Distributive Lattices with Unary Operations

（分配格序代数）

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Foreword

The best-known distributive lattice with a unary operation is of course a Boolean algebra, the unary operation being complementation. Since this notion is very strong, several weaker forms of it have been considered and have given rise to many different algebras. Notable amongst these is the concept of an *Ockham algebra* which may be described simply as an algebra $\mathscr{L} = (L; \wedge, \vee, f, 0, 1)$ of type $\langle 2, 2, 1, 0, 0 \rangle$ in which $(L; \wedge, \vee, 0, 1)$ is a bounded distributive lattice and $f : L \to L$ is a dual endomorphism. The main feature of this generalisation is the retention of the de Morgan laws.

The notion of an Ockham algebra came to prominence in the late 1970s. Since then there have been many papers published concerning this variety of algebras, involving both lattice-theoretic techniques and Priestley duality. The earliest investigations gave rise to the book *Ockham Algebras* by Blyth and Varlet (Oxford University Press, 1994). Following the appearance of this, several new subvarieties of Ockham algebras have been introduced and their properties investigated.

In the present volume, which researchers in the field will welcome as a supplement to the above, Professor Fang has collated many of these interesting new results. In particular, there is emphasis on situations in which the distributive lattice L is endowed with various properties that are related to pseudocomplementation, the resulting algebras being of type $\langle 2, 2, 1, 1, 0, 0 \rangle$. Included are detailed descriptions of the lattices of subvarieties, lattices of congruences, and subdirectly irreducible algebras that arise.

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Preface

With the development of information science and theoretical computer science, lattice-ordered algebraic structure theory has played a more and more important role in theoretical and applied science. Not only is it an important branch of modern mathematics, but it also has broad and important applications in algebra, topology, fuzzy mathematics and other applied sciences such as coding theory, computer programs, multi-valued logic and science of information systems, etc. The research in distributive lattices with unary operations has made great progress in the past three decades, since Joel Berman first introduced the distributive lattices with an additional unary operation in 1978, which were named Ockham algebras by Goldberg a year later. This is due to those researchers who are working on this subject, such as Adams, Beazer, Berman, Blyth, Davey, Goldberg, Priestley, Sankappanavar and Varlet.

The class of distributive lattices with unary operations is a very rich one that contains many important subclasses of the algebras such as Ockham algebras, pseudo-complemented algebras (or called p-algebras), Stone algebras, Boolean algebras, de Morgan algebras and Kleene algebras. In the book *Ockham Algebra* (Oxford University Press, 1994), Blyth and Varlet have further explored Ockham algebras and the related concepts of distributive lattices with unary operations. The author's purpose in writing this text is to provide additional insights and further research results on this theory. In particular, the text includes those algebras such that Ockham algebras, pseudocomplemented Ockham algebras, demi-pseudocomplemented Ockham algebras, the theory of Priestley topological duality and related topics. I hope that readers through this book will learn about research development on the subject of distributive lattices with unary operations since the mid-nineties of the last century.

This book is written for senior-level university students, postgraduates and researchers who are interested in lattice-ordered algebraic structures, lattices and universal algebras, fuzzy mathematics and related fields. It is, to a large extent, based on my research results co-achieved with Professors T. S. Blyth, H. J. Silva and J. C. Varlet. In Chapter 5, some results on pseudocomplemented algebras are derived from the work of George Grätzer and his collaborators; and some others on demi-pseudocomplemented algebras are done from the work of H. P. Sankappanavar. I have tried to make the contents as comprehensive as possible, so that the readers may not have to use other references as aids to understand the book fully. Of course, it will be of great help in grasping the contents of this book if the readers have learnt the basic concepts of lattice theory and universal algebra. In order to attain a more complete understanding of the theory of distributive lattices with unary operations, it would be very helpful if the readers read this text as a companion to the book *Ockham Algebras*.

I gratefully acknowledge Guangdong Polytechnic Normal University for its financial support for the publication of this book. I do appreciate Professor T. S. Blyth very much that he has read the manuscript of this book and kindly written a foreword for it. Furthermore, I am deeply indebted to Professor Huang Yisheng for his very valuable help with the LaTeX. My special thanks also go to Dr. Sun Zhongju, Mr. Wang Leibo, Miss Shen Xiamei and Miss Yang Ting for their proof-reading. In particular, I would like to thank my beloved family. It is hard to imagine that I could engage in teaching and researching in this country with a peaceful mind and would have been able to complete this book without the strong support, trust and love of my family.

Fang Jie
Guangzhou, China
October, 2010

Contents

Chapter 1

Universal Algebra and Lattice-ordered Algebras

1.1 Universal algebra

The fundamental concept of *universal algebra* is the notion of *operation.* If k is a non-negative integer, then a *k-ary operation* on a set A is a mapping $f : A^k \to A$. Specifically, when $k = 0, 1, 2$, we shall say that f is a *nullary operation*, a *unary operation*, and a *binary operation* respectively.

Definition An algebra A of type $\langle k_1, \ldots, k_\alpha \rangle$ is a pair $(A; F)$ where A is a non-empty set and F is an α-tuple $(f_1, \ldots, f_\alpha)$, such that, for each i with $1 \leqslant i \leqslant \alpha$, f_i is a k_i-ary operation on A.

If, in particular, A is of type $\langle 2, 2 \rangle$, the two binary operations being *meet* and *join* and satisfying the following identities:

(L_1) $x \wedge x = x, \quad x \vee x = x$ (idempotent);
(L_2) $x \wedge y = y \wedge x, \quad x \vee y = y \vee x$ (commutativity);
(L_3) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \quad x \vee (y \vee z) = (x \vee y) \vee z$ (associativity);
(L_4) $x \wedge (x \vee y) = x, \quad x \vee (x \wedge y) = x$ (absorption),

then such an algebra A is called *lattice*. Equivalently, a lattice is an ordered set L in which any pair of elements x and y in L have the greatest lower bound which is denoted by $x \wedge y$; and the least upper bound which is denoted by $x \vee y$.

If a lattice L has the smallest element 0 and the greatest element 1, then L is said to be a *bounded lattice*, and it can be regarded as an algebra of type $\langle 2, 2, 0, 0 \rangle$. A lattice L is called to be *distributive* if it satisfies the following distributive law:

(L_5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$; equivalently,
(L_6) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

An ordered set $(E; \leqslant)$ can also be regarded as an algebra with a binary operation (order) $\leqslant$ on E satisfying the following properties:

(E_1) $(\forall x \in E)$ $x \leqslant x$ (reflexive);
(E_2) $(\forall x, y \in E)$ if $x \leqslant y$ and $y \leqslant x$ then $x = y$ (anti-symmetric);
(E_3) $(\forall x, y, z \in E)$ if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$ (transitive).

If now $(A; F)$ is an algebra of type $(k_1, \ldots, k_\alpha)$ with $F = \{f_1, \ldots, f_\alpha\}$ where each f_i is a k_i-ary operation on A, and B is a non-empty subset of A, then $(B; F)$ is called a *subalgebra* of $(A; F)$ if B is closed under the formation of each k_i-ary operation f_i with $1 \leqslant i \leqslant \alpha$, and $x_1, \ldots, x_{k_i} \in B$, $f_i(x_1, \ldots, x_{k_i}) \in B$. If B_i with $i \in I$ is a family of subalgebras of A, then clearly, the intersection $\bigcap\limits_{i \in I} B_i$ is a subalgebra of A, and A is a subalgebra of itself. Given a non-empty subset S of A the intersection of all subalgebras of A which contains S is called the *subalgebra generated by* S that we shall denote by $[S]$. It is obvious that $[S]$ is the smallest subalgebra of A that contains S. We say that A is *finitely generated algebra* if there exists some non-empty set S such that $A = [S]$.

A class $\boldsymbol{C}$ of algebras is said to be *locally finite* if every finitely generated member of $\boldsymbol{C}$ is finite. It is well known that the class $\boldsymbol{D}_{0,1}$ of bounded distributive lattices is locally finite.

An algebra $(A; F)$ where $F = \{f_1, \ldots, f_m\}$ is said to be of *finite range* if for any $x \in A$ and $f_{m_1}, \ldots, f_{m_k} \in F$ there exist positive integers p, q with $p \neq q$ such that

$$(f_{m_1} \cdots f_{m_k})^{p+q}(x) = (f_{m_1} \cdots f_{m_k})^q(x).$$

Theorem 1.1 *Let $\boldsymbol{C}$ be a class of bounded distributive lattices on which are defined finitely many unary operations. If every member $(A; F)$ where $F = \{f_1, \ldots, f_m\}$ of C is of finite range, then $\boldsymbol{C}$ is locally finite.*

Proof Suppose that $\mathcal{A} = (A; F) \in \boldsymbol{C}$ is generated by $\{x_1, \ldots, x_k\}$. Here we consider the case where $F = \{f_1, f_2\}$, and for the general case, it can be obtained by induction. Then there exist natural numbers $m_{i,j}, n_{i,j}$ with $m_{i,j} \neq n_{i,j}$ such that $f_1^{m_{i,j}+n_{i,j}}(x_i) = f_1^{n_{i,j}}(x_i)$ for $i = 1, 2, \ldots, k$ and $j = 1, 2$. Observe first that $f_j^{p+q}(x) = f_j^q(x)$ implies $f_j^{rp+q}(x) = f_j^q(x)$ for all $r \in \mathbb{N}$. Now let $m = \operatorname{lcm}\{m_{i,j} \mid i = 1, 2, \ldots, k; j = 1, 2\}$ and $n = \operatorname{lcm}\{n_{i,j} \mid i = 1, 2, \ldots, k; j = 1, 2\}$, then $m = m_{i,j} r_{i,j}$ and $n = n_{i,j} s_{i,j}$, and for $i = 1, 2, \ldots, k$, $j = 1, 2$,

$$\begin{aligned} f_j^{m+n}(x_i) &= f_j^{m_{i,j} r_{i,j} + n_{i,j} + (s_{i,j}-1) n_{i,j}}(x_i) \\ &= f_j^{n_{i,j} + (s_{i,j}-1) n_{i,j}}(x_i) \\ &= f_j^n(x_i) \quad [= f_j^{2m+n}(x_i)]. \end{aligned}$$

Now, since $\mathcal{A}$ is of finite range by hypothesis, we have in a similar argument that there exist natural numbers p, q with $p \neq q$ such that

$$(f_1 f_2)^{p+q}(x_i) = (f_1 f_2)^q(x_i) \text{ and } (f_2 f_1)^{p+q}(x_i) = (f_2 f_1)^q(x_i) \ (i = 1, 2, \dots, k).$$

Then can see that the set

$$\{g_1^{n_1} g_2^{n_2} \dots g_t^{n_t}(x_i) \mid 1 \leqslant i \leqslant k; g_j \in \{f_1, f_2\}, n_1, \dots, n_t \in \mathbb{N}\}$$

is finite that $\boldsymbol{D}_{0,1}$ generates $\mathcal{A}$. The result now follows from the fact that $\boldsymbol{D}_{0,1}$ is locally finite. □

Definition *A congruence relation* on an algebra A of type $\langle k_1, \dots, k_\alpha \rangle$ is an equivalent relation θ on A that satisfies the *substitution property*: for each k_i-ary operation f_i on A with $1 \leqslant i \leqslant \alpha$,

$$(a_j, b_j) \in \theta \ \ (j = 1, \dots, k_i) \Rightarrow \big(f_i(a_1, \dots, a_{k_i}), f_i(b_1, \dots, b_{k_i})\big) \in \theta. \qquad (1.1)$$

Particularly, in a lattice L, an equivalent relation θ is a congruence if $(a, b) \in \theta$ and $(c, d) \in \theta$ imply $(a \wedge c, b \wedge d) \in \theta$ and $(a \vee c, b \vee d) \in \theta$; and this, by the commutativity of lattice operations $\wedge$ and $\vee$, can be simplified to the following equivalent statement:

$$(a, b) \in \theta \Rightarrow (\forall c \in L) \ \ (a \wedge c, b \wedge c) \in \theta \ \text{ and } \ (a \vee c, b \vee c) \in \theta. \qquad (1.2)$$

The congruences on an algebra A are ordered as equivalence relations

$$\theta \leqslant \theta' \iff (x, y) \in \theta \ \text{ implies } \ (x, y) \in \theta'.$$

If θ_1 and θ_2 are congruences on A, then by the relations defined as following:

$$(x, y) \in \theta_1 \wedge \theta_2 \iff (x, y) \in \theta_1 \ \text{ and } \ (x, y) \in \theta_2,$$

$$(x, y) \in \theta_1 \vee \theta_2 \iff \begin{cases} (\exists \ a_0 = x, a_1, \dots, a_n = y \in A) \\ (a_i, a_{i+1}) \in \theta_1 \ \text{ or } \ (a_i, a_{i+1}) \in \theta_2. \end{cases}$$

We can see that $\theta_1 \wedge \theta_2$ is a congruence on A that is the biggest such that it is contained in θ_1 and θ_2; and $\theta_1 \vee \theta_2$ is a congruence on A that is the smallest such that it contains θ_1 and θ_2. Thus, the ordered set of all congruences on A is a lattice with the smallest element $\omega = \{(a, a) \mid a \in A\}$ and the biggest element $\iota = A \times A$. This lattice is called the *congruence lattice* of A, denoted by $\operatorname{Con} A$. More general, if $\{\theta_i \mid i \in I\} \subseteq \operatorname{Con} A$ ($I \neq \varnothing$ (empty set)) we define

$$(x, y) \in \bigwedge_{i \in I} \theta_i \iff (\forall i \in I) \ \ (x, y) \in \theta_i;$$

$$(x,y) \in \bigvee_{i\in I} \theta_i \iff \begin{cases} (\exists\ a_0, a_1, \ldots, a_n \in A \text{ and } \theta_{i_1}, \ldots, \theta_{i_{j+1}}) \\ \text{such that } (a_j, a_{j+1}) \in \theta_{i_{j+1}} \end{cases}$$

where $j = 0, 1, \ldots, n-1$; $a_0 = x$ and $a_n = y$. Then $\bigwedge_{i\in I} \theta_i$ and $\bigvee_{i\in I} \theta_i$ can be proved to satisfy the substitution property (1.1), and consequently, these are congruences on A. In particular, for $a, b \in A$, $\theta(a,b) = \bigwedge\{\varphi \in \operatorname{Con} A \mid (a,b) \in \varphi\}$ is a congruence on A that we shall call the *principal congruence* generated by $\{a, b\}$.

If a, b are in a lattice L with $a \leqslant b$, then we shall denote by $\theta_{\text{lat}}(a,b)$ the principal lattice congruence generated by a, b. We recall that, in a distributive lattice,

$$(x,y) \in \theta_{\text{lat}}(a,b) \iff x \wedge a = y \wedge a \text{ and } x \vee b = y \vee b,$$

and that the intersection of two principal lattice congruences is again a principal lattice congruence. In fact, if $a \leqslant b$ and $c \leqslant d$, then

$$\theta_{\text{lat}}(a,b) \wedge \theta_{\text{lat}}(c,d) = \theta_{\text{lat}}((a \vee c) \wedge b \wedge d, b \wedge d).$$

If A and B are algebras of the same type $\langle k_1, \ldots, k_\alpha \rangle$, then a mapping $\varphi : A \to B$ is a *morphism* if, for each k_i-ary operation f_i on A with $1 \leqslant i \leqslant \alpha$,

$$f_i(\varphi(a_1), \ldots, \varphi(a_{k_i})) = \varphi(f_i(a_1, \ldots, a_{k_i})), \tag{1.3}$$

whenever $(a_1, \ldots, a_{k_i}) \in A^{k_i}$. If, in addition, the mapping φ is surjective, then φ is called an *epimorphism* with B an *epimorphic image* of A; if φ is injective then it is said to be a *monomorphism*; and if φ is bijective (both of surjective and injective) it is an *isomorphism*. A morphism $\varphi : A \to A$ is said to be an *endomorphism* on A; and an isomorphism $\varphi : A \to A$ is said to be an *automorphism* on A.

In particular, if A and B are ordered sets then a mapping $\varphi : A \to B$ is said to be *isotone* if it is such that

$$(\forall x, y \in A)\ x \leqslant y \Rightarrow \varphi(x) \leqslant \varphi(y)$$

and *antitone* if it is such that

$$(\forall x, y \in A)\ x \leqslant y \Rightarrow \varphi(x) \geqslant \varphi(y).$$

If L and M are lattices, then a mapping $\varphi : L \to M$ is said to be *morphism* if it is such that

$$(\forall x, y \in M)\ \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y) \text{ and } \varphi(x \vee y) = \varphi(x) \vee \varphi(y)$$

and *dual morphism* if it is such that

$$(\forall x, y \in M)\ \varphi(x \wedge y) = \varphi(x) \vee \varphi(y) \text{ and } \varphi(x \vee y) = \varphi(x) \wedge \varphi(y).$$

A bijection morphism is an *isomorphism*, and dually, *dual isomorphism* is a bijection dual morphism.

Theorem 1.2 *Let A and B be algebras with the same type, and let $\varphi : A \to B$ be a morphism. Then the following statements hold:*

(1) If X is a subalgebra of A, then $\varphi(X)$ is a subalgebra of B;

(2) If Y is a subalgebra of B, then $\varphi^{-1}(Y) = \{x \in A \mid \varphi(x) \in Y\}$ is a subalgebra of A.

Proof (1) Given a k_i-ary operation f_i on A and $y_1, \dots, y_{k_i} \in \varphi(X)$, choose $x_1, \dots, x_{k_i} \in X$ such that $\varphi(x_j) = y_j$ $(j = 1, 2, \dots, k_i)$. Since X is a subalgebra of A, then $f_i(x_1, \dots, x_{k_i}) \in X$ and so

$$f_i(y_1, \dots, y_{k_i}) = f_i(\varphi(x_1), \dots, \varphi(x_{k_i})) = \varphi(f_i(x_1, \dots, x_{k_i})) \in \varphi(X).$$

It follows that $\varphi(X)$ is a subalgebra.

(2) For a k_i-ary operation f_i and $x_1, \dots, x_{k_i} \in \varphi^{-1}(Y)$, by the definition of φ^{-1}, each $y_j = \varphi(x_j) \in Y$. Since Y is a subalgebra of B, we have

$$\varphi(f_i(x_1, \dots, x_{k_i})) = f_i(\varphi(x_1), \dots, \varphi(x_{k_i})) = f_i(y_1, \dots, y_{k_i}) \in Y,$$

and so $f_i(x_1, \dots, x_{k_i}) \in \varphi^{-1}(Y)$. Consequently, $\varphi^{-1}(Y)$ is a subalgebra of A. □

The following fundamental result is very useful.

Theorem 1.3 *Let φ be a morphism of an algebra A into an algebra B, and define a relation θ on A by*

$$(x, y) \in \theta \iff \varphi(x) = \varphi(y). \tag{1.4}$$

Then θ is a congruence on A.

Proof Since equality is reflexive, symmetric and transitive, so θ is an equivalence relation. If f_i is a k_i-ary operation and $(x_j, y_j) \in \theta$ $(j = 1, 2, \dots, k_i)$, then $\varphi(x_j) = \varphi(y_j)$, and so

$$\begin{aligned} \varphi(f_i(x_1, \dots, x_{k_i})) &= f_i(\varphi(x_1), \dots, \varphi(x_{k_i})) \\ &= f_i(\varphi(y_1), \dots, \varphi(y_{k_i})) \\ &= \varphi(f_i(y_1, \dots, y_{k_i})). \end{aligned}$$

It follows that $f_i(x_1, \dots, x_{k_i}) \overset{\varphi}{\equiv} f_i(y_1, \dots, y_{k_i})$ and consequently, θ is a congruence. □

The congruence defined by a morphism as in (1.4) is called the *kernel* of the morphism φ, denoted by $\operatorname{Ker}\varphi$.

Let now θ be a congruence on an algebra $(A;F)$, where F is an α-tuple $(f_1,\ldots,f_\alpha)$, and let

$$A/\theta = \{[a]\theta \mid a \in A\},$$

where $[a]\theta$ is the equivalence class of θ on A (i.e. $b \in [a]\theta \iff (a,b) \in \theta$). For each k_i-ary operation $f_i \in F$ we define a k_i-ary operation $\overline{f}_i$ on A/θ by

$$\overline{f}_i([x_1]\theta,\ldots,[x_{k_i}]\theta) = [f_i(x_1,\ldots,x_{k_i})]\theta. \tag{1.5}$$

Then $(A/\theta;\overline{F})$ is an algebra of type $(k_1,\ldots,k_\alpha)$, where $\overline{F} = \{\overline{f}_i \mid f_i \in F\}$. This algebra is called the *quotient algebra* of A. Consider now a mapping $\natural : A \to A/\theta$ by $\natural(x) = [x]\theta$. By (1.5) it is clear that the $\natural$ is an epimorphism from $(A;F)$ to $(A/\theta;\overline{F})$. This epimorphism $\natural$ is called the *natural morphism* from an algebra to its quotient algebra. Conversely, the epimorphic images of any algebra A are those the algebras A/θ defined by the congruence relations θ on A. In fact, if $\psi : A \to B$ is an epimorphism, we let $\theta = \operatorname{Ker}\psi$. Then, by Theorem 1.3, $\theta = \operatorname{Ker}\psi$ is a congruence on A. Define a mapping $\varphi : A/\theta \to B$ by $\varphi([a]\theta) = \psi(a)$, then $\varphi \circ \natural = \psi$, and since ψ and $\natural$ are epimorphisms, it is clear that φ is bijective. By observing that, for each k_i-ary operation $\overline{f}_i$ on A/θ,

$$\begin{aligned}\varphi(\overline{f}_i([a_1]\theta,\ldots,[a_{k_i}]\theta)) &= \varphi([f_i(a_1,\ldots,a_{k_i})]\theta)\\ &= \psi(f_i(a_1,\ldots,a_{k_i}))\\ &= f_i(\psi(a_1),\ldots,\psi(a_{k_i}))\\ &= f_i(\varphi([a_1]\theta),\ldots,\varphi([a_{k_i}]\theta)),\end{aligned}$$

we see that φ satisfies (1.3) and therefore φ is an isomorphism.

Let $(A_i;F)$ $(i \in I)$ be a family of algebras with the same type $(k_1,\ldots,k_\alpha)$, then it can be formed as a direct (cartesian) product $\big(\bigtimes_{i\in I} A_i;F\big)$: $p(i) \in A_i$ for any $p \in \bigtimes_{i\in I} A_i$; and each $f_t \in F$ is given as follows:

$$\big(\forall p_1,\ldots,p_t \in \bigtimes_{i\in I} A_i\big)\ \ f_t(p_1,\ldots,p_{k_t})(i) = f_t(p_1(i),\ldots,p_{k_t}(i)).$$

Consider now the j-th projection $e_j : \bigtimes_{i\in I} A_i \to A_j$ given by $e_j(p) = p(j)$ for $p \in \bigtimes_{i\in I} A_i$. Then each e_j is an epimorphism that can induce a congruence ψ_j given by the following description:

$$\big(\forall p,q \in \bigtimes_{i\in I} A_i\big)\ \ (p,q) \in \psi_j \iff p(j) = q(j); \tag{1.6}$$

and for each $j \in I$, $(\bigtimes_{i\in I} A_i)/\psi_j$ is isomorphic to A_j.

The following result is due to G. Birkhoff.

Theorem 1.4 [20] *Let A and A_i $(i \in I)$ be algebras with the same type $(k_1, \ldots, k_\alpha)$, and let φ_i be a morphism of A to A_i for each $i \in I$. If define a mapping $h : A \to \bigtimes_{i\in I} A_i$ by*

$$(\forall a \in A)(\forall i \in I)\ \ h(a)(i) = \varphi_i(a).$$

Then h is a morphism of A to $\bigtimes_{i\in I} A_i$ and $e_i \circ h = \varphi_i$ for all $i \in I$. □

By Theorem 1.2, $h(A)$ in the above Theorem 1.4 is a subalgebra of $\bigtimes_{i\in I} A_i$. If now A is a subalgebra of the direct product $\bigtimes_{i\in I} A_i$ of a family of algebras A_i $(i \in I)$, we say that A is a *subdirect product* of algebras A_i $(i \in I)$ whenever $e_i(A) = A_i$ for all $i \in I$. This is equivalent to the following property:

for any $a_i \in A$, there exists $p \in A$ such that $p(i) = a_i$.

Let now A be an algebra and B subalgebra of A. If α is a congruence on A, then the equivalence relation β, the restriction α to B (this being denoted by $\alpha|_B = \beta$) is a congruence on B. Let ψ_i be denoted as in (1.5), and $\theta_i = \psi_i|_A$. Then the following results are also due to G. Birkhoff.

Theorem 1.5 [20] *If A is a subdirect product of a family of algebras A_i $(i \in I)$ with the same type, then the following statements hold:*

(1) *$A/\theta_i \simeq A_i$ (isomorphic);*

(2) $\bigwedge_{i\in I} \theta_i = \omega$. □

Theorem 1.6 [20] *Let A be an algebra, and let $\{\theta_i \mid i \in I\}$ be a family of congruences on A such that $\bigwedge_{i\in I} \theta_i = \omega$. Then A is isomorphic to a subdirect product of the algebras A/θ_i $(i \in I)$.* □

An important notion in universal algebras is so called *subdirectly irreducible.* We say that an algebra A is *subdirectly irreducible* if $\bigwedge_{i\in I} \theta_i = \omega$ $(\theta_i \in \operatorname{Con} A)$ implies that there exists at least one $\theta_i \in \operatorname{Con} A$ such that $\theta_i = \omega$. This condition is equivalent to that A has a smallest non-trivial congruence; i.e. a congruence α such that $\theta \geqslant \alpha$ for all $\theta \in \operatorname{Con} A$ with $\theta \neq \omega$. Such a congruence α is called the *monolith* of $\operatorname{Con} A$, and dually, the *comonolith* $\beta \in \operatorname{Con} A$ if $\beta \geqslant \theta$ for all $\theta \in \operatorname{Con} A$ with $\theta \neq \iota$. A particular important case of a subdirectly irreducible algebra is a *simple algebra*, namely one for which the congruence lattice is the two-element chain $\omega < \iota$.

A class of algebras is *equational* (or call *variety*) if it is closed under the formation of subalgebras, epimorphic images, and direct products. As illustrated in the following result, which is a classic theorem of Birkhoff, subdirectly irreducible algebras play a very important role in study of equational algebras.

Theorem 1.7 [20] *Every algebra in a variety of algebras is isomorphic to a direct product of subdirectly irreducible algebras.* □

A subclass of a variety $\boldsymbol{V}$ which is also a variety is called a *subvariety* of $\boldsymbol{V}$. The subvarieties of $\boldsymbol{V}$ form a lattice which we denote by $\Lambda(\boldsymbol{V})$, in which the meet $\boldsymbol{A} \wedge \boldsymbol{B}$ of two subvarieties $\boldsymbol{A}$ and $\boldsymbol{B}$ is their intersection, and the join $\boldsymbol{A} \vee \boldsymbol{B}$ is the smallest subvariety of $\boldsymbol{V}$ that contains $\boldsymbol{A} \cup \boldsymbol{B}$. A classic theorem of B. Jónsson[86] states that if $\boldsymbol{V}$ is a variety every algebra of which has a distributive congruence lattice then $\Lambda(\boldsymbol{V})$ is distributive. A fundamental theorem for a congruence-distributive variety was established by B. A. Davey[55] that is stated as follows.

Theorem 1.8 [55] *Let $\boldsymbol{K} = \boldsymbol{V}(S)$ be a congruence-distributive variety generated by a finite set S of finite algebras, and order the set $Si(\boldsymbol{K})$ of subdirectly irreducible algebras in $\boldsymbol{K}$ by*

$$A \leqslant B \iff A \text{ is a homomorphic image of a subalgebra of } B.$$

Then $\Lambda(\boldsymbol{K})$ is a finite distributive lattice and is isomorphic to $\mathcal{O}(Si(\boldsymbol{K}))$, the set of down-sets of $Si(\boldsymbol{K})$. Moreover, $\boldsymbol{A}$ is a join-irreducible element of $\Lambda(\boldsymbol{K})$ if and only if $\boldsymbol{A} = \boldsymbol{V}(A)$ for some $A \in Si(\boldsymbol{K})$. □

A class $\boldsymbol{K}$ of algebras is said to enjoy the (principal) *congruence extension property* if, for all $A, B \in \boldsymbol{K}$ with A a subalgebra of B, every (principal) congruence θ on A is the restriction $\varphi|_A$ of some congruence φ on B. We say that B is a *strong extension* of A if every congruence on A has at most one extension to B, in which case A is said to be a *strongly large subalgebra* of B.

As shown by A. Day, in an equational class of algebras $\boldsymbol{K}$, it enjoys congruence extension property if and only if it enjoys principal congruence extension property which is equivalent to the following condition.

for all subalgebras A of B and all $a, b \in A$, $\theta_A(a,b) = \theta_B(a,b)|_A$.

The following result shall prove to be useful.