NONLINEAR
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SCIENCE

Michal Fečkan

Bifurcation and Chaos in Discontinuous and Continuous Systems

不连续及连续系统中的分岔和混沌

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不连续及连续动力系统中的分岔和混沌

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To my beloved family

Preface

This book is devoted to the comprehensive bifurcation theory of chaos in nonlinear dynamical systems with applications to mechanics and vibrations. Precise and complete proofs of derived mathematical results are presented with many stimulating and illustrative examples. I study bifurcations of chaotic solutions for perturbed problems from either homoclinic or heteroclinic orbits of unperturbed ones. This method is also known as the Melnikov-type approach. Certainly there are many interesting books in this direction, but all results of this book have not yet been published in any book, since I have collected some results of mine together with my coauthors appeared only in articles and manuscripts. So I hope that this book is a useful contribution to a rapidly developing theory of chaos and it is a good continuation of my recently published book in Springer with similar topics.

The book is intended to be used by scientists interested in the theory of chaos and its applications, like mathematicians, physicists, or engineers. It can also serve as a textbook for a class of nonlinear oscillations and dynamical systems.

Here is a brief outline of each chapter.

Chapter 1 is an introduction to the topic of the book by presenting two well-known chaotic models: damped and driven Duffing and pendulum equations.

To make this book as self-contained as possible, some basic preliminary results are included in Chapter 2.

Chapter 3 studies chaotic bifurcations of discrete dynamical systems including: nonautonomous difference equations; diffeomorphisms; perturbed singular and singularly perturbed impulsive ordinary differential equations (ODEs); and inflated dynamical systems arising in computer assisted proofs and in other numerical methods in dynamical systems, so an extension of Smale horseshoe to inflated dynamical systems is presented.

Chapter 4 deals with proving chaos for parameterized ODEs in arbitrary dimensions. It is shown that if the Melnikov function is identically zero the second order Melnikov function must be derived. I consider a broad variety of ODEs: coupled nonresonant ODEs, resonant systems of ODEs investigated with the help of averaging theory; singularly perturbed ODEs; and inflated ODEs. I also show that the structure of chaotic parameters is related to the Morin singularity of smooth mappings. I end this chapter with infinite dimensional ODEs on lattices by considering a model of two one-dimensional interacting sublattices of harmonically coupled protons and heavy ions.

Chapter 5 shows chaotic vibrations of partial differential equations (PDEs): slowly periodically perturbed and weakly nonlinear beams on elastic bearings; periodically forced and nonresonant buckled elastic beams; and periodically forced compressed beams at resonance.

Chapter 6 is devoted to the study of chaotic oscillations of discontinuous (nonsmooth) differential equations (DDEs). First I consider the case when the homoclinic orbit of the unperturbed DDE transversally crosses discontinuity surfaces. Then I study a chaos for time-perturbed DDEs. I apply our general results to quasiperiodic piecewise linear systems in $\mathbb{R}^3$, and to piecewise smooth forced planar DDEs. Then I extend those result to sliding homoclinic bifurcations, when a part of the homoclinic orbit of the unperturbed DDE lies on a discontinuity surface. A rigorous proof of the existence of chaos for stick-slip systems is presented. I utilize general theoretical results to planar and 3-dimensional sliding homoclinic cases.

In Chapter 7, first I investigate the Melnikov function in general by computing its Fourier coefficients. These computations allow me to find examples when the Melnikov function is ether identically zero or not. I also derive the second order Melnikov function when the (first order) Melnikov function is identically zero. For construction of concrete examples, I solve an inverse problem when the homoclinic orbit is given and a second order ODE is found so that it possesses that homoclinic orbit. The second part of this chapter is devoted to showing chaos near transversal heteroclinic orbits. The third part deals with the blue sky catastrophe for periodic orbits.

In all chapters, derived bifurcation conditions for the existence of chaos are expressed as simple zeroes of corresponding Melnikov functions. Functional analytic approaches are used which are roughly based on a concept of exponential dichotomy together with Lyapunov-Schmidt method. Numerical computations described by figures are given with the help of a computational software program *Mathematica*.

The author is indebted to the coauthors for some results mentioned in this book: Jan Awrejcewicz, Flaviano Battelli, Giovanni Colombo, Matteo Franca, Barnabás M. Garay, Joseph Gruendler, Paweł Olejnik, Weiyao Zeng. Partial support of Grants VEGA-SAV 2/0124/10, VEGA-MS 1/0098/08, an award from Literárny fond and by the Slovak Research and Development Agency under the contract No. APVV-0414-07 are also appreciated.

Michal Fečkan
Bratislava, Slovakia
June 2010

Contents

Chapter 1
Introduction

Many problems in the natural and engineering sciences can be modeled as evolution processes. Mathematically this leads to either discrete or continuous dynamical systems, i.e. to either difference or differential equations. Usually such dynamical systems are nonlinear or even discontinuous and depend on parameters. Consequently the study of qualitative behaviour of their solutions is very difficult. Rather effective method for handling dynamical systems is the bifurcation theory, when the original problem is a perturbation of a solvable problem, and we are interested in qualitative changes of properties of solutions for small parameter variations. Nowadays the bifurcation and perturbation theories are well developed and methods applied by these theories are rather broad including functional-analytical tools and numerical simulations as well [1–13].

Next, one of the fascinating behaviour of nonlinear dynamical systems which may occur is their sensitive dependence on the initial value conditions, which results in a chaotic time behaviour. Chaos is by no means exceptional but a typical property of many dynamical systems in periodically stimulated cardiac cells, in electronic circuits, in chemical reactions, in lasers, in mechanical devices, and in many other models of biology, meteorology, economics and physics. In spite of the fact that it is very difficult to show chaos for general evolution equations, the bifurcation theory based on perturbation methods is a powerful tool for concluding chaos in a rather wide class of parameterized nonlinear dynamical systems. Especially functional-analytical methods are very convenient to show rigorously the existence of chaos in concrete dynamical systems [14–20].

Now we show two well-known simple chaotic mechanical models. First, we consider a periodically forced and damped Duffing equation

$$\dot{x} = y, \quad \dot{y} - x + 2x^3 + \mu_1 y = \mu_2 \cos t \tag{1.0.1}$$

with μ_1, μ_2 being small. Note

$$\ddot{x} + \mu_1 \dot{x} - x + 2x^3 = \mu_2 \cos t$$

describes dynamics of a buckled beam, when only one mode of vibration is considered (cf Section 5.2 and [21]). Particularly, an experimental apparatus in [4, pp. 83–84] is a slender steel beam clamped to a rigid framework which supports two magnets, when x is the beam tip displacement. The apparatus is periodically forced using electromagnetic vibration generator (Figure 1.1).

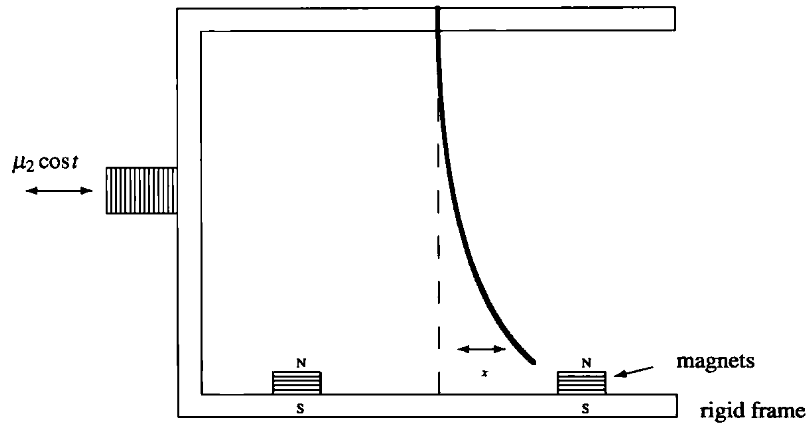

Fig. 1.1 The magneto-elastic beam.

Next, the phase portrait of

$$\dot{x} = y, \quad \dot{y} - x + 2x^3 = 0 \tag{1.0.2}$$

is simply found analytically by analyzing the level sets $\dot{x}^2 - x^2 + x^4 = c \in \mathbb{R}$ [1,4,13]. Here $\mathbb{R}$ denotes the set of real numbers. There are three equilibria: $(0,0)$ is hyperbolic and $(\pm\sqrt{2}/2, 0) \doteq (\pm 0.707107, 0)$ are centers. There is also a symmetric *homoclinic cycle* $\pm\tilde{\gamma}_d(t)$ with $\tilde{\gamma}_d(t) = (\gamma_d(t), \dot{\gamma}_d(t))$ and $\gamma_d(t) = \operatorname{sech} t$. The rest are all periodic solutions. These results are consistent with the above experimental model without damping and external forcing as follows: When attractive forces of the magnets overcome the elastic force of the beam, the beam settles with its tip close to one or more of the magnets: these are centers of (1.0.2). There is also an unstable central equilibrium position of the beam at which the magnetic forces are canceled: this is the unstable equilibrium of (1.0.2) (Figure 1.2).

When $\mu_{1,2}$ are small and not identically zero, in spite of the fact that (1.0.1) is a simply looking equation, its dynamics is very difficult. This is demonstrated in Figure 1.3. We see that there are random oscillations of the beam tip between the two magnets. These chaotic vibrations are also observed in the experimental apparatus of Figure 1.1 as shown in [4, p. 84]. Theoretically it is justified by Lemma 7.2.4. Note that almost all trajectories of the damped case $\mu_1 > 0$, $\mu_2 = 0$ tend to one of the stable equilibria $(\pm\sqrt{2}/2, 0)$ (cf case A of Figure 1.3).

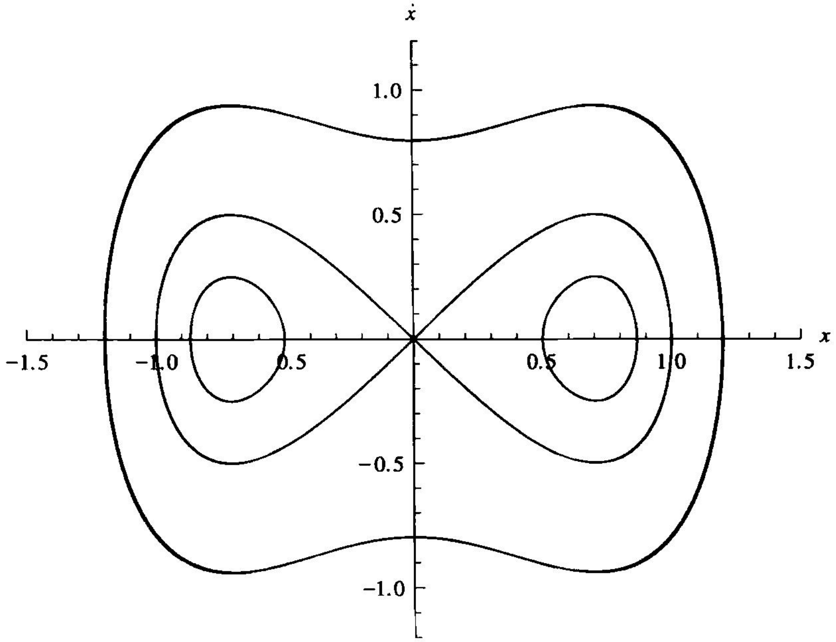

Fig. 1.2 The phase portrait of the Duffing equation (1.0.2).

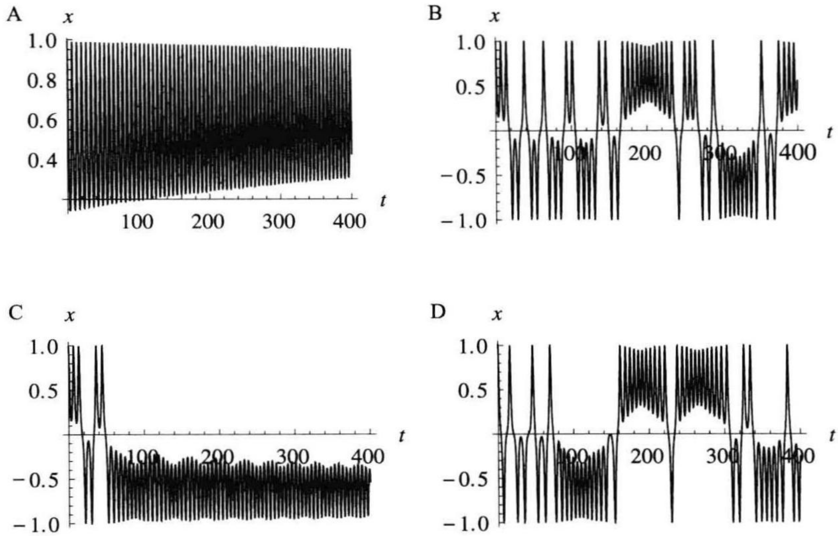

Fig. 1.3 The solution $x(t)$, $0 \leq t \leq 400$ of (1.0.1) for A: $\mu_1 = 0.001$, $\mu_2 = 0$, $x(0) = 0.99$, $\dot{x}(0) = 0$; B: $\mu_1 = 0$, $\mu_2 = 0.01$, $x(0) = 0.99$, $\dot{x}(0) = 0$; C: $\mu_1 = 0.001$, $\mu_2 = 0.01$, $x(0) = 0.99$, $\dot{x}(0) = 0$; D: $\mu_1 = 0.001$, $\mu_2 = 0.01$, $x(0) = 1.01$, $\dot{x}(0) = 0$.

The aim of this book is to show chaos in (1.0.1) analytically. This is presented in Section 4.1 and Subsection 5.2.6: now the *Melnikov function* is given by

$$M(\alpha) = \int_{-\infty}^{\infty} \dot{\gamma}_d(t)\,(\mu_2 \cos(\alpha + t) - \mu_1 \dot{\gamma}_d(t))\,dt = \mu_2 \pi \operatorname{sech}\frac{\pi}{2}\sin\alpha - \mu_1 \frac{2}{3}.$$

When μ_1, μ_2 satisfy

$$|\mu_1| < |\mu_2| \frac{3\pi}{2} \operatorname{sech}\frac{\pi}{2} \doteq 1.87806|\mu_2|, \tag{1.0.3}$$

clearly there is a *simple zero* α_0 of M, i.e. $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$. Hence by Remark 4.1.6, (1.0.1) is chaotic for μ_1, μ_2 sufficiently small fulfilling (1.0.3). Note (1.0.3) holds for cases B, C, D of Figure 1.3.

The second popular example of chaotic physical model is a damped and forced pendulum consisting of a mass attached to a vertically oscillating pivot point by means of mass-less and inextensible wire described by ODE ([1, p. 278] and [22, p. 216])

$$\ddot{\phi} + \mu_1 \dot{\phi} + \sin\phi = \mu_2 \cos t \sin\phi\,, \tag{1.0.4}$$

where μ_1, μ_2 are parameters (Figure 1.4).

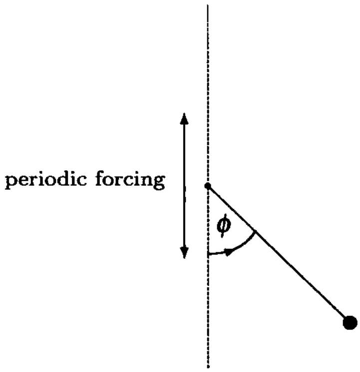

Fig. 1.4 The damped and forced pendulum (1.0.4).

The unperturbed ODE is given by

$$\ddot{\phi} + \sin\phi = 0 \tag{1.0.5}$$

with the phase portrait in Figure 1.5.

Note that $(2k\pi, 0)$ are centers and $((2k+1)\pi, 0)$ are hyperbolic equilibria of (1.0.5) for $k \in \mathbb{Z}$. Here $\mathbb{Z}$ denotes the set of integer numbers. Moreover, $(-\pi, 0)$ and $(\pi, 0)$ are joined by the upper *separatrix* or *heteroclinic orbit* $\widetilde{\gamma}_p(t)$ with $\widetilde{\gamma}_p(t) = (\gamma_p(t), \dot{\gamma}_p(t))$ and $\gamma_p(t) = 2\arctan(\sinh t)$. The lower separatrix is $-\widetilde{\gamma}_p(t)$.

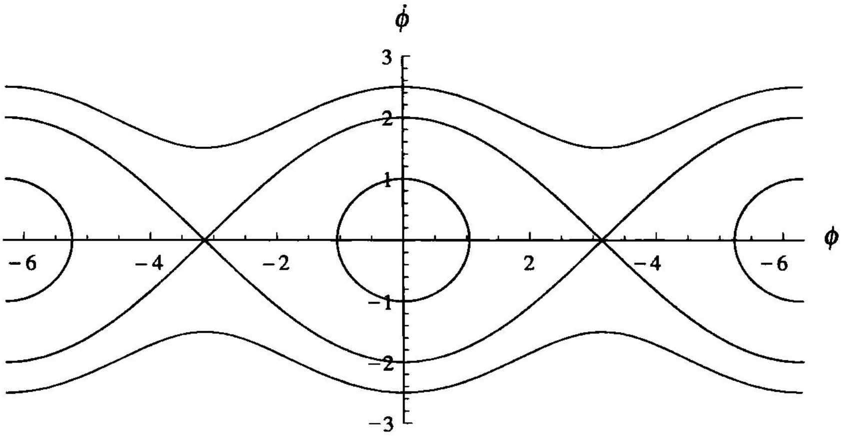

Fig. 1.5 The phase portrait of the pendulum equation (1.0.5).

When $\mu_{1,2}$ are small and not identically zero, (1.0.4) has very difficult dynamics. This is demonstrated in Figure 1.6.

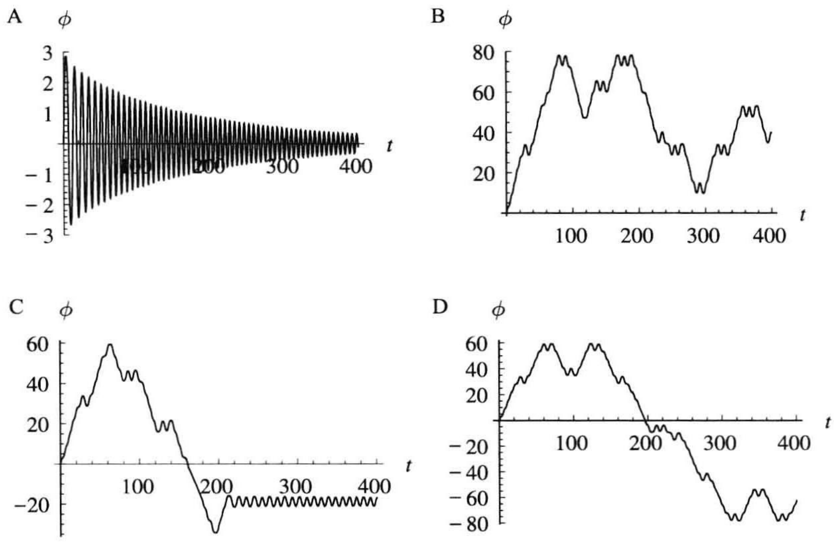

Fig. 1.6 The solution $\phi(t)$, $0 \leq t \leq 400$ of (1.0.4) for A: $\mu_1 = 0.01$, $\mu_2 = 0$, $\phi(0) = 0$, $\dot{\phi}(0) = 2$; B: $\mu_1 = 0.001$, $\mu_2 = 0.1$, $\phi(0) = 0$, $\dot{\phi}(0) = 1.998$; C: $\mu_1 = 0.001$, $\mu_2 = 0.1$, $\phi(0) = 0$, $\dot{\phi}(0) = 2$; D: $\mu_1 = 0.001$, $\mu_2 = 0.1$, $\phi(0) = 0$, $\dot{\phi}(0) = 2.002$.

Now the Melnikov function is given by [1, p. 467]

$$M(\alpha) = \int_{-\infty}^{\infty} \dot{\gamma}_p(t)\,(\mu_2 \cos(\alpha + t)\sin\gamma_p(t) - \mu_1 \dot{\gamma}_p(t))\,dt = -2\pi\mu_2 \operatorname{csch}\frac{\pi}{2}\sin\alpha - 8\mu_1\,.$$

When μ_1, μ_2 satisfy

$$|\mu_1| < |\mu_2| \frac{\pi}{4} \operatorname{csch} \frac{\pi}{2} \doteq 0.341285|\mu_2|, \tag{1.0.6}$$

clearly there is a simple zero α_0 of M, i.e. $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$. Hence by Remark 4.1.6, (1.0.4) is chaotic for μ_1, μ_2 sufficiently small fulfilling (1.0.6). Note that (1.0.6) holds for cases B, C, D of Figure 1.6. Note that almost all trajectories of the damped case $\mu_1 > 0$, $\mu_2 = 0$ tend to the one of the stable equilibria $(2k\pi, 0)$, $k \in \mathbb{Z}$ (cf case A of Figure 1.6).

In summary, examples (1.0.1) and (1.0.4) have the following common features: they are simply looking equations with unpredictable dynamics. But deriving their Melnikov functions, it is easy to show their chaotic behaviour. Consequently, the aim of this book is to present many different discrete and continuous dynamical systems defined on spaces with arbitrarily high dimensions including infinite ones when this Melnikov type analysis is shown to be useful, and then we demonstrate abstract results on concrete examples.

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