

国外数学名著系列

(影印版) 68

Dmitry Kozlov

Combinatorial Algebraic
Topology

组合代数拓扑



科学出版社

国外数学名著系列(影印版) 68

Combinatorial Algebraic Topology

组合代数拓扑

Dmitry Kozlov



科学出版社

北京

图字: 01-2011-3328

Dmitry Kozlov: Combinatorial Algebraic Topology
© Springer-Verlag Berlin Heidelberg 2008

This reprint has been authorized by Springer-Verlag (Berlin/Heidelberg/New York) for sale in the People's Republic of China only and not for export therefrom.

本书英文影印版由德国施普林格出版公司授权出版。未经出版者书面许可,不得以任何方式复制或抄袭本书的任何部分。本书仅限在中华人民共和国销售,不得出口。版权所有,翻印必究。

图书在版编目(CIP)数据

组合代数拓扑= Combinatorial Algebraic Topology / (德)科兹洛夫
(Kozlov, D.)编著. —影印版. —北京: 科学出版社, 2011
(国外数学名著系列; 68)

ISBN 978-7-03-031383-6

I. ①组… II. ①科… III. ①代数拓扑-英文 IV. ①O189.2

中国版本图书馆 CIP 数据核字(2011) 第 105009 号

责任编辑:赵彦超 李欣/责任印刷:钱玉芬/封面设计:陈敬

科学出版社 出版

北京东黄城根北街16号

邮政编码:100717

<http://www.sciencep.com>

双青印刷厂 印刷

科学出版社发行 各地新华书店经销

*

2011年6月第一版 开本: B5(720×1000)

2011年6月第一次印刷 印张: 26

印数: 1—2 000 字数: 496 000

定价: 98.00 元

(如有印装质量问题, 我社负责调换)

《国外数学名著系列》(影印版)序

要使我国的数学事业更好地发展起来,需要数学家淡泊名利并付出更艰苦地努力。另一方面,我们也要从客观上为数学家创造更有利的发展数学事业的外部环境,这主要是加强对数学事业的支持与投资力度,使数学家有较好的工作与生活条件,其中也包括改善与加强数学的出版工作。

从出版方面来讲,除了较好较快地出版我们自己的成果外,引进国外的先进出版物无疑也是十分重要与必不可少的。从数学来说,施普林格(Springer)出版社至今仍然是世界上最具权威的出版社。科学出版社影印一批他们出版的好的新书,使我国广大数学家能以较低的价格购买,特别是在边远地区工作的数学家能普遍见到这些书,无疑是对推动我国数学的科研与教学十分有益的事。

这次科学出版社购买了版权,一次影印了 23 本施普林格出版社出版的数学书,就是一件好事,也是值得继续做下去的事情。大体上分一下,这 23 本书中,包括基础数学书 5 本,应用数学书 6 本与计算数学书 12 本,其中有些书也具有交叉性质。这些书都是很新的,2000 年以后出版的占绝大部分,共计 16 本,其余的也是 1990 年以后出版的。这些书可以使读者较快地了解数学某方面的前沿,例如基础数学中的数论、代数与拓扑三本,都是由该领域大数学家编著的“数学百科全书”的分册。对从事这方面研究的数学家了解该领域的前沿与全貌很有帮助。按照学科的特点,基础数学类的书以“经典”为主,应用和计算数学类的书以“前沿”为主。这些书的作者多数是国际知名的大数学家,例如《拓扑学》一书的作者诺维科夫是俄罗斯科学院的院士,曾获“菲尔兹奖”和“沃尔夫数学奖”。这些大数学家的著作无疑将会对我国的科研人员起到非常好的指导作用。

当然,23 本书只能涵盖数学的一部分,所以,这项工作还应该继续做下去。更进一步,有些读者面较广的好书还应该翻译成中文出版,使之有更大的读者群。

总之,我对科学出版社影印施普林格出版社的部分数学著作这一举措表示热烈的支持,并盼望这一工作取得更大的成绩。

王 元

2005 年 12 月 3 日

Dedicated to my family

Preface

The intent of this book is to introduce the reader to the beautiful world of Combinatorial Algebraic Topology. While the main purpose is to describe the modern research tools and latest applications of this field, an attempt has been made to keep the presentation as self-contained as possible.

A book to teach from

The text is divided into three major parts, which provide several options for adoption for course purposes, depending on the time available to the instructor.

The first part furnishes a brisk walk through some of the basic concepts of algebraic topology. While it is in no way meant to replace a standard course in that field, it could prove helpful at the beginning of the lectures, in case the audience does not have much prior knowledge of algebraic topology or would like to focus on refreshing those notions that will be needed in subsequent chapters. The first part can be read by itself, or used as a blueprint with a standard textbook in algebraic topology such as [Mun84] or [Hat02] as additional reading. Alternatively, it could also be used for an independent course or for a student seminar.

If the audience is sufficiently familiar with algebraic topology, then one could start directly with the second part. This is suitable for a graduate or advanced undergraduate course whose purpose would be to learn contemporary tools of Combinatorial Algebraic Topology and to see them in use on some examples. At the end of the course, a successful student should be able to conduct independent research on this topic.

The third and last part of the book is a foray into one specific realm of a present-day application: the topology of complexes of graph homomorphisms. It fits well at the end of the envisioned graduate course, and is meant as a source of illustrations of various techniques developed in the second part. Another possibility would be to use it as material for a reading seminar.

What is different in our presentation

In the second part we lay the foundations of Combinatorial Algebraic Topology. In particular, we survey many of the tools that have been used in research in topological combinatorics over the last 20 years. However, our approach is at times quite different from the one prevailing in some of the literature.

Perhaps the major novelty is the general shift of focus from the category of posets to the category of acyclic categories. Correspondingly, the entire Chapter 10 is devoted to the development of the fundamental theory of acyclic categories and of the topology of their nerves, which in turn are no longer abstract simplicial complexes, but rather regular triangulated spaces.

Also, Chapter 11 is designed to give quite a different take on discrete Morse theory. The theory is broken into three major branches: combinatorial, topological, and algebraic; each one with its own specifics. A very new feature here is the recasting of discrete Morse theory for posets in terms of *poset maps with small fibers*. This, together with the existence of a universal object associated to every acyclic matching and the Patchwork Theorem allows for a structural understanding of the techniques that have been used until now.

There are further novelties scattered in the remaining four chapters of the second part. In Chapter 13 we connect the notion of evasiveness with monotone poset maps, and introduce the notion of NE-reduction. After that, the importance of colimits in Combinatorial Algebraic Topology is emphasized. We look at regular colimits and their relation with group actions in Chapter 14, and at homotopy colimits in Chapter 15. We provide complete proofs for all the statements in Chapter 15, based on the previous groundwork pertaining to cofibrations in Chapter 7. Finally, in Chapter 16, we take a daring step of counting the machinery of spectral sequences to the core methods of Combinatorial Algebraic Topology.

Let us also comment briefly on our citation policy. As far as possible we have tried to avoid citations directly in the text, choosing to present material in the way that appeared to us to be most coherent from the contemporary point of view. Instead, each chapter in the second and third parts ends with a detailed bibliographic account of the contents of that chapter. Since the mathematics of the first part is much more classical, we skip bibliographic information there almost completely, giving only general references to the existing textbooks. An exception is provided by Chapter 8, where the material is slightly less standard, thus justifying making some reading suggestions.

Acknowledgments

Many organizations as well as individuals have made it possible for this book project to be completed. To start with, it certainly would not have materialized without the generous financial support of the Swiss National Science Foundation and of the Institute of Theoretical Computer Science at the Swiss Federal Institute of Technology in Zürich. Furthermore, a major part of this work has been done while the author was in residence as a research professor at the Mathematical Science Research Institute in Berkeley, whose hospitality, as well as the collegiality of the organizers of the special program during the fall term 2006, is warmly appreciated. The last academic institution that the author would like to thank is the University of Bremen, which has generously granted him a research leave, so that in particular this project could be completed.

The staff at Springer has been most encouraging and helpful indeed. Many thanks go to Martin Peters and Ruth Allewelt, who have managed to circumvent all the clever excuses that I kept fabricating for not being able to finish the writing.

This text has grown from a one-year graduate course that I have given at ETH Zürich to an enthusiastic group of students. Their comments have been most welcome and have led to substantial improvements. Special thanks go to Peter Csorba, whose additional careful proofreading of the text has revealed many inconsistencies both notational and mathematical.

The head of the Institute of Theoretical Computer Science at ETH Zürich, Emo Welzl, has been a congenial host and a thoughtful mentor during my sojourn as an assistant professor here. For this he has my deepest gratitude.

Much of the material in this book is based on joint research with my long-time collaborator Eric Babson, from UC Davis. Without him there would be no book. He is the spiritual coauthor and I thank him for this.

Writing a text of this length can be a daunting task, and it is invaluable when someone's support is guaranteed come rain or come shine. During this work, I was in a singularly fortunate situation of having my mathematical collaborator and my wife, Eva-Maria Feichtner, by my side, to help me to persevere when it seemed all but futile. There is no way I can thank her enough for all the advice, comfort, and reassurance that she lent me.

Finally, I would like to thank my daughter, Esther Yael Feichtner, who was born in the middle of this project and immediately introduced an element of randomness into the timetable. The future looks bright for her, as the opportunities for this welcome sabotage abound.

Contents

1	Overture	1
----------	-----------------------	----------

Part I Concepts of Algebraic Topology

2	Cell Complexes	7
2.1	Abstract Simplicial Complexes	7
2.1.1	Definition of Abstract Simplicial Complexes and Maps Between Them	7
2.1.2	Deletion, Link, Star, and Wedge	10
2.1.3	Simplicial Join	12
2.1.4	Face Posets	12
2.1.5	Barycentric and Stellar Subdivisions	13
2.1.6	Pulling and Pushing Simplicial Structures	15
2.2	Polyhedral Complexes	16
2.2.1	Geometry of Abstract Simplicial Complexes	16
2.2.2	Geometric Meaning of the Combinatorial Constructions	19
2.2.3	Geometric Simplicial Complexes	23
2.2.4	Complexes Whose Cells Belong to a Specified Set of Polyhedra	25
2.3	Trisps	28
2.3.1	Construction Using the Gluing Data	28
2.3.2	Constructions Involving Trisps	30
2.4	CW Complexes	33
2.4.1	Gluing Along a Map	33
2.4.2	Constructive and Intrinsic Definitions	34
2.4.3	Properties and Examples	35
3	Homology Groups	37
3.1	Betti Numbers of Finite Abstract Simplicial Complexes	37
3.2	Simplicial Homology Groups	39

3.2.1	Homology Groups of Trisps with Coefficients in \mathbb{Z}_2	39
3.2.2	Orientations	41
3.2.3	Homology Groups of Trisps with Integer Coefficients . . .	41
3.3	Invariants Connected to Homology Groups	44
3.3.1	Betti Numbers and Torsion Coefficients	44
3.3.2	Euler Characteristic and the Euler–Poincaré Formula . .	45
3.4	Variations	46
3.4.1	Augmentation and Reduced Homology Groups	46
3.4.2	Homology Groups with Other Coefficients	47
3.4.3	Simplicial Cohomology Groups	47
3.4.4	Singular Homology	49
3.5	Chain Complexes	51
3.5.1	Definition and Homology of Chain Complexes	51
3.5.2	Maps Between Chain Complexes and Induced Maps on Homology	52
3.5.3	Chain Homotopy	53
3.5.4	Simplicial Homology and Cohomology in the Context of Chain Complexes	54
3.5.5	Homomorphisms on Homology Induced by Trisp Maps .	54
3.6	Cellular Homology	56
3.6.1	An Application of Homology with Integer Coefficients: Winding Number	56
3.6.2	The Definition of Cellular Homology	57
3.6.3	Cellular Maps and Properties of Cellular Homology . . .	58
4	Concepts of Category Theory	59
4.1	The Notion of a Category	59
4.1.1	Definition of a Category, Isomorphisms	59
4.1.2	Examples of Categories	60
4.2	Some Structure Theory of Categories	63
4.2.1	Initial and Terminal Objects	63
4.2.2	Products and Coproducts	64
4.3	Functors	68
4.3.1	The Category Cat	68
4.3.2	Homology and Cohomology Viewed as Functors	70
4.3.3	Group Actions as Functors	70
4.4	Limit Constructions	71
4.4.1	Definition of Colimit of a Functor	71
4.4.2	Colimits and Infinite Unions	72
4.4.3	Quotients of Group Actions as Colimits	73
4.4.4	Limits	74
4.5	Comma Categories	74
4.5.1	Objects Below and Above Other Objects	74
4.5.2	The General Construction and Further Examples	75

5	Exact Sequences	77
5.1	Some Structure Theory of Long and Short Exact Sequences ...	77
5.1.1	Construction of the Connecting Homomorphism	77
5.1.2	Exact Sequences	79
5.1.3	Deriving Long Exact Sequences from Short Ones	81
5.2	The Long Exact Sequence of a Pair and Some Applications ...	82
5.2.1	Relative Homology and the Associated Long Exact Sequence	82
5.2.2	Applications	84
5.3	Mayer–Vietoris Long Exact Sequence	85
6	Homotopy	89
6.1	Homotopy of Maps	89
6.2	Homotopy Type of Topological Spaces	90
6.3	Mapping Cone and Mapping Cylinder	91
6.4	Deformation Retracts and Collapses	93
6.5	Simple Homotopy Type	95
6.6	Homotopy Groups	96
6.7	Connectivity and Hurewicz Theorems	97
7	Cofibrations	101
7.1	Cofibrations and the Homotopy Extension Property	101
7.2	NDR-Pairs	103
7.3	Important Facts Involving Cofibrations	105
7.4	The Relative Homotopy Equivalence	107
8	Principal Γ-Bundles and Stiefel–Whitney Characteristic Classes	111
8.1	Locally Trivial Bundles	111
8.1.1	Bundle Terminology	111
8.1.2	Types of Bundles	112
8.1.3	Bundle Maps	113
8.2	Elements of the Principal Bundle Theory	114
8.2.1	Principal Bundles and Spaces with a Free Group Action	114
8.2.2	The Classifying Space of a Group	116
8.2.3	Special Cohomology Elements	119
8.2.4	\mathbb{Z}_2 -Spaces and the Definition of Stiefel–Whitney Classes	120
8.3	Properties of Stiefel–Whitney Classes	122
8.3.1	Borsuk–Ulam Theorem, Index, and Coindex	122
8.3.2	Stiefel–Whitney Height	123
8.3.3	Higher Connectivity and Stiefel–Whitney Classes	123
8.3.4	Combinatorial Construction of Stiefel–Whitney Classes	124
8.4	Suggested Reading	125

Part II Methods of Combinatorial Algebraic Topology

9	Combinatorial Complexes Melange	129
9.1	Abstract Simplicial Complexes	129
9.1.1	Simplicial Flag Complexes	129
9.1.2	Order Complexes	130
9.1.3	Complexes of Combinatorial Properties	133
9.1.4	The Neighborhood and Lovász Complexes	133
9.1.5	Complexes Arising from Matroids	134
9.1.6	Geometric Complexes in Metric Spaces	134
9.1.7	Combinatorial Presentation by Minimal Nonsimplices ..	136
9.2	Prodsimplicial Complexes	138
9.2.1	Prodsimplicial Flag Complexes	138
9.2.2	Complex of Complete Bipartite Subgraphs	138
9.2.3	Hom Complexes	140
9.2.4	General Complexes of Morphisms	141
9.2.5	Discrete Configuration Spaces of Generalized Simplicial Complexes	144
9.2.6	The Complex of Phylogenetic Trees	144
9.3	Regular Trisps	145
9.4	Chain Complexes	147
9.5	Bibliographic Notes	148
10	Acyclic Categories	151
10.1	Basics	151
10.1.1	The Notion of Acyclic Category	151
10.1.2	Linear Extensions of Acyclic Categories	152
10.1.3	Induced Subcategories of \mathbf{Cat}	153
10.2	The Regular Trisp of Composable Morphism Chains in an Acyclic Category	153
10.2.1	Definition and First Examples	153
10.2.2	Functoriality	155
10.3	Constructions	156
10.3.1	Disjoint Union as a Coproduct	156
10.3.2	Stacks of Acyclic Categories and Joins of Regular Trisps	156
10.3.3	Links, Stars, and Deletions	158
10.3.4	Lattices and Acyclic Categories	159
10.3.5	Barycentric Subdivision and Δ -Functor	160
10.4	Intervals in Acyclic Categories	161
10.4.1	Definition and First Properties	161
10.4.2	Acyclic Category of Intervals and Its Structural Functor	164
10.4.3	Topology of the Category of Intervals	167

10.5	Homeomorphisms Associated with the Direct Product	
	Construction	168
10.5.1	Simplicial Subdivision of the Direct Product	168
10.5.2	Further Subdivisions	171
10.6	The Möbius Function	173
10.6.1	Möbius Function for Posets	173
10.6.2	Möbius Function for Acyclic Categories	174
10.7	Bibliographic Notes	178
11	Discrete Morse Theory	179
11.1	Discrete Morse Theory for Posets	179
11.1.1	Acyclic Matchings in Hasse Diagrams of Posets	179
11.1.2	Poset Maps with Small Fibers	182
11.1.3	Universal Object Associated to an Acyclic Matching	183
11.1.4	Poset Fibrations and the Patchwork Theorem	185
11.2	Discrete Morse Theory for CW Complexes	187
11.2.1	Attaching Cells to Homotopy Equivalent Spaces	187
11.2.2	The Main Theorem of Discrete Morse Theory for CW Complexes	189
11.2.3	Examples	192
11.3	Algebraic Morse Theory	201
11.3.1	Acyclic Matchings on Free Chain Complexes and the Morse Complex	201
11.3.2	The Main Theorem of Algebraic Morse Theory	203
11.3.3	An Example	205
11.4	Bibliographic Notes	208
12	Lexicographic Shellability	211
12.1	Shellability	211
12.1.1	The Basics	211
12.1.2	Shelling Induced Subcomplexes	214
12.1.3	Shelling Nerves of Acyclic Categories	215
12.2	Lexicographic Shellability	216
12.2.1	Labeling Edges as a Way to Order Chains	216
12.2.2	EL-Labeling	217
12.2.3	General Lexicographic Shellability	219
12.2.4	Lexicographic Shellability and Nerves of Acyclic Categories	223
12.3	Bibliographic Notes	224
13	Evasiveness and Closure Operators	225
13.1	Evasiveness	225
13.1.1	Evasiveness of Graph Properties	225
13.1.2	Evasiveness of Abstract Simplicial Complexes	229

13.2	Closure Operators	232
13.2.1	Collapsing Sequences Induced by Closure Operators	232
13.2.2	Applications	234
13.2.3	Monotone Poset Maps	236
13.2.4	The Reduction Theorem and Implications	237
13.3	Further Facts About Nonevasiveness	238
13.3.1	NE-Reduction and Collapses	238
13.3.2	Nonevasiveness of Noncomplemented Lattices	240
13.4	Other Recursively Defined Classes of Complexes	242
13.5	Bibliographic Notes	243
14	Colimits and Quotients	245
14.1	Quotients of Nerves of Acyclic Categories	245
14.1.1	Desirable Properties of the Quotient Construction	245
14.1.2	Quotients of Simplicial Actions	245
14.2	Formalization of Group Actions and the Main Question	248
14.2.1	Definition of the Quotient and Formulation of the Main Problem	248
14.2.2	An Explicit Description of the Category C/G	249
14.3	Conditions on Group Actions	250
14.3.1	Outline of the Results and Surjectivity of the Canonical Map	250
14.3.2	Condition for Injectivity of the Canonical Projection	251
14.3.3	Conditions for the Canonical Projection to be an Isomorphism	252
14.3.4	Conditions for the Categories to be Closed Under Taking Quotients	255
14.4	Bibliographic Notes	257
15	Homotopy Colimits	259
15.1	Diagrams over Trisps	259
15.1.1	Diagrams and Colimits	259
15.1.2	Arrow Pictures and Their Nerves	260
15.2	Homotopy Colimits	262
15.2.1	Definition and Some Examples	262
15.2.2	Structural Maps Associated to Homotopy Colimits	263
15.3	Deforming Homotopy Colimits	265
15.4	Nerves of Coverings	266
15.4.1	Nerve Diagram	266
15.4.2	Projection Lemma	267
15.4.3	Nerve Lemmas	269
15.5	Gluings Spaces	271
15.5.1	Gluings Lemma	271
15.5.2	Quillen Lemma	272
15.6	Bibliographic Notes	273

16 Spectral Sequences	275
16.1 Filtrations	275
16.2 Contriving Spectral Sequences	276
16.2.1 The Objects to be Constructed	276
16.2.2 The Actual Construction	278
16.2.3 Questions of Convergence and Interpretation of the Answer	280
16.2.4 An Example	280
16.3 Maps Between Spectral Sequences	281
16.4 Spectral Sequences and Nerves of Acyclic Categories	283
16.4.1 A Class of Filtrations	283
16.4.2 Möbius Function and Inequalities for Betti Numbers ...	285
16.5 Bibliographic Notes	288

Part III Complexes of Graph Homomorphisms

17 Chromatic Numbers and the Kneser Conjecture	293
17.1 The Chromatic Number of a Graph	293
17.1.1 The Definition and Applications	293
17.1.2 The Complexity of Computing the Chromatic Number .	294
17.1.3 The Hadwiger Conjecture	295
17.2 State Graphs and the Variations of the Chromatic Number ...	298
17.2.1 Complete Graphs as State Graphs	298
17.2.2 Kneser Graphs as State Graphs and Fractional Chromatic Number	298
17.2.3 The Circular Chromatic Number	300
17.3 Kneser Conjecture and Lovász Test	301
17.3.1 Formulation of the Kneser Conjecture	301
17.3.2 The Properties of the Neighborhood Complex	302
17.3.3 Lovász Test for Graph Colorings	303
17.3.4 Simplicial and Cubical Complexes Associated to Kneser Graphs	304
17.3.5 The Vertex-Critical Subgraphs of Kneser Graphs	306
17.3.6 Chromatic Numbers of Kneser Hypergraphs	307
17.4 Bibliographic Notes	307
18 Structural Theory of Morphism Complexes	309
18.1 The Scope of Morphism Complexes	309
18.1.1 The Morphism Complexes and the Prodsimplicial Flag Construction	309
18.1.2 Universality	311
18.2 Special Families of Hom Complexes	312
18.2.1 Coloring Complexes of a Graph	312

18.2.2	Complexes of Bipartite Subgraphs and Neighborhood Complexes	313
18.3	Functoriality of $\text{Hom}(-, -)$	315
18.3.1	Functoriality on the Right	315
18.3.2	$\text{Aut}(G)$ Action on $\text{Hom}(T, G)$	316
18.3.3	Functoriality on the Left	316
18.3.4	$\text{Aut}(T)$ Action on $\text{Hom}(T, G)$	318
18.3.5	Commuting Relations	318
18.4	Products, Compositions, and Hom Complexes	320
18.4.1	Coproducts	320
18.4.2	Products	320
18.4.3	Composition of Hom Complexes	322
18.5	Folds	323
18.5.1	Definition and First Properties	323
18.5.2	Proof of the Folding Theorem	324
18.6	Bibliographic Notes	326
19	Characteristic Classes and Chromatic Numbers	327
19.1	Stiefel–Whitney Characteristic Classes and Test Graphs	327
19.1.1	Powers of Stiefel–Whitney Classes and Chromatic Numbers of Graphs	327
19.1.2	Stiefel–Whitney Test Graphs	328
19.2	Examples of Stiefel–Whitney Test Graphs	329
19.2.1	Complexes of Complete Multipartite Subgraphs	329
19.2.2	Odd Cycles as Stiefel–Whitney Test Graphs	334
19.3	Homology Tests for Graph Colorings	337
19.3.1	The Symmetrizer Operator and Related Structures	338
19.3.2	The Topological Rationale for the Tests	338
19.3.3	Homology Tests	340
19.3.4	Examples of Homology Tests with Different Test Graphs	341
19.4	Bibliographic Notes	346
20	Applications of Spectral Sequences to Hom Complexes	349
20.1	Hom_+ Construction	349
20.1.1	Various Definitions	349
20.1.2	Connection to Independence Complexes	351
20.1.3	The Support Map	352
20.1.4	An Example: $\text{Hom}_+(C_m, K_n)$	353
20.2	Setting up the Spectral Sequence	354
20.2.1	Filtration Induced by the Support Map	354
20.2.2	The 0th and the 1st Tableaux	355
20.2.3	The First Differential	355
20.3	Encoding Cohomology Generators by Arc Pictures	356
20.3.1	The Language of Arcs	356
20.3.2	The Corresponding Cohomology Generators	356

20.3.3	The First Reduction	357
20.4	Topology of the Torus Front Complexes	358
20.4.1	Reinterpretation of $H^*(A_t^*, d_1)$ Using a Family of Cubical Complexes $\{\Phi_{m,n,g}\}$	358
20.4.2	The Torus Front Interpretation	360
20.4.3	Grinding	362
20.4.4	Thin Fronts	364
20.4.5	The Implications for the Cohomology Groups of $\text{Hom}(C_m, K_n)$	366
20.5	Euler Characteristic Formula	367
20.6	Cohomology with Integer Coefficients	368
20.6.1	Fixing Orientations on Hom and Hom_+ Complexes	368
20.6.2	Signed Versions of Formulas for Generators $[\sigma_V^S]$	370
20.6.3	Completing the Calculation of the Second Tableau	371
20.6.4	Summary: the Full Description of the Groups $\tilde{H}^*(\text{Hom}(C_m, K_n); \mathbb{Z})$	374
20.7	Bibliographic Notes and Conclusion	376
References		377
Index		385