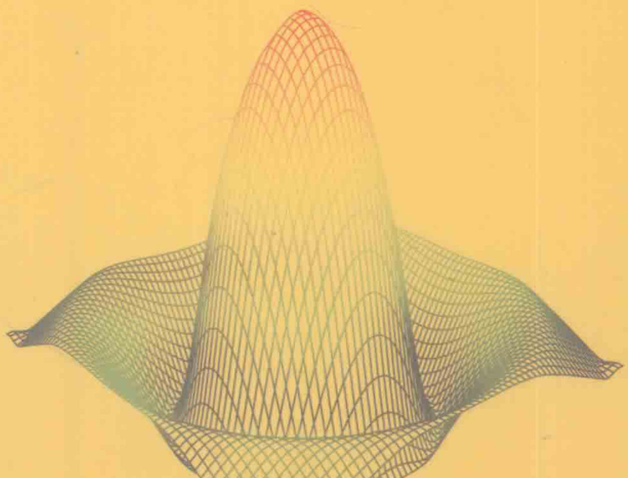
Number Theory and Special Functions

(数论与特殊函数)

Hailong Li



SCIENCE PRESS
Beijing

Hailong Li

Number Theory and Special Functions

(数论与特殊函数)

The monograph is supported by the following funds:
Academic Publication Fund of Weinan Teachers University
The Special Fund for Mathematics Key Discipline of Shaanxi Province

SCIENCE PRESS
Beijing

Responsible Editors: Wang Liping and Ma Shouquan

Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, P. R. China

Printed in Beijing

ISBN 978-7-03-030911-2

Preface

The purpose of this book is two-fold. Firstly, it gives some basics of complex function theory and special functions and secondly, it assembles most important results in my own research made in the last several years under the guidance of Professor S. Kanemitsu, known as Jin Guangzi in China. The author would like to thank Mr. Y. -L. Lu for helping with typesetting.

Thus a beginner reader can use this book as a quick introduction to complex analysis and special functions, and an advanced reader can use it as a source book of many research problems. E.g. in the study of the Euler integral which appeared in a generalization of Jensen's formula, there are possible new results obtained, likewise the study of Catalan's constant and Kummer's Fourier series would be a rich arsenal for future studies.

Also, many of the results are presented so that the reader can get used to the use of the zeta-symmetry, the ultimate power in analytic number theory, in the same spirit as Vinogradov's book *Elements of Number Theory* which makes the reader be familiar with exponential sums, leading the reader to the rich world of modular relations.

A somewhat new device is to consider a part of the series of the integrals as the case may be, which speaks out for the whole, i.e. in Chapter 2, the partial sum $L_u(x, a)$ for the Hurwitz zeta-function plays a fundamental role and so does the partial integral $I_\kappa(x)$ in Section 3.5.

Here is a rough description of the contents.

Chapter 1 gives a quick introduction to complex analysis to such an extent that is needed to go through this book and can be skipped by an advanced reader.

Chapter 2 assembles those results which I obtained in the paper [LT] and some material for X. -H. Wang's Master's thesis, elaborating some results in the book of Srivastava and Choi [SC].

Chapter 3 incorporates those results in the paper [DLS] and my Ph.D thesis [Li], connected with arithmetic properties of Laurent coefficients of important zeta-functions including the power of the Riemann zeta-function.

Chapter 4 incorporates the results connected with Mikolás' integral representations for the Hurwitz and the product of Hurwitz zeta-functions, and is based on the paper [LHK1].

Chapter 5 has the title "zeta-value relations", meaning the linear relations between values of a class of zeta-functions at rational arguments. In the first half, we present the method of Yamamoto of finite Fourier series and obtain the generalization of the Eisenstein's formula, based on [LHK2]. In the last half, we make full use of the intrinsic properties of the Lipcshitz-Lerch zeta-functions, showing an easier and clearer approach to such problems through special functions, based on [CKL].

Chapter 6 assembles two important summation formulas, i.e. that of Poisson and of Plana after proving the Poisson summation formula for C^1-functions, we establish the theta transformation formula, which is then applied to derive the functional equation for the Riemann zeta-function following one of two methods of Riemann, based on the book of Rademacher. In the second-half, we establish the Plana summation formula by proving at the same time the integral representation for the Lipschitz-Lerch zeta-function, based on [LY].

Finally, in Chapter 7 we collect some results on the modular relation: One is a new derivation of the Fourier series for the periodic Bernoulli polynomial ([LMZ]) and the other is a vast generalization of Katsurada's results ([LKT]) which are in themselves vast generalizations of previous results.

Hailong Li
2011.2.26

Contents

Chapter 1

A quick introduction to complex analysis

In this chapter I will assemble some basic facts from complex analysis to such an extent that will be needed to follow the argument that follows. I will state the results in the simplest possible way so that even a beginner can go through the flow of the thoughts and argument.

1.1 Introduction

In this book I am going to present what I have obtained in the last several years in the intermediate field of **number theory and special functions**.

However, to have a better grasp of the results, it is indispensable to have a good command of complex function theory. Therefore, I will assemble some of the most basic facts of the theory as a quick introduction in this chapter, which I hope will serve as teaching material for a short course on function theory.

Naturally complex function theory consists of two main streams, **Weierstrass theory** of power series and their **analytic continuation** and **Cauchy's theory** of complex integration.

First I will state rudiments of the theory of power series which are very useful in extending the real theory into complex theory. I will state manifestations of the **Cauchy integral theorem**, the **Cauchy integral formula**, which may be regarded as a consequence of the **Cauchy residue theorem**, and then I will follow Ahlfors [Ahl] to present a generalization of the **Gauss mean value theorem**, in which use is made of the Euler integral (1.3.5).

The most essential results in my research depend on the use of the **functional equation** for the relevant **zeta-functions**, i.e. the **zeta symmetry**, and so I will try to introduce this rich realm of symmetries and the world of modular relation supremacy gradually to the readers (cf. [KT2]). I will state another consequence of the Cauchy residue theorem, the partial fraction expansion for the cotangent function, since this is known as part of function theory. I will eventually show that it is a manifestation of the functional equation. This partial fraction expansion has turned out to be very useful, e.g. in deducing an integral representation for

Kinkelin's formula (2.5.33) in Section 1.5. For many other important results, cf. [KT1], [CKT].

1.2 A quick introduction to complex analysis

1.2.1 Complex number system

This section is quite elementary and an advanced reader can skip this chapter and go on to the next chapter. I begin with an introduction of complex numbers since there is still a confusion existing regarding the meaning of these imaginary numbers which sound non-existing. But they do exist as plane vectors as we shall see.

We know that two dimensional vectors (plane vectors) $\boldsymbol{z} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ form a **vector space** under the addition (translation)

$$\boldsymbol{z} + \boldsymbol{z}' = \begin{pmatrix} x + x' \\ y + y' \end{pmatrix}, \quad \boldsymbol{z}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

and the scalar multiplication ($c \in \mathbb{R}$)

$$c\boldsymbol{z} = \begin{pmatrix} cx \\ cy \end{pmatrix}.$$

There are multiplications defined.

Scalar product:

$$\boldsymbol{z} \cdot \boldsymbol{z}' = xx' + yy' \in \mathbb{R}.$$

Vector product:

$$\boldsymbol{z} \times \boldsymbol{z}' = \begin{vmatrix} x & x' \\ y & y' \end{vmatrix} = xy' - x'y \in \mathbb{R},$$

where the middle term indicates the determinant.

We introduce the vector $\bar{\boldsymbol{z}} = \begin{pmatrix} x \\ -y \end{pmatrix}$, which is a reflection of $\boldsymbol{z}$ with respect to the x-axis, and combine these two multiplications in the following $*$-operation due to Gauss:

$$\boldsymbol{z} * \boldsymbol{z}' = \begin{pmatrix} \bar{\boldsymbol{z}} \cdot \boldsymbol{z}' \\ \bar{\boldsymbol{z}} \times \boldsymbol{z}' \end{pmatrix} = \begin{pmatrix} xx' - yy' \\ xy' + x'y \end{pmatrix} \in \mathbb{R}^2.$$

E.g. if we label $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ by $\boldsymbol{i}$, then we have

$$\boldsymbol{i}^2 = \boldsymbol{i} * \boldsymbol{i} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} * \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad \mathrm{i}^2 = -1.$$

In view of

$$\boldsymbol{z} = \begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix} = x\boldsymbol{e}_1 + y\boldsymbol{i},$$

where $\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we think of the vector $\boldsymbol{z}$ as a "number", a **complex number**, $z = x + \mathrm{i}y$:

$$\mathbb{R}^2 \ni \boldsymbol{z} = \begin{pmatrix} x \\ y \end{pmatrix} \longleftrightarrow z = x + \mathrm{i}y \in \mathbb{C}. \tag{1.2.1}$$

Since we just denote the basis vector $\boldsymbol{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ as i, the above correspondence is 1:1 and moreover, it turns out that the star product is the same as ordinary multiplication of numbers with i^2 replaced by -1:

$$zz' = z \cdot z' = (x + \mathrm{i}y)(x' + \mathrm{i}y') = xx' - yy' + \mathrm{i}(xy' + x'y) \longleftrightarrow \boldsymbol{z} * \boldsymbol{z}'.$$

We can easily prove that the system $(\mathbb{R}^2, +, *)$ forms a field, which we denote by $\mathbb{C} = (\mathbb{C}, +, \cdot)$ and refer to it as the **field of complex numbers**.

We recall the length (norm, absolute value) of a vector

$$|\boldsymbol{z}| = \sqrt{x^2 + y^2}$$

(the Pythagorean theorem).

By (1.2.1) we introduce the **absolute value** of the complex number $z = x + \mathrm{i}y$ by

$$|z| = |x + \mathrm{i}y| = |\boldsymbol{z}| = \sqrt{x^2 + y^2}.$$

Don't do this: $|1 + 3\mathrm{i}| = \sqrt{1 + (3\mathrm{i})^2} = \sqrt{-8}$.

We may of course define the distance between two numbers z, z' by

$$d(z, z') = |z - z'| = \sqrt{(x - x')^2 + (y - y')^2}.$$

Then $(\mathbb{C}, d)$ is a metric space which is **complete** because $\mathbb{R}^2$ is so. These are simplest examples of the **Hilbert spaces**. Hence we may develop analysis on it, **complex analysis**.

1.2.2 Cauchy-Riemann equation and inverse functions

Theorem 1.1 *A function $w = f(z) = u + \mathrm{i}v$ is analytic on the domain $D \subset \mathbb{C} \Leftrightarrow u$, v are totally differentiable on $D \subset \mathbb{R}^2$ and satisfy the* **Cauchy-Riemann equation** $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. *In this case we have* $f'(z) = \dfrac{\partial f}{\partial x} = u_x + \mathrm{i}v_x$ *(Also, substituting the Cauchy-Riemann equation in this formula, we may express it as* $f'(z) = \dfrac{\partial f}{\partial(\mathrm{i}y)} = \dfrac{1}{\mathrm{i}}(u_y + \mathrm{i}v_y) = v_y - \mathrm{i}u_y$*).*

Proof ($\Rightarrow$) The differentiability of f at a point z_0 means

$$f(z) - f(z_0) = f'(z_0)(z - z_0) + o(|z - z_0|). \tag{1.2.2}$$

Substituting the value $\alpha = f'(z_0) = P + \mathrm{i}Q$ and comparing the real and imaginary parts, we obtain

$$\begin{aligned} u(x,y) - u(x_0,y_0) &= P(x - x_0) - Q(y - y_0) + o(|z - z_0|) \\ &= (P, -Q)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(|z - z_0|) \end{aligned}$$

and

$$\begin{aligned} v(x,y) - v(x_0,y_0) &= Q(x - x_0) + P(y - y_0) + o(|z - z_0|) \\ &= (Q, P)\begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} + o(|z - z_0|). \end{aligned}$$

Hence,

$$\begin{aligned} u(\boldsymbol{z}) - u(\boldsymbol{z}_0) &= (P, -Q)(\boldsymbol{z} - \boldsymbol{z}_0) + o(|\boldsymbol{z} - \boldsymbol{z}_0|), \\ v(\boldsymbol{z}) - v(\boldsymbol{z}_0) &= (Q, P)(\boldsymbol{z} - \boldsymbol{z}_0) + o(|\boldsymbol{z} - \boldsymbol{z}_0|), \end{aligned}$$

where $\boldsymbol{z} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\boldsymbol{z}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ are the real vectors corresponding to $z = x + \mathrm{i}y$, $z_0 = x_0 + \mathrm{i}y_0$. This means that u, v are both totally differentiable at $\boldsymbol{z}_0$ and $P = u_x$, $Q = -u_y$; $Q = v_x$, $P = v_y$ holds. Hence, in particular, it follows that $u_x = P = v_y$, $u_y = -Q = -v_x$ whence the Cauchy-Riemann equation.

($\Leftarrow$) We may trace back the above proof in the reverse direction. With

$$P = u_x = v_y, \quad Q = -u_y = v_x, \quad \boldsymbol{h} = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \to \mathbf{0} \quad (\Leftrightarrow h := h_1 + \mathrm{i}h_2 \to 0),$$

we have

$$u(x + h_1, y + h_2) - u(x,y) = (u_x, u_y)\boldsymbol{h} + o(|\boldsymbol{h}|) = Ph_1 - Qh_2 + o(|\boldsymbol{h}|),$$

$$v(x + h_1, y + h_2) - v(x,y) = (v_x, v_y)\boldsymbol{h} + o(|\boldsymbol{h}|) = Qh_1 + Ph_2 + o(|\boldsymbol{h}|),$$

so that

$$\begin{aligned} f(\boldsymbol{z} + \boldsymbol{h}) - f(\boldsymbol{z}) &= f(x + h_1, y + h_2) - f(x,y) \\ &= Ph_1 - Qh_2 + \mathrm{i}(Qh_1 + Ph_2) + o(|\boldsymbol{h}|) \\ &= (P + \mathrm{i}Q)(h_1 + \mathrm{i}h_2) + o(|\boldsymbol{h}|). \end{aligned}$$

Hence it follows that f is differentiable and $f'(z) = u_x + \mathrm{i}v_x$. □

Remark 1.1 (i) *To remember the Cauchy-Riemann equation, notice the alphabetical order of letters in the variables* $w = u + \mathrm{i}v$, $z = x + \mathrm{i}y$ *and remember first* $u_x = v_y$, *which is in alphabetical order and then changing the order to replace the real and imaginary parts, then we have a sign change to get* $u_y = -v_x$. *Also* $f'(z)$ *is first to think of* $f(z)$ *as a formal sum* $u+\mathrm{i}v$ *of two real functions and to partially differentiate it with respect to* x : *to find* $\dfrac{\partial f}{\partial x}$.

(ii) *The total differentiability in Theorem* 1.1 *is assured by the coninutity of* $\dfrac{\partial u}{\partial x}, \cdots, \dfrac{\partial v}{\partial y}$, *which is then assured roughly by their differentiability. Thus, if assume that* u, v *are of* C^2*-class, then we may appeal to Theorem* 1.1.

Corollary 1.1 (The **inverse function theorem** for complex variables) *If a function* $w = f(z)$ *is analytic in a domain* D *and* $f'(z_0) \neq 0$ *at* $z_0 \in D$, *then the inverse function* $g(w) = f^{-1}(w)$ *exists in the neighborhood of* $w_0 = f(z_0)$ *and is analytic at* w_0, *with the derivative given by*

$$g'(w) = \frac{1}{f'(z)}, \quad \frac{\mathrm{d}z}{\mathrm{d}w} = \frac{\mathrm{d}w}{\mathrm{d}z}.$$

Proof When we view the complex function $u + \mathrm{i}v = w = f(z)$ as the vector-valued function $\boldsymbol{w} = \begin{pmatrix} u \\ v \end{pmatrix}$, $u = u(\boldsymbol{z})$, $v = v(\boldsymbol{z})$, $\boldsymbol{z} = \begin{pmatrix} x \\ y \end{pmatrix}$, then the Jacobian $J_{\boldsymbol{w}} = \dfrac{\partial(u,v)}{\partial(x,y)}$ is $|f'(z)|^2$:

$$J(x,y) = J_{\boldsymbol{w}}(\boldsymbol{z}) = |f'(z)|^2. \tag{1.2.3}$$

Now we may appeal to the inverse function theorem below for several variable real functions completing the proof.

Inverse function theorem *If a vector-valued function* $\boldsymbol{w}$ *is differentiable at a point* $\boldsymbol{z}_0$ *and* $J_{\boldsymbol{w}}(\boldsymbol{z}_0) \neq 0$ *in the neighborhood, then there exits a local inverse function of* $\boldsymbol{w}$ *and is locally* C^1. □

Example 1.1 *The (complex)* **logarithm function** $\log w$ *is a multi-valued function given as the inverse function of the exponential function* $w = \mathrm{e}^z$:

$$\log w = z = x + \mathrm{i}y = \log|w| + \mathrm{i}\arctan\frac{v}{u} = \log|w| + \mathrm{i}\arg w \tag{1.2.4}$$

and for any branch, we have $(\log w)' = \dfrac{1}{w}$, $w \neq 0$. *We note that the multi-valuedness of the* arctan *function gives rise to that of the logarithm function.*

Indeed, the Jacobian is $J(x,y) = |\mathrm{e}^z|^2 = \mathrm{e}^{2x} \neq 0$ for $z \neq 0$, so that the inverse function exists. To prove (1.2.4), we write (cf. (1.2.18) below)

$$u + \mathrm{i}v = w = \mathrm{e}^z = \mathrm{e}^x \mathrm{e}^{\mathrm{i}y}.$$

Then we have

$$\begin{cases} u = \mathrm{e}^x \cos y, \\ v = \mathrm{e}^x \sin y \end{cases} \quad \Leftrightarrow \quad z = \log w$$

or $|w| = \mathrm{e}^x$ and $\tan y = \dfrac{v}{u}$, whence

$$x = \log|w| \quad \text{and} \quad y = \arctan\frac{v}{u}.$$

Since $\arctan\dfrac{v}{u}$ is nothing but $\arg w$, we conclude (1.2.4).

In what follows, we choose the **principal branch** $\log z$ of the logarithm once and for all, where the principal branch means that the argument lies between $-\pi$ and π:

$$\log z = \log|z| + \mathrm{i}\arg z, \quad -\pi < \arg z < \pi.$$

For the complex variable $s = \sigma + it$ and $n \in \mathbb{N}$, we define the power function n^{-s} by

$$n^{-s} = \mathrm{e}^{-s\log n} = n^{-\sigma}(\cos t \log n - \mathrm{i}\sin t \log n), \tag{1.2.5}$$

with $\log n$ designating the real number.

1.2.3 A rough description of complex analysis

The most fundamental ingredients in complex functions theory are **differentiation** and integration, the former of which is defined in the same way as with real functions, while integration is more complicated.

We say that a complex function is **analytic** (or sometimes **holomorphic** or **regular**) in a **domain** if it is **differentiable** at each point of the domain, where differentiability means the existence of the **derivative** as given by (1.2.2) above, and a domain (sometimes referred to as a **region**) mathematically means a connected set, we simply understand a domain to be a certain plane figure ($\subset \mathbb{C}$) with interior and with the boundary curve. We usually assume that domains are arcwise connected. Typical domains are the rectangles (parallelopipeds) and circles and there is no need to worry about what domains are.

We assume throughout that a curve is a **piecewise smooth (Jordan) curve** described by the parametric expression

$$z = z(t) = x(t) + \mathrm{i}y(t), \quad t \in [a,b]. \tag{1.2.6}$$

E.g., the unit circle C: $|z| = 1$ is given by

$$z = z(t) = \mathrm{e}^{2\pi \mathrm{i} t}, \quad t \in [0, 1] \tag{1.2.7}$$

or by

$$C : z = z(t) = \mathrm{e}^{\mathrm{i}t}, \quad t \in [0, 2\pi].$$

By the Jordan curve theorem, such a curve encircles a domain D. In this context, we denote the curve by ∂D and refer to it as its boundary.

Note that this is a **positively-oriental curve** to the effect that if you traverse the curve, you'll see the inside on your left. We assume all the curves are positively oriented. The expression (1.2.6) for the unit circle is in counter-clockwise direction. The same curve oriented in the opposite direction, denoted $-C$, is given by

$$-C : z = z(t) = \mathrm{e}^{2\pi \mathrm{i}(1-t)}, \quad t \in [0, 1].$$

The notion of analytic functions is indispensably connected with their domains, and we say that a function is **analytic at a point** or **on a curve** if it is analytic in a domain containing the point or the curve.

There are two main streams of ideas in complex analysis. One is due to Weierstrass and appeals to the power series expansion. The other is due to Cauchy and describes the analyticity in terms of contour integrals. Complex integrals are all line (or contour) integrals and are slightly different from the ordinary Riemann integrals $\int_a^b f(x)\,\mathrm{d}x$. However we may think of it as performing the change of variable $\mathrm{d}z = z'(t)\,\mathrm{d}t$:

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t, \tag{1.2.8}$$

if the curve C is given by (1.2.6). We note that

$$\int_{-C} f(z)\,\mathrm{d}z = -\int_C f(z)\,\mathrm{d}z.$$

Weierstrass's main theorem says that a function $f(z)$ is analytic in a domain D if and only if it is expanded into a **power series** (Taylor series) at each point z_0 of D:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{1.2.9}$$

If f is analytic, then it has derivatives of all orders and (1.2.9) holds, which is very distinct from real analytic functions.

This implies the **consistency** (or **unicity**) **theorem** to the effect that *if two functions are analytic in a domain D and they coincide on a subset S of D having* *an accumulation point in D, then they must coincide on the whole D.* The consistency theorem is a basis for the **principle of analytic continuation** to the effect that if analytic functions $f(z)$ and $g(z)$ coincide e.g. on a segment of the real axis, then one is an analytic continuation of the other. cf. the passages after Remark 1.2.

The Cauchy integral theorem says that a function $f(z)$ is analytic in a domain D if and only if for any curve C within D, we have

$$\int_C f(z)\,\mathrm{d}z = 0. \tag{1.2.10}$$

Example 1.2 *The circle with center at α and radius r, $|z-\alpha| = r$, can be expressed as*

$$C:\ z = \alpha + r\mathrm{e}^{2\pi\mathrm{i}t}, \quad t \in [0,1]. \tag{1.2.11}$$

Evaluate the integral

$$I = \int_C \frac{1}{z-\alpha}\,\mathrm{d}z.$$

Substituting (1.2.11), *we get*

$$I = \int_0^1 \frac{1}{r\mathrm{e}^{2\pi\mathrm{i}t}} r\mathrm{e}^{2\pi\mathrm{i}t}(2\pi\mathrm{i})\,\mathrm{d}t = 2\pi\mathrm{i}.$$

How about $I_n = \int_C \frac{1}{(z-\alpha)^n}\,\mathrm{d}z$, $n \geqslant 2$? *Similarly, we have*

$$I_n = \int_0^1 \frac{r\mathrm{e}^{2\pi\mathrm{i}t}}{(r\mathrm{e}^{2\pi\mathrm{i}t})^n} 2\pi\mathrm{i}\,\mathrm{d}t = \frac{2\pi\mathrm{i}}{r^{n-1}} \int_0^1 \mathrm{e}^{2\pi\mathrm{i}(n-1)t}\,\mathrm{d}t = 0.$$

By the Cauchy integral theorem,

$$\int_C (z-\alpha)^n\,\mathrm{d}z = 0, \quad n \geqslant 0.$$

Hence we have evaluated the integral

$$\int_C (z-\alpha)^n\,\mathrm{d}z = \begin{cases} 0, & n \neq -1, \\ 2\pi\mathrm{i}, & n = -1, \end{cases} \tag{1.2.12}$$

where C is a circle enclosing the point $z = \alpha$.

If $f(z)$ is analytic in a domain D except for a point $z = \alpha \in D$, then we may reduce the integral along a curve C containing α to a circle c containing α.

For we connect c and C by two lines L_1 and L_2 oriented in opposite way. The curve $C_1 = -c \cup L_1 \cup L_2$ encircles a domain, where $f(z)$ is analytic, so that

$$0 = \int_{C_1} f(z)\mathrm{d}z = \int_{-c} f(z)\,\mathrm{d}z + \int_{L_1} f(z)\mathrm{d}z + \int_{L_2} f(z)\mathrm{d}z + \int_C f(z)\,\mathrm{d}z,$$

which gives

$$\int_C f(z)\mathrm{d}z = \int_c f(z)\,\mathrm{d}z.$$

Thus we find that complex analysis is very geometric (topological).

Example 1.3 *Consider the function* $f(z) = \dfrac{1}{z^2+1}$ *which has simple poles at* $z = \pm \mathrm{i}$. *A simple pole means that we have a denominator* $z - \mathrm{i}$ *and* $z + \mathrm{i}$ *(for more details, see below). Let*

$$C_i : z - \mathrm{i} = \frac{1}{2}\mathrm{e}^{2\pi \mathrm{i} t}, \quad t \in [0,1].$$

Then

$$\int_{C_i} f(z)\,\mathrm{d}z = 2\mathrm{i}\int_C \left(\frac{1}{z-\mathrm{i}} + \frac{1}{z+\mathrm{i}}\right)\mathrm{d}z,$$

where we applied the "partial fraction expansion". The first integral is already in (1.2.12) *and is* -4π, *while for the second, if we substitute the parametric expression, then we are to face the integral*

$$2\pi \mathrm{i}\int_0^1 \frac{\mathrm{e}^{2\pi \mathrm{i} t}}{\mathrm{e}^{2\pi \mathrm{i} t} + 4\mathrm{i}}\,\mathrm{d}t,$$

which is $2\pi\mathrm{i} + 8\pi^2 \displaystyle\int_0^1 \frac{\cos 2\pi t - (\sin 2\pi t + 4)\mathrm{i}}{8\sin 2\pi t + 17}\,\mathrm{d}t$, *and we don't want to go on.*

We should apply (1.2.10) *to conclude that it is* 0. *Hence*

$$\int_{C_i} \frac{1}{z^2+1}\,\mathrm{d}z = \pi.$$

Similarly, if C_{-i} *is the circle with center at* $z = -\mathrm{i}$ *and with radius* $\frac{1}{2}$, *then*

$$\int_{C_{-i}} \frac{1}{z^2+1}\,\mathrm{d}z = -\pi,$$

while for any circle C *with center at the origin and radius* $0 < r < 1$,

$$\int_C \frac{1}{z^2+1}\,\mathrm{d}z = 0.$$

Remark 1.2 *Here we notice a big difference between the real analytic function* $f(x) = \dfrac{1}{x^2+1}$ *and the complex analytic function* $f(z) = \dfrac{1}{z^2+1}$. *Indeed,* $f(x)$ *is a very obedient function, and you may pay no attention to the fact that although it has the Maclaurin expansion*

$$f(x) = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty}(-1)^n x^{2n},$$

it is absolutely convergent **only** *in* $|x| < 1$. *Why is there such a restriction? It is because of the singularities at* $z = \pm\, \mathrm{i}$ *which prevent the complex power series being convergent outside of* $|z| < 1$.

Weierstrass's main theorem implies that *if a function is analytic in a domain D containing a point* α, *then it can be expanded into a Taylor series around* α *in the maximum disc that is contained in D, and moreover, the real analytic series for* $f(x)$ *can be uniquely continued analytically to* $f(z)$. Therefore inside the domain of analyticity, we may simply write z for x and get an analytic function which involves the real analytic function as a special case

$$f(x) = \frac{1}{x^2+1} \to f(z) = \frac{1}{z^2+1}, \quad |z| < 1.$$

1.2.4 Power series

A **power series** is like a polynomial of infinite degree as given by Remark 1.2 and is of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

a_n being called the n-th coefficient. They are uniquely determined, i.e. if $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} b_n z^n$ in some region, then $a_n = b_n$. Recall that the **geometric series**

$$f(z) = \sum_{n=0}^{\infty} z^n$$

is absolutely and uniformly convergent in $|z| < 1$, divergent in $|z| > 1$ and on the circle $|z| = 1$ it is (conditionally) convergent except for $z = 1$. The last because on the unit circle, we have $z = \mathrm{e}^{2\pi \mathrm{i} x}$, $x \in \mathbb{R}$, so that the common ratio is 1 if and only if $x \in \mathbb{Z}$. It turns out that the region of convergence of power series is always a circle (finite or infinite) and the threshold circle as above is called the **circle of convergence** and its radius r is called the radius of convergence. It can be most easily determined by the D'Alembert test:

$$r = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided that $a_n \neq 0$.

- Within the circle of convergence, power series behave exactly like ordinary polynomials, i.e. we may sum, subtract, multiply and divide (provided that the denominator $\neq 0$).