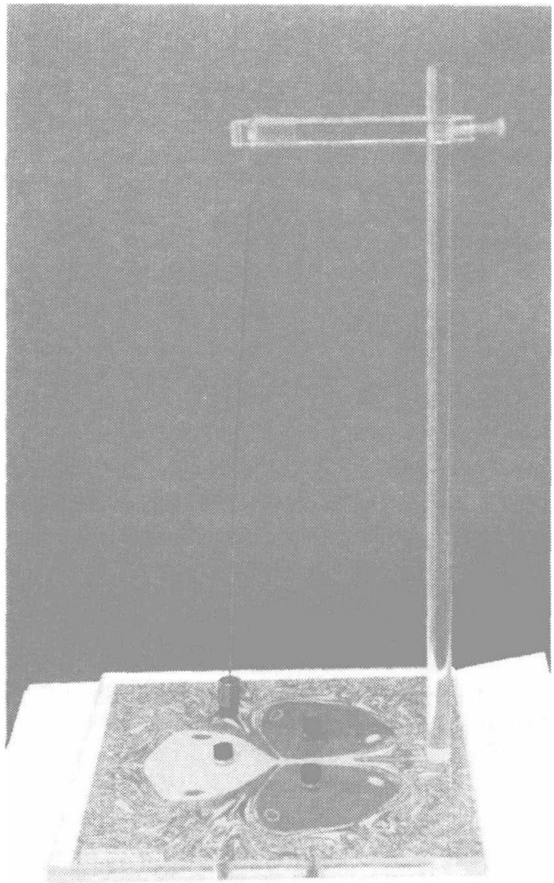

Fig. 1.1. Photograph of the pendulum with three magnets (The design is due to U. Smilansky.)

been proven for the simpler example of Fig. 3.6, and we conjecture the same result for the pendulum.)

The subject of this course is a generic understanding of such phenomena. While this example is not as clear as the one of the "crab" of Fig. 3.6, it displays a feature which will follow us throughout: **instability**. For the case at hand, there is exactly one unstable point in the problem, namely the center, and it is indicated in yellow in Fig. 1.3. Whenever the pendulum comes close to this point, it will have to "decide" on which side it will go: It may creep over the point, and the most minute change in the initial condition might change the final target color the pendulum will reach.

The aim of this book is to give an account of central concepts used to understand, describe, and analyze this kind of phenomena. We will describe the tools with which mathematicians and physicists study chaotic systems.

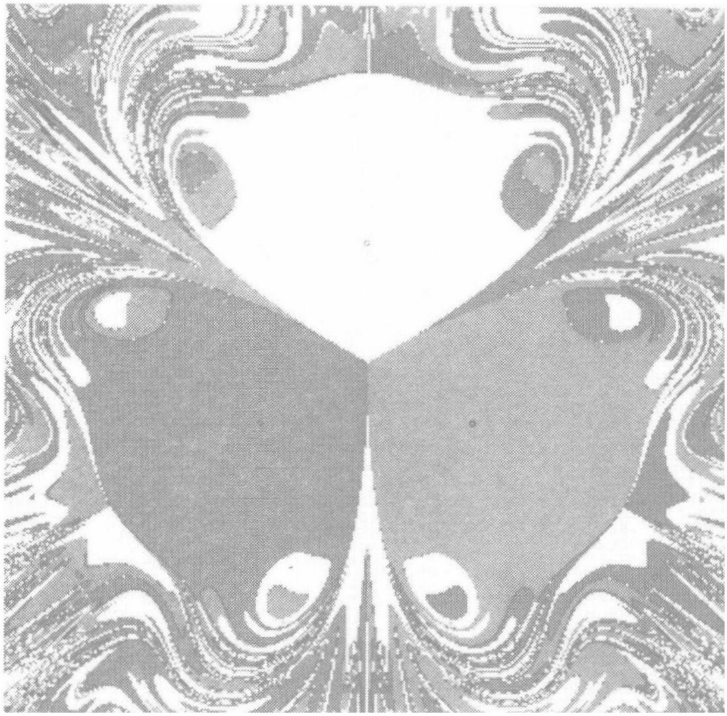

Fig. 1.2. The basins of attraction of the three magnets, color coded. The coordinates are the two components of the initial position: $q = (q_1, q_2)$. The three circles show the positions of the fixed magnets

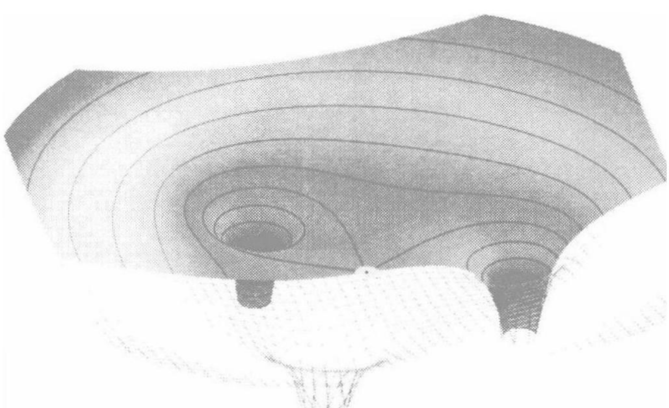

Fig. 1.3. A typical potential for the pendulum of Fig. 1.1. The equation used for the drawing is given in (1.1). The coordinates are the two components of the initial position: $q = (q_1, q_2)$, the height is the value of the potential

2

Dynamical Systems

2.1 Basics of Mechanical Systems

While we assume some familiarity with elementary mechanics, we begin here with the nonlinear pendulum in 1 dimension, to have some basis of discussion of phase space and the like. The nonlinear pendulum is a mathematical pendulum, under the influence of gravity, and with a damping $\gamma \geq 0$. Normalizing all the coefficients to 1, we consider the coordinates of this problem: momentum p and the angle φ. The pendulum can go over the top, and thus φ varies in $[0, 2\pi)$ while the momentum varies in $(-\infty, \infty)$. (We use here the notation $[a, b)$ to denote the half-open interval $a \leq x < b$.) Thus, the phase space Ω is actually a cylinder (since we identify $\varphi = 2\pi$ and $\varphi = 0$). The equations of motion for this problem are

$$\begin{aligned} \dot{\varphi} &= p \, , \\ \dot{p} &= -\sin(\varphi) - \gamma p \, , \end{aligned} \tag{2.1}$$

where both φ and p are functions of the time t, and $\dot{\varphi} = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)$, $\dot{p} = \frac{\mathrm{d}}{\mathrm{d}t}p(t)$.

This problem has 2 fixed points, where the pendulum does not move, $p = 0$, with $\varphi = 0$ or π. The fixed point $\varphi = \pi$ is unstable, and the pendulum will fall down under any small perturbation (of its initial position, when the initial velocity is 0) while $\varphi = 0$ is stable. The nature of stability near $\varphi = 0$ changes with γ, as can be seen by linearizing (2.1) around $\varphi = p = 0$.

i) For $\gamma = 0$ the system is Hamiltonian, and the flow is shown in Fig. 2.1. Such pictures are called *phase portraits*, because they describe the phase space, to be defined more precisely in Sect. 2.2. The vertical axis is p, the circumference of the cylinder is the variable φ. The lower equilibrium point is in the front (left) of the cylinder and the upper equilibrium point is in the back, at the crossing of the red curves. These curves are called *homoclinic orbits*, they correspond to the pendulum leaving the top at zero (infinitesimally small) speed and returning to it after an infinite time. One can see two such curves, one for each direction of rotation.

ii) For $\gamma \in (0, \gamma_{\mathrm{crit}})$, with $\gamma_{\mathrm{crit}} = 2$ (*critical damping*), the orbits are as shown in Fig. 2.2. The red line is the stable manifold of the unstable fixed point: A pendulum with initial conditions on the red line will make a number of full turns and stop finally (after infinite time) at the unstable equilibrium. All other initial conditions lead to orbits spiraling into the stable fixed point. The blue line is called the unstable manifold of the unstable fixed point and it also spirals into the stable fixed point. We treat this in detail in Sect. 4.1.1.

iii) For $\gamma = \gamma_{\mathrm{crit}}$ the phase portrait is shown in Fig. 2.3. This value of γ is the smallest for which the pendulum, when started at the top ($\varphi = \pi, p = \varepsilon, \varepsilon \to 0$), will move to the stable fixed point $\varphi = 0$ without oscillating around it.

iv) For $\gamma > \gamma_{\mathrm{crit}}$, which is usually called the *supercritical* case, the phase portrait is shown in Fig. 2.4. In this figure, the red and blue lines are again the stable and unstable manifolds of the unstable fixed point, while the green line is the *strongly stable manifold* of the stable fixed point. (The tangent flow of the stable fixed point has 2 stable (eigen-)directions and the strongly stable one is the direction which attracts faster.)

v) When $\gamma = 0$, there is no friction and the flow for the pendulum is area preserving. We illustrate this in Fig. 2.5. Note that the Poincaré recurrence theorem (see also Theorem 8.2) tells us that any open set contained in a compact invariant set must return infinitely often to itself. This is clearly seen for both the red and the blue region which eventually intersect infinitely often the original ellipse from which the motion started. Note that Poincaré's theorem does *not* say that a given point must return close to itself, just that the regions must intersect. Note furthermore how the regions come close to, but avoid the unstable fixed point (since the original region did not contain that fixed point).

Some final remarks on general flows in $\mathbb{R}^d$ are in order. They are all described by differential equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}(t) = \mathbf{F}(\mathbf{x}(t)) , \tag{2.2}$$

with $\mathbf{x} : \mathbb{R} \to \mathbb{R}^d$ and $\mathbf{F} : \mathbb{R}^d \to \mathbb{R}^d$ is called a *vector field*.

Theorem 2.1. *If* $\mathbf{F}$ *is Lipschitz continuous, the solution of (2.2) either exists for all times or diverges at some finite time.*

Definition 2.2. *If* $\mathbf{F}(\mathbf{x}_*) = 0$ *one calls* $\mathbf{x}_*$ *a fixed point.*

Theorem 2.3. *Outside a fixed point (that is in any small open set not containing a fixed point) any smooth flow is locally trivial, in the sense that there exists a coordinate change for which (2.2) takes the form* $\dot{\mathbf{y}} = \mathbf{A}$*, where* $\mathbf{A}$ *is the constant column vector with d components:* $\mathbf{A} = (1, 0, \ldots, 0)$*. (Furthermore, if (2.2) is a Hamiltonian equation, the coordinate change can be chosen to be canonical (Darboux' theorem).)*

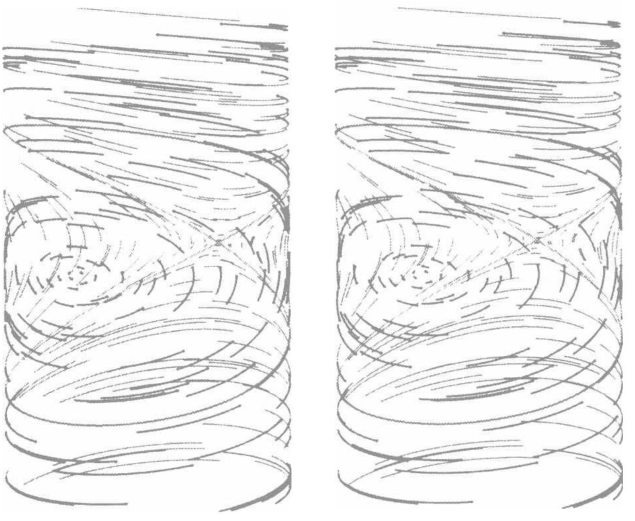

Fig. 2.1. Stereo picture for the flow of (2.1) for $\gamma = 0$. These pictures give a 3-dimensional effect if one "stares" at them to bring images of the two cylinders to convergence. The stable equilibrium point is in the front (to the left) and the unstable one is in the back of the cylinder. The black lines show short pieces of orbit. These pieces start at the dot and extend along the short lines

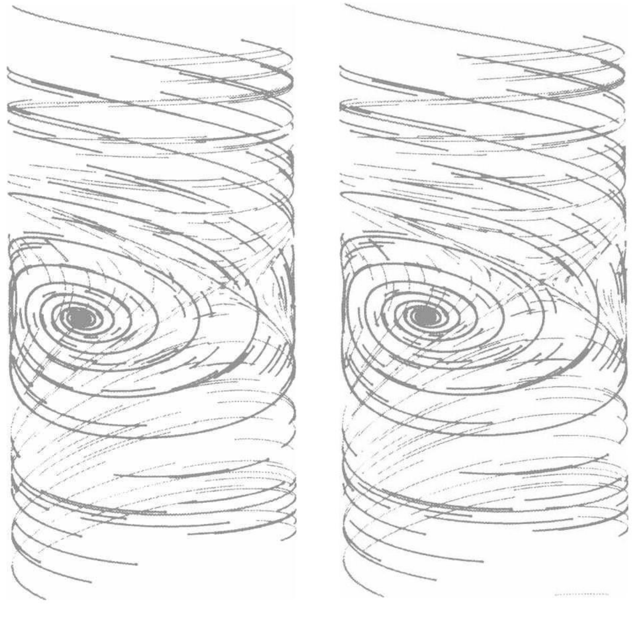

Fig. 2.2. The same view for a subcritical γ. The value is $\gamma = 0.25$

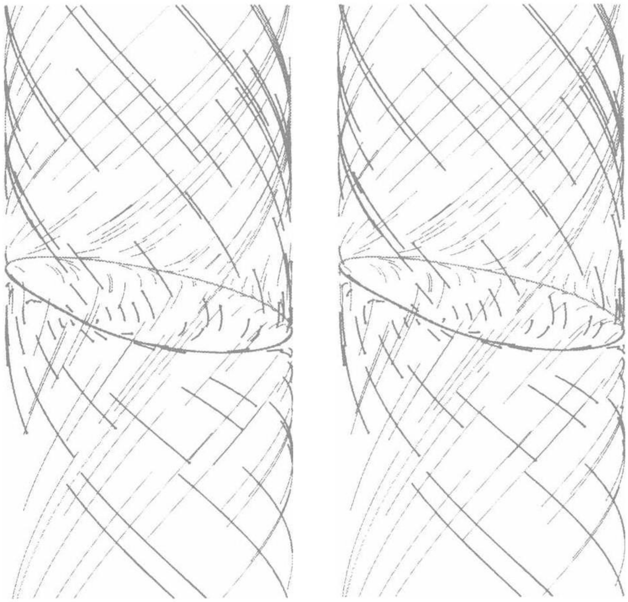

Fig. 2.3. Critical damping. The value is $\gamma = 2$

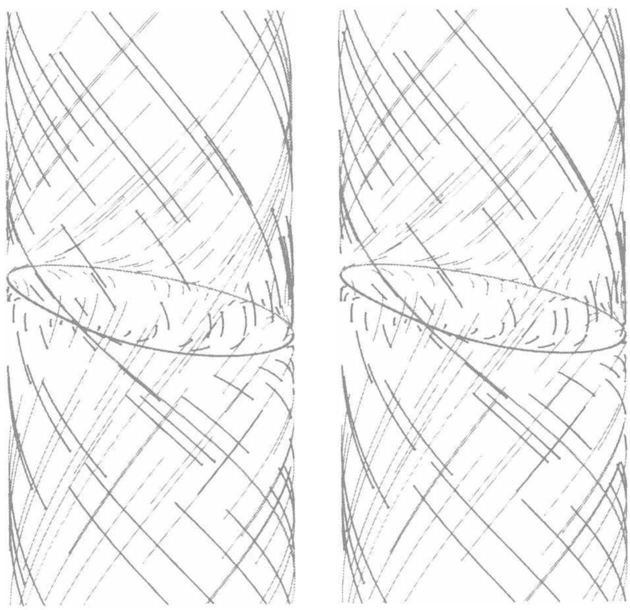

Fig. 2.4. Supercritical damping. The value is $\gamma = 2.2$