

NONLINEAR
PHYSICAL
SCIENCE

Michal Fečkan

Bifurcation and Chaos in Discontinuous and Continuous Systems

不连续及连续系统中的
分岔和混沌



高等教育出版社

HIGHER EDUCATION PRESS

World Scientific
Publishers
Singapore

Michael Thera

Bifurcation and Chaos in Discontinuous and Continuous Systems

不連續及連續系統中的
分岔和混沌

World Scientific

Michal Fečkan

Bifurcation and Chaos in Discontinuous and Continuous Systems

不连续及连续系统中的分岔和混沌

BU LIANXU JI LIANXU XITONG ZHONG DE FENCHA HE HUNDUN

With 30 figures

 高等教育出版社·北京
HIGHER EDUCATION PRESS BEIJING

Author

Michal Fečkan

Department of Mathematical Analysis and Numerical Mathematics
Faculty of Mathematics, Physics and Informatics, Comenius University
Mlynská dolina, 842 48 Bratislava, Slovakia
E-mail: Michal.Feckan@fmph.uniba.sk

© 2011 Higher Education Press, 4 Dewai Dajie, 100120, Beijing, P.R. China

图书在版编目(CIP)数据

不连续及连续系统中的分岔和混沌=Bifurcation and Chaos
in Discontinuous and Continuous Systems: 英文/(斯洛伐克)
费坎(Fečkan, M.)著. — 北京: 高等教育出版社, 2011.3
(非线性物理科学/罗朝俊,(瑞典)伊布拉基莫夫主编)
ISBN 978-7-04-031533-2

I. ①不… II. ①费… III. ①非线性系统(自动化)—研究—英文 IV. ①TP271

中国版本图书馆 CIP 数据核字(2011)第 012530 号

关键词: 分岔, 混沌, 振动, 不连续动力系统, 脉冲方程

策划编辑 王丽萍 责任编辑 王丽萍 封面设计 杨立新
责任校对 杨凤玲 责任印制 朱学忠

出版发行	高等教育出版社	购书热线	010-58581118
社 址	北京市西城区德外大街 4 号	咨询电话	400-810-0598
邮政编码	100120	网 址	http://www.hep.edu.cn http://www.hep.com.cn
经 销	蓝色畅想图书发行有限公司	网上订购	http://www.landrac.com http://www.landrac.com.cn
印 刷	涿州市星河印刷有限公司	畅想教育	http://www.widedu.com
开 本	787 × 1092 1/16	版 次	2011 年 3 月第 1 版
印 张	24.5	印 次	2011 年 3 月第 1 次印刷
字 数	550 000	定 价	89.00 元

本书如有缺页、倒页、脱页等质量问题, 请到所购图书销售部门联系调换。

版权所有 侵权必究

物料号 31533-00

Sales only inside the mainland of China

{ 仅限中国大陆地区销售 }

本书海外版由 Springer 负责在中国大陆地区以外区域销售, ISBN 为 978-3-642-18268-6

To my beloved family

Preface

This book is devoted to the comprehensive bifurcation theory of chaos in nonlinear dynamical systems with applications to mechanics and vibrations. Precise and complete proofs of derived mathematical results are presented with many stimulating and illustrative examples. I study bifurcations of chaotic solutions for perturbed problems from either homoclinic or heteroclinic orbits of unperturbed ones. This method is also known as the Melnikov-type approach. Certainly there are many interesting books in this direction, but all results of this book have not yet been published in any book, since I have collected some results of mine together with my coauthors appeared only in articles and manuscripts. So I hope that this book is a useful contribution to a rapidly developing theory of chaos and it is a good continuation of my recently published book in Springer with similar topics.

The book is intended to be used by scientists interested in the theory of chaos and its applications, like mathematicians, physicists, or engineers. It can also serve as a textbook for a class of nonlinear oscillations and dynamical systems.

Here is a brief outline of each chapter.

Chapter 1 is an introduction to the topic of the book by presenting two well-known chaotic models: damped and driven Duffing and pendulum equations.

To make this book as self-contained as possible, some basic preliminary results are included in Chapter 2.

Chapter 3 studies chaotic bifurcations of discrete dynamical systems including: nonautonomous difference equations; diffeomorphisms; perturbed singular and singularly perturbed impulsive ordinary differential equations (ODEs); and inflated dynamical systems arising in computer assisted proofs and in other numerical methods in dynamical systems, so an extension of Smale horseshoe to inflated dynamical systems is presented.

Chapter 4 deals with proving chaos for parameterized ODEs in arbitrary dimensions. It is shown that if the Melnikov function is identically zero the second order Melnikov function must be derived. I consider a broad variety of ODEs: coupled nonresonant ODEs, resonant systems of ODEs investigated with the help of averaging theory; singularly perturbed ODEs; and inflated ODEs. I also show that the structure of chaotic parameters is related to the Morin singularity of smooth map-

pings. I end this chapter with infinite dimensional ODEs on lattices by considering a model of two one-dimensional interacting sublattices of harmonically coupled protons and heavy ions.

Chapter 5 shows chaotic vibrations of partial differential equations (PDEs): slowly periodically perturbed and weakly nonlinear beams on elastic bearings; periodically forced and nonresonant buckled elastic beams; and periodically forced compressed beams at resonance.

Chapter 6 is devoted to the study of chaotic oscillations of discontinuous (non-smooth) differential equations (DDEs). First I consider the case when the homoclinic orbit of the unperturbed DDE transversally crosses discontinuity surfaces. Then I study a chaos for time-perturbed DDEs. I apply our general results to quasiperiodic piecewise linear systems in \mathbb{R}^3 , and to piecewise smooth forced planar DDEs. Then I extend those result to sliding homoclinic bifurcations, when a part of the homoclinic orbit of the unperturbed DDE lies on a discontinuity surface. A rigorous proof of the existence of chaos for stick-slip systems is presented. I utilize general theoretical results to planar and 3-dimensional sliding homoclinic cases.

In Chapter 7, first I investigate the Melnikov function in general by computing its Fourier coefficients. These computations allow me to find examples when the Melnikov function is either identically zero or not. I also derive the second order Melnikov function when the (first order) Melnikov function is identically zero. For construction of concrete examples, I solve an inverse problem when the homoclinic orbit is given and a second order ODE is found so that it possesses that homoclinic orbit. The second part of this chapter is devoted to showing chaos near transversal heteroclinic orbits. The third part deals with the blue sky catastrophe for periodic orbits.

In all chapters, derived bifurcation conditions for the existence of chaos are expressed as simple zeroes of corresponding Melnikov functions. Functional analytic approaches are used which are roughly based on a concept of exponential dichotomy together with Lyapunov-Schmidt method. Numerical computations described by figures are given with the help of a computational software program *Mathematica*.

The author is indebted to the coauthors for some results mentioned in this book: Jan Awrejcewicz, Flaviano Battelli, Giovanni Colombo, Matteo Franca, Barnabás M. Garay, Joseph Gruendler, Paweł Olejnik, Weiyao Zeng. Partial support of Grants VEGA-SAV 2/0124/10, VEGA-MS 1/0098/08, an award from Literárny fond and by the Slovak Research and Development Agency under the contract No. APVV-0414-07 are also appreciated.

Michal Fečkan
Bratislava, Slovakia
June 2010

Contents

1	Introduction	1
	References	6
2	Preliminary Results	9
2.1	Linear Functional Analysis	9
2.2	Nonlinear Functional Analysis	11
2.2.1	Banach Fixed Point Theorem	11
2.2.2	Implicit Function Theorem	11
2.2.3	Lyapunov-Schmidt Method	12
2.2.4	Brouwer Degree	13
2.2.5	Local Invertibility	13
2.2.6	Global Invertibility	14
2.3	Multivalued Mappings	14
2.4	Differential Topology	15
2.4.1	Differentiable Manifolds	15
2.4.2	Vector Bundles	16
2.4.3	Tubular Neighbourhoods	16
2.5	Dynamical Systems	17
2.5.1	Homogenous Linear Equations	17
2.5.2	Chaos in Diffeomorphisms	18
2.5.3	Periodic ODEs	19
2.5.4	Vector Fields	20
2.5.5	Global Center Manifolds	22
2.5.6	Two-Dimensional Flows	22
2.5.7	Averaging Method	23
2.5.8	Carathéodory Type ODEs	24
2.6	Singularities of Smooth Maps	24
2.6.1	Jet Bundles	24
2.6.2	Whitney C^∞ Topology	25
2.6.3	Transversality	25
2.6.4	Malgrange Preparation Theorem	26

2.6.5	Complex Analysis	26
References	28
3	Chaos in Discrete Dynamical Systems	29
3.1	Transversal Bounded Solutions	29
3.1.1	Difference Equations	29
3.1.2	Variational Equation	30
3.1.3	Perturbation Theory	35
3.1.4	Bifurcation from a Manifold of Homoclinic Solutions	38
3.1.5	Applications to Impulsive Differential Equations	40
3.2	Transversal Homoclinic Orbits	44
3.2.1	Higher Dimensional Difference Equations	44
3.2.2	Bifurcation Result	45
3.2.3	Applications to McMillan Type Mappings	51
3.2.4	Planar Integrable Maps with Separatrices	54
3.3	Singular Impulsive ODEs	55
3.3.1	Singular ODEs with Impulses	55
3.3.2	Linear Singular ODEs with Impulses	56
3.3.3	Derivation of the Melnikov Function	64
3.3.4	Examples of Singular Impulsive ODEs	68
3.4	Singularly Perturbed Impulsive ODEs	70
3.4.1	Singularly Perturbed ODEs with Impulses	70
3.4.2	Melnikov Function	71
3.4.3	Second Order Singularly Perturbed ODEs with Impulses	72
3.5	Inflated Deterministic Chaos	73
3.5.1	Inflated Dynamical Systems	73
3.5.2	Inflated Chaos	74
References	83
4	Chaos in Ordinary Differential Equations	87
4.1	Higher Dimensional ODEs	87
4.1.1	Parameterized Higher Dimensional ODEs	87
4.1.2	Variational Equations	88
4.1.3	Melnikov Mappings	90
4.1.4	The Second Order Melnikov Function	93
4.1.5	Application to Periodically Perturbed ODEs	95
4.2	ODEs with Nonresonant Center Manifolds	97
4.2.1	Parameterized Coupled Oscillators	97
4.2.2	Chaotic Dynamics on the Hyperbolic Subspace	98
4.2.3	Chaos in the Full Equation	100
4.2.4	Applications to Nonlinear ODEs	105
4.3	ODEs with Resonant Center Manifolds	108
4.3.1	ODEs with Saddle-Center Parts	108
4.3.2	Example of Coupled Oscillators at Resonance	109
4.3.3	General Equations	121

4.3.4	Averaging Method	127
4.4	Singularly Perturbed and Forced ODEs	131
4.4.1	Forced Singular ODEs	131
4.4.2	Center Manifold Reduction	132
4.4.3	ODEs with Normal and Slow Variables	135
4.4.4	Homoclinic Hopf Bifurcation	135
4.5	Bifurcation from Degenerate Homoclinics	136
4.5.1	Periodically Forced ODEs with Degenerate Homoclinics	136
4.5.2	Bifurcation Equation	137
4.5.3	Bifurcation for 2-Parametric Systems	138
4.5.4	Bifurcation for 4-Parametric Systems	144
4.5.5	Autonomous Perturbations	147
4.6	Inflated ODEs	150
4.6.1	Inflated Carathéodory Type ODEs	150
4.6.2	Inflated Periodic ODEs	151
4.6.3	Inflated Autonomous ODEs	154
4.7	Nonlinear Diatomic Lattices	156
4.7.1	Forced and Coupled Nonlinear Lattices	156
4.7.2	Spatially Localized Chaos	157
	References	163
5	Chaos in Partial Differential Equations	167
5.1	Beams on Elastic Bearings	167
5.1.1	Weakly Nonlinear Beam Equation	167
5.1.2	Setting of the Problem	168
5.1.3	Preliminary Results	171
5.1.4	Chaotic Solutions	191
5.1.5	Useful Numerical Estimates	215
5.1.6	Lipschitz Continuity	217
5.2	Infinite Dimensional Non-Resonant Systems	220
5.2.1	Buckled Elastic Beam	220
5.2.2	Abstract Problem	224
5.2.3	Chaos on the Hyperbolic Subspace	224
5.2.4	Chaos in the Full Equation	226
5.2.5	Applications to Vibrating Elastic Beams	227
5.2.6	Planer Motion with One Buckled Mode	227
5.2.7	Nonplaner Symmetric Beams	230
5.2.8	Nonplaner Nonsymmetric Beams	235
5.2.9	Multiple Buckled Modes	238
5.3	Periodically Forced Compressed Beam	242
5.3.1	Resonant Compressed Equation	242
5.3.2	Formulation of Weak Solutions	242
5.3.3	Chaotic Solutions	243
	References	247

6	Chaos in Discontinuous Differential Equations	249
6.1	Transversal Homoclinic Bifurcation	249
6.1.1	Discontinuous Differential Equations	249
6.1.2	Setting of the Problem	250
6.1.3	Geometric Interpretation of Nondegeneracy Condition	255
6.1.4	Orbits Close to the Lower Homoclinic Branches	257
6.1.5	Orbits Close to the Upper Homoclinic Branch	263
6.1.6	Bifurcation Equation	265
6.1.7	Chaotic Behaviour	287
6.1.8	Almost and Quasiperiodic Cases	293
6.1.9	Periodic Case	294
6.1.10	Piecewise Smooth Planar Systems	295
6.1.11	3D Quasiperiodic Piecewise Linear Systems	299
6.1.12	Multiple Transversal Crossings	310
6.2	Sliding Homoclinic Bifurcation	312
6.2.1	Higher Dimensional Sliding Homoclinics	312
6.2.2	Planar Sliding Homoclinics	319
6.2.3	Three-Dimensional Sliding Homoclinics	321
6.3	Outlook	332
	References	332
7	Concluding Related Topics	335
7.1	Notes on Melnikov Function	335
7.1.1	Role of Melnikov Function	335
7.1.2	Melnikov Function and Calculus of Residues	336
7.1.3	Second Order ODEs	340
7.1.4	Applications and Examples	347
7.2	Transverse Heteroclinic Cycles	361
7.3	Blue Sky Catastrophes	369
7.3.1	Symmetric Systems with First Integrals	370
7.3.2	D'Alembert and Penalized Equations	371
	References	373
	Index	375

Chapter 1

Introduction

Many problems in the natural and engineering sciences can be modeled as evolution processes. Mathematically this leads to either discrete or continuous dynamical systems, i.e. to either difference or differential equations. Usually such dynamical systems are nonlinear or even discontinuous and depend on parameters. Consequently the study of qualitative behaviour of their solutions is very difficult. Rather effective method for handling dynamical systems is the bifurcation theory, when the original problem is a perturbation of a solvable problem, and we are interested in qualitative changes of properties of solutions for small parameter variations. Nowadays the bifurcation and perturbation theories are well developed and methods applied by these theories are rather broad including functional-analytical tools and numerical simulations as well [1–13].

Next, one of the fascinating behaviour of nonlinear dynamical systems which may occur is their sensitive dependence on the initial value conditions, which results in a chaotic time behaviour. Chaos is by no means exceptional but a typical property of many dynamical systems in periodically stimulated cardiac cells, in electronic circuits, in chemical reactions, in lasers, in mechanical devices, and in many other models of biology, meteorology, economics and physics. In spite of the fact that it is very difficult to show chaos for general evolution equations, the bifurcation theory based on perturbation methods is a powerful tool for concluding chaos in a rather wide class of parameterized nonlinear dynamical systems. Especially functional-analytical methods are very convenient to show rigorously the existence of chaos in concrete dynamical systems [14–20].

Now we show two well-known simple chaotic mechanical models. First, we consider a periodically forced and damped Duffing equation

$$\dot{x} = y, \quad \dot{y} - x + 2x^3 + \mu_1 y = \mu_2 \cos t \quad (1.0.1)$$

with μ_1, μ_2 being small. Note

$$\ddot{x} + \mu_1 \dot{x} - x + 2x^3 = \mu_2 \cos t$$

describes dynamics of a buckled beam, when only one mode of vibration is considered (cf Section 5.2 and [21]). Particularly, an experimental apparatus in [4, pp. 83–84] is a slender steel beam clamped to a rigid framework which supports two magnets, when x is the beam tip displacement. The apparatus is periodically forced using electromagnetic vibration generator (Figure 1.1).

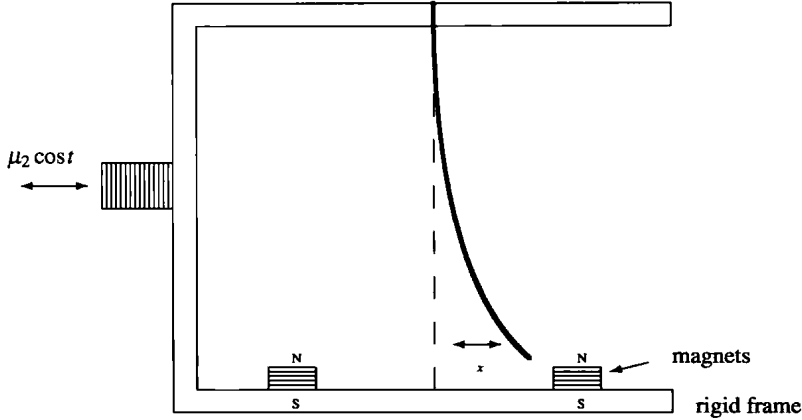


Fig. 1.1 The magneto-elastic beam.

Next, the phase portrait of

$$\dot{x} = y, \quad \dot{y} - x + 2x^3 = 0 \quad (1.0.2)$$

is simply found analytically by analyzing the level sets $\dot{x}^2 - x^2 + x^4 = c \in \mathbb{R} [1, 4, 13]$. Here \mathbb{R} denotes the set of real numbers. There are three equilibria: $(0, 0)$ is hyperbolic and $(\pm\sqrt{2}/2, 0) \doteq (\pm 0.707107, 0)$ are centers. There is also a symmetric *homoclinic cycle* $\pm \tilde{\gamma}_d(t)$ with $\tilde{\gamma}_d(t) = (\gamma_d(t), \dot{\gamma}_d(t))$ and $\gamma_d(t) = \operatorname{sech} t$. The rest are all periodic solutions. These results are consistent with the above experimental model without damping and external forcing as follows: When attractive forces of the magnets overcome the elastic force of the beam, the beam settles with its tip close to one or more of the magnets: these are centers of (1.0.2). There is also an unstable central equilibrium position of the beam at which the magnetic forces are canceled: this is the unstable equilibrium of (1.0.2) (Figure 1.2).

When $\mu_{1,2}$ are small and not identically zero, in spite of the fact that (1.0.1) is a simply looking equation, its dynamics is very difficult. This is demonstrated in Figure 1.3. We see that there are random oscillations of the beam tip between the two magnets. These chaotic vibrations are also observed in the experimental apparatus of Figure 1.1 as shown in [4, p. 84]. Theoretically it is justified by Lemma 7.2.4. Note that almost all trajectories of the damped case $\mu_1 > 0, \mu_2 = 0$ tend to one of the stable equilibria $(\pm\sqrt{2}/2, 0)$ (cf case A of Figure 1.3).

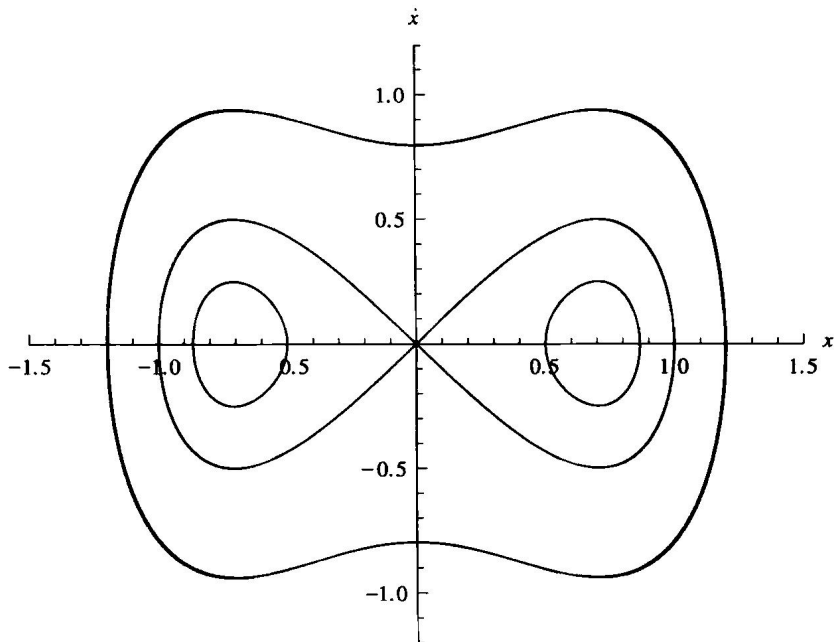


Fig. 1.2 The phase portrait of the Duffing equation (1.0.2).

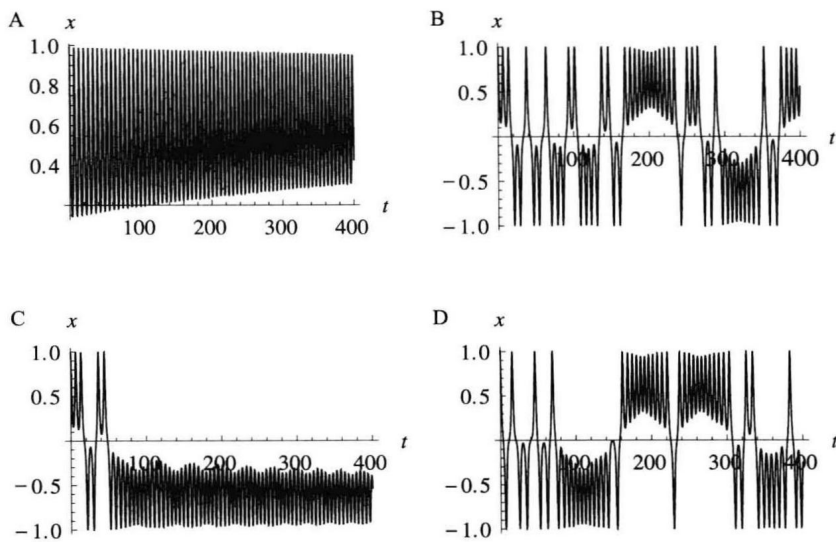


Fig. 1.3 The solution $x(t)$, $0 \leq t \leq 400$ of (1.0.1) for A: $\mu_1 = 0.001, \mu_2 = 0, x(0) = 0.99, \dot{x}(0) = 0$; B: $\mu_1 = 0, \mu_2 = 0.01, x(0) = 0.99, \dot{x}(0) = 0$; C: $\mu_1 = 0.001, \mu_2 = 0.01, x(0) = 0.99, \dot{x}(0) = 0$; D: $\mu_1 = 0.001, \mu_2 = 0.01, x(0) = 1.01, \dot{x}(0) = 0$.

The aim of this book is to show chaos in (1.0.1) analytically. This is presented in Section 4.1 and Subsection 5.2.6: now the *Melnikov function* is given by

$$M(\alpha) = \int_{-\infty}^{\infty} \dot{\gamma}_d(t) (\mu_2 \cos(\alpha + t) - \mu_1 \dot{\gamma}_d(t)) dt = \mu_2 \pi \operatorname{sech} \frac{\pi}{2} \sin \alpha - \mu_1 \frac{2}{3}.$$

When μ_1, μ_2 satisfy

$$|\mu_1| < |\mu_2| \frac{3\pi}{2} \operatorname{sech} \frac{\pi}{2} \doteq 1.87806 |\mu_2|, \quad (1.0.3)$$

clearly there is a *simple zero* α_0 of M , i.e. $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$. Hence by Remark 4.1.6, (1.0.1) is chaotic for μ_1, μ_2 sufficiently small fulfilling (1.0.3). Note (1.0.3) holds for cases B, C, D of Figure 1.3.

The second popular example of chaotic physical model is a damped and forced pendulum consisting of a mass attached to a vertically oscillating pivot point by means of mass-less and inextensible wire described by ODE ([1, p. 278] and [22, p. 216])

$$\ddot{\phi} + \mu_1 \dot{\phi} + \sin \phi = \mu_2 \cos t \sin \phi, \quad (1.0.4)$$

where μ_1, μ_2 are parameters (Figure 1.4).

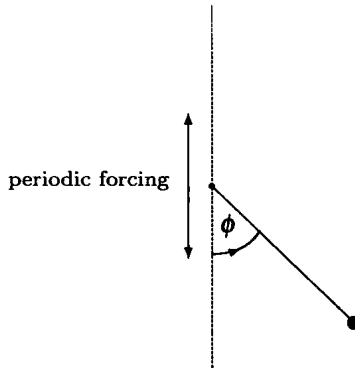


Fig. 1.4 The damped and forced pendulum (1.0.4).

The unperturbed ODE is given by

$$\ddot{\phi} + \sin \phi = 0 \quad (1.0.5)$$

with the phase portrait in Figure 1.5.

Note that $(2k\pi, 0)$ are centers and $((2k+1)\pi, 0)$ are hyperbolic equilibria of (1.0.5) for $k \in \mathbb{Z}$. Here \mathbb{Z} denotes the set of integer numbers. Moreover, $(-\pi, 0)$ and $(\pi, 0)$ are joined by the upper *separatrix* or *heteroclinic orbit* $\tilde{\gamma}_p(t)$ with $\tilde{\gamma}_p(t) = (\gamma_p(t), \dot{\gamma}_p(t))$ and $\gamma_p(t) = 2 \arctan(\sinh t)$. The lower separatrix is $-\tilde{\gamma}_p(t)$.

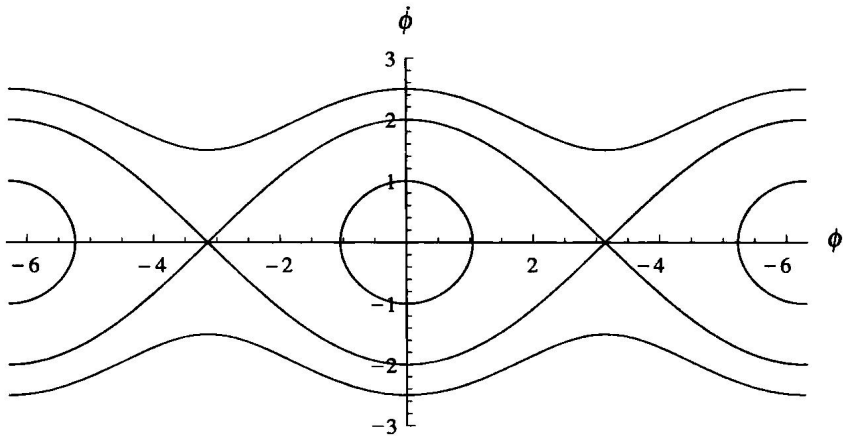


Fig. 1.5 The phase portrait of the pendulum equation (1.0.5).

When $\mu_{1,2}$ are small and not identically zero, (1.0.4) has very difficult dynamics. This is demonstrated in Figure 1.6.

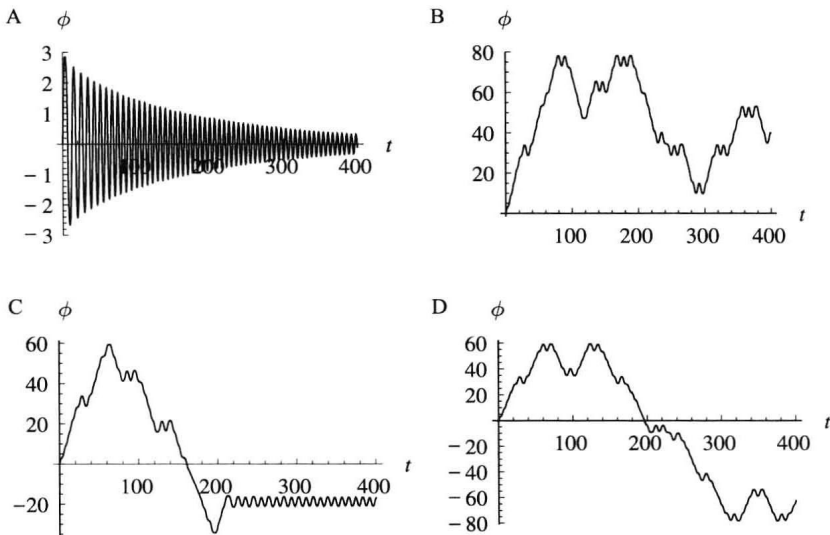


Fig. 1.6 The solution $\phi(t)$, $0 \leq t \leq 400$ of (1.0.4) for A: $\mu_1 = 0.01$, $\mu_2 = 0$, $\phi(0) = 0$, $\dot{\phi}(0) = 2$; B: $\mu_1 = 0.001$, $\mu_2 = 0.1$, $\phi(0) = 0$, $\dot{\phi}(0) = 1.998$; C: $\mu_1 = 0.001$, $\mu_2 = 0.1$, $\phi(0) = 0$, $\dot{\phi}(0) = 2$; D: $\mu_1 = 0.001$, $\mu_2 = 0.1$, $\phi(0) = 0$, $\dot{\phi}(0) = 2.002$.

Now the Melnikov function is given by [1, p. 467]

$$M(\alpha) = \int_{-\infty}^{\infty} \dot{\gamma}_p(t) (\mu_2 \cos(\alpha + t) \sin \gamma_p(t) - \mu_1 \dot{\gamma}_p(t)) dt = -2\pi\mu_2 \operatorname{csch} \frac{\pi}{2} \sin \alpha - 8\mu_1.$$

When μ_1, μ_2 satisfy

$$|\mu_1| < |\mu_2| \frac{\pi}{4} \operatorname{csch} \frac{\pi}{2} \doteq 0.341285 |\mu_2|, \quad (1.0.6)$$

clearly there is a simple zero α_0 of M , i.e. $M(\alpha_0) = 0$ and $M'(\alpha_0) \neq 0$. Hence by Remark 4.1.6, (1.0.4) is chaotic for μ_1, μ_2 sufficiently small fulfilling (1.0.6). Note that (1.0.6) holds for cases B, C, D of Figure 1.6. Note that almost all trajectories of the damped case $\mu_1 > 0, \mu_2 = 0$ tend to the one of the stable equilibria $(2k\pi, 0)$, $k \in \mathbb{Z}$ (cf case A of Figure 1.6).

In summary, examples (1.0.1) and (1.0.4) have the following common features: they are simply looking equations with unpredictable dynamics. But deriving their Melnikov functions, it is easy to show their chaotic behaviour. Consequently, the aim of this book is to present many different discrete and continuous dynamical systems defined on spaces with arbitrarily high dimensions including infinite ones when this Melnikov type analysis is shown to be useful, and then we demonstrate abstract results on concrete examples.

References

1. C. CHICONE: *Ordinary Differential Equations with Applications*, Texts in Applied Mathematics, **34**, Springer, New York, 2006.
2. S. ELAYDI: *An Introduction to Difference Equations*, Springer, New York, 2005.
3. M. FARKAS: *Periodic Motions*, Springer, New York, 1994.
4. J. GUCKENHEIMER & P. HOLMES: *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Springer, New York, 1983.
5. J. K. HALE & H. KOÇAK: *Dynamics and Bifurcations*, Springer, New York, 1991.
6. M.C. IRWIN: *Smooth Dynamical Systems*, Academic Press, London, 1980.
7. Y.A. KUZNETSOV: *Elements of Applied Bifurcation Theory*, Applied Mathematical Sciences, **112**, Springer, New York, 2004.
8. M. MEDVED: *Fundamentals of Dynamical Systems and Bifurcation Theory*, Adam Hilger, Bristol, 1992.
9. C. ROBINSON: *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRC Press, Boca Raton, 1998.
10. M. RONTO & A.M. SAOMOILENKO: *Numerical-Analytic Methods in the Theory of Boundary-Value Problems*, World Scientific Publishing Co., Singapore, 2001.
11. J.J. STOKER: *Nonlinear Vibrations in Mechanical and Electrical Systems*, Interscience, New York, 1950.
12. A. STUART & A.R. HUMPHRIES: *Dynamical Systems and Numerical Analysis*, Cambridge University Press, Cambridge, 1999.
13. M. TABOR: *Chaos and Integrability in Nonlinear Dynamics: An Introduction*, Wiley-Interscience, New York, 1989.
14. J. AWREJCWICZ & M.M. HOLICKE: *Smooth and Nonsmooth High Dimensional Chaos and the Melnikov-Type Methods*, World Scientific Publishing Co., Singapore, 2007.
15. M. CENCINI, F. CECCONI & A. VULPIANI: *Chaos: from Simple Models to Complex Systems*, World Scientific Publishing Co., Singapore, 2009.
16. R. CHACÓN: *Control of Homoclinic Chaos by Weak Periodic Perturbations*, World Scientific Publishing Co., Singapore, 2005.
17. S.N. CHOW & J.K. HALE: *Methods of Bifurcation Theory*, Springer-Verlag, New York, 1982.